# TGD as a Generalized Number Theory I: p-Adicization Program

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## Contents

**Abstract**

The vision about a number theoretic formulation of quantum TGD is based on the gradual accumulation of wisdom coming from different sources. The attempts to find a formulation allowing to understand real and p-adic physics as aspects of some more general scenario have been an important stimulus and generated a lot of, not necessarily mutually consistent ideas, some of which might serve as building blocks of the final formulation.

The first part of the 3-part chapter is devoted to the p-adicization program attempting to construct physics in various number fields as an algebraic continuation of physics in the field of rationals (or appropriate extension of rationals). The program involves in essential manner the generalization of number concept obtained by fusing reals and p-adic number fields to a larger structure by gluing them together along common rationals. Highly non-trivial number theoretic conjectures are an i outcome of the program.

*1. Real and p-adic regions of the space-time as geometric correlates of matter and mind*

The solutions of the equations determining space-time surfaces are restricted by the requirement that the imbedding space coordinates are real. When this is not the case, one might apply instead of a real completion with some rational-adic or p-adic completion: this is how rational-adic p-adic physics could emerge from the basic equations of the theory. One could interpret the resulting rational-adic or p-adic regions as geometrical correlates for 'mind stuff'.

p-Adic non-determinism implies extreme flexibility and therefore makes the identification of the p-adic regions as seats of cognitive representations very natural. Unlike real completion, p-adic completions preserve the information about the algebraic extension of rationals and algebraic coding of quantum numbers must be associated with 'mind like' regions of space-time. p-Adics and reals are in the same relationship as map and territory.

The implications are far-reaching and consistent with TGD inspired theory of consciousness: p-adic regions are present even at elementary particle level and provide some kind of model of 'self' and external world. In fact, p-adic physics must model the p-adic cognitive regions representing real elementary particle regions rather than elementary particles themselves!

*2. The generalization of the notion of number*

The unification of real physics of material work and p-adic physics of cognition and intentionality leads to the generalization of the notion of number field. Reals and various p-adic number fields are glued along their common rationals (and common algebraic numbers too) to form a fractal book like structure. Allowing all possible finite-dimensional extensions of p-adic numbers brings additional pages to this "Big Book".

At space-time level the book like structure corresponds to the decomposition of space-time surface to real and p-adic space-time sheets. This has deep implications for the view about cognition. For instance, two points infinitesimally near p-adically are infinitely distant in real sense so that cognition becomes a cosmic phenomenon.

*3. p-Adicization by algebraic continuation*

One general idea which results as an outcome of the generalized notion of number is the idea of a universal function continuable from a function mapping rationals to rationals or to a finite extension of rationals to a function in any number field. This algebraic continuation is analogous to the analytical continuation of a real analytic function to the complex plane. Rational functions with rational coefficients are obviously functions satisfying this constraint. Algebraic functions with rational coefficients satisfy this requirement if appropriate finite-dimensional algebraic extensions of p-adic numbers are allowed. Exponent function is such a function.

For instance, residue calculus might be generalized so that the value of an integral along the real axis could be calculated by continuing it instead of the complex plane to any number field via its values in the subset of rational numbers forming the rim of the book like structure having number fields as its pages. If the poles of the continued function in the finitely extended number field allow interpretation as real numbers it might be possible to generalize the residue

formula. One can also imagine of extending residue calculus to any algebraic extension. An interesting situation arises when the poles correspond to extended p-adic rationals common to different pages of the "great book". Could this mean that the integral could be calculated at any page having the pole common. In particular, could a p-adic residue integral be calculated in the ordinary complex plane by utilizing the fact that in this case numerical approach makes sense.

Algebraic continuation is the basic tool of p-adicization program. Entire physics of the TGD Universe should be algebraically continuable to various number fields. Real number based physics would define the physics of matter and p-adic physics would describe correlates of cognition and intentionality. The basic stumbling block of this program is integration and algebraic continuation should allow to circumvent this difficulty. Needless to say, the requirement that the continuation exists must pose immensely tight constraints on the physics.

Due to the fact that real and p-adic topologies are fundamentally different, ultraviolet and infrared cutoffs in the set of rationals are unavoidable notions and correspond to a hierarchy of different physical phases on one hand and different levels of cognition on the other hand. Two types of cutoffs are predicted: p-adic length scale cutoff and a cutoff due to phase resolution. The latter cutoff seems to correspond naturally to the hierarchy of algebraic extensions of p-adic numbers and Beraha numbers $B_n = 4cos^2(\pi/n)$, $n \geq 3$ related closely to the hierarchy of quantum groups, braid groups, and $II_1$ factors of von Neumann algebra. This cutoff hierarchy seems to relate closely to the hierarchy of cutoffs defined by the hierarchy of subalgebras of the super-canonical algebra defined by the hierarchy of sets $(z_1, ...z_n)$, where $z_i$ are the first $n$ non-trivial zeros of Riemann Zeta. Hence there are good hopes that the p-adicization program might unify apparently unrelated branches of mathematics.

*4. Number theoretic democracy*

The interpretation allows all finite-dimensional extensions of p-adic number fields and even infinite-P p-adics. The notion arithmetic quantum theory generalizes to include Gaussian and Eisenstein variants of infinite primes and corresponding arithmetic quantum field theories. Also the notion of p-adicity generalizes: it seems that one can indeed assign to Gaussian and Eisenstein primes what might be called G-adic and E-adic numbers. These number fields seem to be tailor made for modelling logarithmic spirals which represent the basic fractal like structures in a living matter and excitable media.

p-Adicization by algebraic continuation gives hopes of continuing quantum TGD from reals to various p-adic number fields. The existence of this continuation poses extremely strong constraints on theory and has already now inspired several number theoretic conjectures.

# 1 Introduction

The vision about a number theoretic formulation of quantum TGD is based on the gradual accumulation of wisdom coming from different sources. The attempts to find a formulation allowing to understand real and p-adic physics as aspects of some more general scenario have been an important stimulus and generated a lot of, not necessarily mutually consistent ideas, some of which might serve as building blocks of the final formulation. The original chapter representing the number theoretic vision as a consistent narrative grew so massive that I decided to divide it to three parts.

The first part is devoted to the p-adicization program attempting to construct physics in various number fields as an algebraic continuation of physics in the field of rationals (or appropriate extension of rationals). The program involves in essential manner the generalization of number concept obtained by fusing reals and p-adic number fields to a larger structure by gluing them together along common rationals. Highly non-trivial number theoretic conjectures are an i outcome of the program.

Second part focuses on the idea that the tangent spaces of space-time and imbedding space can be regarded as 4- *resp.* 8-dimensional algebras such that space-time tangent space defines sub-algebra of imbedding space. The basic candidates for the pair of algebras are hyper-quaternions and hyper-octonions.

The great idea is that space-time surfaces $X^4$ correspond to hyper-quaternionic or co-hyper-quaternionic sub-manifolds of $HO = M^8$. The possibility to assign to $X^4$ a surface in $M^4 \times CP_2$ means a number theoretic analog for spontaneous compactification. Of course, nothing dynamical is involved: a dual relation between totally different descriptions of the physical world are in question. In the spirit of generalized algebraic geometry one can ask whether hyper-quaternionic space-time surfaces and their duals could be somehow assigned to hyper-octonion analytic maps $HO \rightarrow HO$, and there are good arguments suggesting that this is the case.

The third part is devoted to infinite primes. Infinite primes are in one-one correspondence with the states of super-symmetric arithmetic quantum field theories. The infinite-primes associated with hyper-quaternionic and hyper-octonionic numbers are the most natural ones physically because of the underlying Lorentz invariance, and the possibility to interpret them as momenta with mass squared equal to prime. Most importantly, the polynomials associated with hyper-octonionic infinite primes have automatically space-time surfaces as representatives so that space-time geometry becomes a representative for the quantum states.

## 1.1 The painting is the landscape

The work with TGD inspired theory of consciousness has led to a vision about the relationship of mathematics and physics. Physics is not in this view a model of reality but objective reality itself: painting is the landscape. One can also equate mathematics and physics in a well defined sense and the often implicitly assumed Cartesian theory-world division disappears. Physical realities are mathematical ideas represented by configuration space spinor fields (quantum histories) and quantum jumps between quantum histories give rise to consciousness and to the subjective existence of mathematician.

The concrete realization for the notion algebraic hologram based on the notion of infinite prime is a second new element. The notion of infinite rationals leads to the generalization of also the notion of finite number since infinite-dimensional space of real units obtained from finite rational valued ratios $q$ of infinite integers divided by $q$. These units are not units in p-adic sense. The generalization to the quaternionic and octonionic context means that ordinary space-time points become infinitely structured and space-time point is able to represent even the quantum physical state of the Universe in its algebraic structure. Single space-time point becomes the Platonia not visible at the level of real physics but essential for mathematical cognition.

In this view evolution becomes also evolution of mathematical structures, which become more and more self-conscious quantum jump by quantum jump. The notion of p-adic evolution is indeed a basic prediction of quantum TGD but even this vision might be generalized by allowing rational-adic topologies for which topology is defined by a ring with unit rather than number field.

## 1.2 Real and p-adic regions of the space-time as geometric correlates of matter and mind

The solutions of the equations determining space-time surfaces are restricted by the requirement that the components of quaternions are real. When this is not the case, one might apply instead of a real completion with some rational-adic or p-adic completion: this is how rational-adic p-adic physics emerges from basic equations of the theory. One can interpret the resulting rational-adic or p-adic regions as geometrical correlates for 'mind stuff'.

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The implications are far-reaching and consistent with TGD inspired theory of consciousness: p-adic regions are present even at elementary particle level and provide some kind of model of 'self' and external world. In fact, p-adic physics must model the p-adic cognitive regions representing real elementary particle regions rather than elementary particles themselves!

## 1.3 The generalization of the notion of number

The unification of real physics of material work and p-adic physics of cognition and intentionality leads to the generalization of the notion of number field. Reals and various p-adic number fields are glued along their common rationals (and common algebraic numbers too) to form a fractal book like structure. Allowing all possible finite-dimensional extensions of p-adic numbers brings additional pages to this "Big Book".

At space-time level the book like structure corresponds to the decomposition of space-time surface to real and p-adic space-time sheets. This has deep implications for the view about cognition. For instance, two points infinitesimally near p-adically are infinitely distant in real sense so that cognition becomes a cosmic phenomenon.

## 1.4 Zero energy ontology, cognition, and intentionality

One could argue that conservation laws forbid p-adic-real phase transitions in practice so that cognitions (intentions) realized as real-to-padic (p-adic-to-real) transitions would not be possible. The situation changes if one accepts what might be called zero energy ontology [C1, C2].

### 1.4.1 Zero energy ontology classically

In TGD inspired cosmology [D5] the imbeddings of Robertson-Walker cosmologies are vacuum extremals. Same applies to the imbeddings of Reissner-Nordström solution [D3] and in practice to all solutions of Einstein's equations imbeddable as extremals of Kähler action. Since four-momentum currents define a collection of vector fields rather than a tensor in TGD, both positive and negative signs for energy corresponding to two possible assignments of the arrow of the geometric time to a given space-time surface are possible. This leads to the view that all physical states have vanishing net energy classically and that physically acceptable universes are creatable from vacuum.

The result is highly desirable since one can avoid unpleasant questions such as "What are the net values of conserved quantities like rest mass, baryon number, lepton number, and electric charge for the entire universe?", "What were the initial conditions in the big bang?", "If only single solution of field equations is selected, isn't the notion of physical theory meaningless since in principle it is not possible to compare solutions of the theory?". This picture fits also nicely with the view that entire universe understood as quantum counterpart 4-D space-time is recreated in each quantum jump and allows to understand evolution as a process of continual re-creation.

### 1.4.2 Zero energy ontology at quantum level

Also the construction of S-matrix [C2] leads to the conclusion that all physical states possess vanishing conserved quantum numbers. Furthermore, the entanglement coefficients between positive and negative energy components of the state define a unitary S-matrix. S-matrix thus becomes a property of the zero energy state and physical states code by their structure what is usually identified as quantum dynamics.

Also the transitions between zero energy states are possible but general arguments lead to the conclusion that the corresponding S-matrix is almost trivial. This finding, which actually forced the new view about S-matrix, is highly desirable since it explains why positive energy ontology works so well if one forgets effects related to intentional action.

At space-time level this would mean that positive energy component and negative energy component are at a temporal distance characterized by an appropriate p-adic time scale and the integer characterizing the value of Planck constant for the state in question. The scale in question would also characterize the geometric duration of quantum jump and the size scale of space-time region contributing to the contents of conscious experience. The interpretation in terms of a mini bang followed by a mini crunch suggests itself also.

### 1.4.3 Hyper-finite factors of type $\mathrm{II}_1$ and new view about S-matrix

The representation of S-matrix as unitary entanglement coefficients would not make sense in ordinary quantum theory but in TGD the von Neumann algebra in question is not a type I factor as for quantum mechanics or a type III factor as for quantum field theories, but what is called hyper-finite factor of type $\mathrm{II}_1$ [C6]. This algebra is an infinite-dimensional algebra with the almost defining, and at the first look very strange, property that the infinite-dimensional unit matrix has unit trace. The infinite dimensional Clifford algebra spanned by the configuration space gamma matrices (configuration space understood as the space of 3-surfaces, the "world of classical worlds") is indeed very naturally algebra of this kind since infinite-dimensional Clifford algebras provide a canonical representations for hyper-finite factors of type $\mathrm{II}_1$.

### 1.4.4 The new view about quantum measurement theory

This mathematical framework leads to a new kind of quantum measurement theory. The basic assumption is that only a finite number of degrees of freedom can be quantum measured in a given measurement and the rest remain untouched. What is known as Jones inclusions $\mathcal{N} \subset \mathcal{M}$ of von Neumann algebras allow to realize mathematically this idea [C6]. $\mathcal{N}$ characterizes measurement resolution and quantum measurement reduces the entanglement in the non-commutative quantum space $\mathcal{M}/\mathcal{N}$. The outcome of the quantum measurement is still represented by a unitary S-matrix but in the space characterized by $\mathcal{N}$. It is not possible to end up with a pure state with a finite sequence of quantum measurements.

The obvious objection is that the replacement of a universal S-matrix coding entire physics with a state dependent unitary entanglement matrix is too heavy a price to be paid for the resolution of the above mentioned paradoxes. Situation could be saved if the S-matrices have fractal structure. The quantum criticality of TGD Universe indeed implies fractality. The possibility of an infinite sequence of Jones inclusions for hyperfinite type $II_1$ factors isomorphic as von Neumann algebras expresses this fractal character algebraically. Thus one can hope that the S-matrix appearing as entanglement coefficients is more or less universal in the same manner as Mandelbrot fractal looks more or less the same in all length scales and for all resolutions. Whether this kind of universality must be posed as an additional condition on entanglement coefficients or is an automatic consequence of unitarity in type $\mathrm{II}_1$ sense is an open question.

### 1.4.5 The S-matrix for p-adic-real transitions makes sense

In zero energy ontology conservation laws do not forbid p-adic-real transitions and one can develop a relatively concrete vision about what happens in these kind of transitions. The starting point is the generalization of the number concept obtained by gluing p-adic number fields and real numbers along common rationals (expressing it very roughly). At the level of the imbedding space this means that p-adic and real space-time sheets intersect only along common rational points of the imbedding space and transcendental p-adic space-time points are infinite as real numbers so that they can be said to be infinite distant points so that intentionality and cognition become cosmic phenomena.

In this framework the long range correlations characterizing p-adic fractality can be interpreted as being due to a large number of common rational points of imbedding space for real space-time sheet and p-adic space-time sheet from which it resulted in the realization of intention in quantum jump. Thus real physics would carry direct signatures about the presence of intentionality. Intentional behavior is indeed characterized by short range randomness and long range correlations.

One can even develop a general vision about how to construct the S-matrix elements characterizing the process [C2]. The basic guideline is the vision that real and various p-adic physics as well as their hybrids are continuable from the rational physics. This means that these S-matrix elements must be characterizable using data at rational points of the imbedding space shared by p-adic and real space-time sheets so that more or less same formulas describe all these S-matrix elements. Note that also $p_1 \rightarrow p_2$ p-adic transitions are possible.

## 1.5 p-Adicization by algebraic continuation

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For instance, residue calculus might be generalized so that the value of an integral along the real axis could be calculated by continuing it instead of the complex plane to any number field via its values in the subset of rational numbers forming the rim of the book like structure having number fields as its pages. If the poles of the continued function in the finitely extended number field allow interpretation as real numbers it might be possible to generalize the residue formula. One can also imagine of extending residue calculus to any algebraic extension. An interesting situation arises when the poles correspond to extended p-adic rationals common to different pages of the "great book". Could this mean that the integral could be calculated at any page having the pole common. In particular, could a p-adic residue integral be calculated in the ordinary complex plane by utilizing the fact that in this case numerical approach makes sense.

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# 2 How p-adic numbers emerge from algebraic physics?

The new algebraic vision leads to several generalization of the p-adic philosophy. Besides p-adic topologies more general rational-adic topologies are possible. Topology is purely dynamically determined and -adic topologies are quite 'real'. There is a physics oriented review article by Brekke and Freund [27]. The books of Gouvêa and Khrennikov give more mathematics-oriented views about p-adics [28, 26].

This section is written before the discovery that it is possible to generalize the notion of the number field by the fusion reals and various p-adic numbers fields and their extensions together along common rationals (and also common algebraic numbers) to form a book like structure. The interpretation of p-adic physics as physics of intention and cognition removes interpretational problems. This vision provides immediately an answer to many questions raised in the text. In particular, it leads naturally to a complete algebraic democracy. The introduction of infinite primes, which are discussed in next chapter, extends the algebraic democracy even further and gives hopes of describing mathematically also mathematical cognition.

## 2.1 Basic ideas and questions

It is good to list the basic ideas and pose the basic question before more detailed considerations.

### 2.1.1 Topology is dynamical

The dynamical emergence of p-adicity is strongly supported both by the applications of p-adic and algebraic physics. The solutions of polynomial equations involving more than one variable involve roots of polynomials. Only roots in the real algebraic extensions of rationals are allowed since the components of quaternions must be real numbers. When the root is complex in real topology, one can however introduce p-adic topology such that the root exists as a number in a real extension of p-adics. In p-adic context only a finite-dimensional algebraic extension of rational numbers is needed. The solutions of the derivative conditions guaranteing Lagrange manifold property involve p-adic pseudo constants so that the p-adic solutions are non-deterministic. The interpretation is that real roots of polynomials correspond to geometric correlates of matter whereas p-adic regions are geometric correlates of mind in consistency with the p-adic non-determinism.

Does this picture imply the physically attractive working hypothesis stating that the decomposition of infinite prime into primes of lower level corresponds to a decomposition of the space-time surface to various p-adic regions appearing in the definition of the infinite prime? Generating infinite primes correspond to quaternionic rationals and these rationals contain powers of quaternionic primes defining the infinite prime. The convergence of the power series solution of the polynomial equations defining space-time surface might depend crucially on the norms of these rationals in the p-adic topology used. This could actually force in a given space-time region p-adic topology associated with some prime involved in the expansion. This is in complete accordance with the idea that p-adic topologies are topologies of sensory experience and real topology is the topology of reality.

### 2.1.2 Various generalizations of p-adic topologies

p-Adicized quaternions is not a number field anymore. One could allow also rational-adic extensions [26] for which pinary expansions are replaced by expansions in powers of rational. These extensions give rise to rings with unit but not to number fields. In this appoach p-adic, or more generally rational-adic, topology determined by the algebraic number field on a given space-time sheet would be absolutely 'real' rather than mere effective topology. Space-time surface decomposes into regions which look like fractal dust when seen by an observer characterized by different number field unless the observer uses some resolution.

This approach suggests even further generalizations. The original observation stimulated by the work with Riemann hypothesis was that the primes associated with the algebraic extensions of rationals, in particular Gaussian primes and Eisenstein primes, have very attractive physical interpretation. Quaternionic primes and rationals might in turn define what might be regarded as noncommutative generalization of the p-adic and rational-adic topology.

Also infinite-P p-adic topologies are a precisely defined concept using the correspondence between infinite primes and polynomials and infinite-P p-adic numbers are apart from infinitesimals equivalent with reals under canonical identification.

### 2.1.3 ...-Adic topology measures the complexity of the quantum state

The higher the degree of the polynomial, and thus the number of particles in the physical state and its complexity, the higher the algebraic dimension of the rational quaternions. A complete algebraic and quaternion and octonion-dimensional democracy would prevail. Accordingly, space-time topology would be completely dynamical in the sense that space-time contains both rational-adic, p-adic regions, infinite-P p-adic and real regions. Physical evolution could be seen as evolution of mathematical structures in this framework: p-adic topologies would be obviously winners over rational-adic topologies and p-adic length scale hypothesis would select the surviving p-adic topologies. For instance, Gaussian-adic and Eisenstein-adic topologies would in turn be higher level survivers possibly associated with biological systems. Infinite-P p-adic topologies are effectively real topologies and would represent higher levels of consciousness: perhaps mathematical consciousness could be assigned to infinite-P p-adics.

Dimensional democracy would be realized in the sense that one can regard the space-time sheets defining $n$-sheeted topological condensate also as a $4n$-dimensional surface in $H^n$. This hypothesis fixes the interactions associated with the topological condensation, and the hierarchical structure of the topological condensate conforms with the hierarchical ordering of the quaternionic arguments of the polynomials to which infinite primes are mapped. Polynomials (infinite integers) at a given level of hierarchy in turn can be interpreted in terms of formation of bound states by the formation of join along boundaries bonds.

### 2.1.4 Is adelic principle consistent with the dynamical topology?

There is competing, and as it seems, almost diametrically opposite view. Just like adelic formula allows to express the norm of a rational number as product of its p-adic norms, various algebraic number fields and even more general structures such as quaternions allowing the notion of prime, provide a collection of incomplete but hopefully calculable views about physics. The net description gives rise to quantum TGD formulated using real numbers. These descriptions would be like summary over all experiences about world of conscious experiencers characterized by p-adic completions of various four-dimensional algebraic number rationals. What is important is that the descriptions using algebraic number fields or their generalization might be calculable. This view need not be conflict with the dynamical view and one could indeed claim that the p-adic physics associated with various algebraic extensions of rational quaternions provide a model about physics constructed by various conscious observers. For a given quantum state there would be however minimal algebraic extension containing all points of the space-time surface in it.

## 2.2 Are more general adics indeed needed?

The considerations related to Riemann hypothesis inspired the notion of G- and E-adic numbers in which rational prime $p$ is replaced with Gaussian or Eisenstein prime. The notion of Eisenstein prime is so attractive because it makes possible to circumvent the complexification of p-adic numbers for $p\ mod\ 4 = 1$ for which $\sqrt{-1}$ exists as a p-adic number. What forces to take the notion of G-adics very seriously is that Gaussian Mersennes correspond to the p-adic length scale of atomic nucleus and to important biological length scales in the range between 10 nanometers and few micrometers. Also the key role of Golden Mean $\tau$ in biology and self-organizing systems could be understood if $Q(\tau, i)$ defines D-adic topology. Thus there is great temptation to believe that the notion of p-adic number generalizes in these sense that any irreducible associated with real or complex algebraic extension defines generalization of p-adic numbers and that these extensions appear in the algebraic extensions of quaternions.

Thus one must consider seriously also generalized p-adic numbers, D-adics as they were called in the chapter "TGD and Number theory: Riemann Hypothesis". D-adics would correspond to powers series of a prime belonging to a complex algebraic extension of rationals. Quaternions decompose naturally in longitudinal and transversal part and transversal part can be interpreted as a complex algebraic extension of rationals in the case of both $M^4$ and $CP_2$. Thus some irreducibles of this complex extension could define a generalization of p-adic numbers used to define the algebraic extension of rational quaternions reduced to a pair of complex coordinates.

Perhaps one could go even further: quaternion-adics defined as power series of quaternionic primes of norm $p$ suggest themselves. What would be nice that this prime could perhaps be interpreted as a representation for the momentum of corresponding space-time sheets. The components of the prime belong to algebraic extension of rationals and would even code information about external world if the proposed interpretations are correct. One can also ask whether quaternionic primes could define what might be called quaternion-adic algebras and whether these algebras might be a basic element of algebraic physics.

This would mean that space-time topology would code information about the quantum numbers of a physical state. Rings with unit rather than number fields are in question since the p-adic counterparts of quaternionic integers in general fail to have inverse. It must be emphasized that the field property might not be absolutely essential. For instance 'rational-adics' [26], for which prime $p$ is replaced with a rational $q$ such that norm comes as a power of $q$, exists as rings with unit and define topology. Rational-adic topologies could have also quaternionic counterparts.

The idea of q-rational topologies is supported by the physical picture about the correspondence between Fock states and space-time sheets. Single 3-surface can in principle carry arbitrarily high fermion and boson numbers but is unstable to a topological decay to 3-surfaces carrying single fermion and boson states. The translation of this statement to ...-adic context would be that the Fock states associated with infinite primes which correspond to rational-adic quaternionic topologies are unstable against decay to states described by polynomial primes in which each factor corresponds to prime (bosons) or its inverse (fermions) in algebraic extension of quaternions. This tendency to evolve to prime-adic topologies could be seen also as a manifestation of p-adic evolution and self-organization. Rational-adic topologies would be simply losers in the fight for survival against topologies defining number fields. Since also quaternion-adic topologies fail to define number fields they are expected to be losers in the fight for survival. Winners would be ...-adic topologies defining number fields. At the level of Fock states this would mean the unstability of states which contain more than one prime: that this is indeed the case, is one of the basic assumptions of quantum TGD forced by the experimental fact that elementary particles correspond to simplest Fock states associated with configuration space spinors.

## 2.3 Why completion to p-adics necessarily occurs?

There is rather convincing argument in favor of ...-adic physics. Typically one must find zeros of rational functions of several variables. Simplifying somewhat, at the first level one must find zeros of polynomials $P(x_1, x_2)$. Newton's theorem states that the monic polynomial $P_n(y, x) =$

$y^n + a_{n-1}x^{n-1} + ..$ allows a factorization in an algebraically closed number field

$$P(y,x^m) = \prod_k (y - f_k(x)) \ . \tag{1}$$

Here $f_k$ are polynomials and $m$ is integer which divides $n$ and equals to $n$ for an irreducible polynomial $P$. Since the multiplication of $x$ by m:th root of unity ($\zeta_m$) leaves left hand side invariant it must permute the factors on right hand side. Thus one can express the formula also as

$$P(y,x) = \prod_{k=1,..m} (y - f_k(\zeta_m^k x^{1/m})) \ . \tag{2}$$

When number field is not algebraically closed this means that one must introduce an algebraic extension by $m$:th roots of all rationals.

The problem is that these roots are not real in general and one cannot solve the problem by using a completion to complex numbers since only real extensions for the components of quaternion are possible. Only in the region where some of the roots of the polynomial are real, this is possible. The only manner to achieve consistency with the reality requirement is to allow p-adic topology or possibly rational-adic topology: in this case also the algebraic extension allowing m:th roots is always finite-dimensional. For instance, for $m = 2$ p-adic extension of rationals would be 4-dimensional for $p > 2$. The situation is similar for rational-adic topology.

If this argument is correct, one can conclude that real topology is possible only in the regions where real roots of the polynomial equation are possible: in the regions where all roots are complex, p-adicization gives rise to roots in the algebraic extension of p-adics and p-adic topology emerges naturally. This picture provides a precise view about how the space-time surface defined by the polynomial of quaternions decomposes to real and p-adic regions. Also a connection with catastrophe theory [30] emerges: the boundaries of the catastrophe regions where some roots coincide, serve also as boundaries between ...-adic and real regions.

## 2.4 Decomposition of space-time to ...-adic regions

Number-theoretic constraints are important in determining which ...-adic topologies are possible in a given space-time region. There is no hope of building any unique vision unless one poses some general principles. Complete algebraic and topological democracy and the generalization of the notion of p-adic evolution to what might be called rational-adic evolution allow to build plausible and sufficiently general working hypothesis not requiring too much adhoc assumptions and allowing at least mathematical testing. A further natural principle states that the topology for a given region is such that complex extension of rationals is not needed and that the series defining the normal quaternionic coordinate as function of the space-time quaternionic coordinate converges and gives rise to a smooth surface.

### 2.4.1 The power series defining solutions of polynomial equations must converge in some topology

The roots of polynomials of several variables can be expressed as Taylor series. When the root is complex, real topology is not possible and some p-adic topology must be considered. This suggests a very attractive dynamical mechanism of p-adicization. In the regions where the root belongs to a complex extension of rationals in the real topology, one could find those values of $p$ for which the series converges p-adically. The rational numbers characterizing the polynomials associated with the generating infinite primes certainly determine the convergence and the primes for which p-adic convergence occurs are certainly functions of these rationals. Hence it could occur that the p-adic topologies for which convergence occurs correspond to the primes appearing as factors in these rationals.

In this approch topology is a result of dynamics. Note that also the notion of symmetry depends on the region of space-time. Contrary to the basic working hypothesis, ...-adic topology of a given space-time sheet is its 'real' topology rather than being only an effective topology and the topology of space-time is completely dynamical being dictated by algebraic physics and smoothness requirement.

It is also possible that convergence does not occur with respect to any ...-adic topology and in this case the topology would be discrete. This situation would correspond to primordial chaos but still the algebraic formulation and Fock space description of the theory would make sense.

### 2.4.2 Space-time surfaces must be smooth in the completion

The completion must give rise to a smooth or at least continuous ....-adic or real surface satisfying absolute minimization of Kähler action. This requirement might allow only finite number of...-adic topologies for a given space-time region. If the completion involves functions expandable in powers of a (possibly quaternionic) rational $q = m/n$, then the prime factors of $m$ define natural p-adic number fields for which completion is possible. Also $q$ itself could define rational-adic topology. Since the space-time surface decomposes into regions labelled by rationals in an algebraic extension of rationals $q_1$, there is interesting possibility that $q_1$ as such defines the rational-adic topology so that there would be no need to understand why the space-time region labelled by $q$ decomposes into space-time sheets labelled by the prime factors of $q$.

Whatever the details of the coding are, the coding would mean that the quantum numbers associated with the space-time sheet would determine the generalized ...-adic topology associated with it. The information about quantum systems would be mapped to space-time physics and the coding of quantum numbers to ...-adic topology would solve at a general level the problem how the information about quantum state is coded into the structure of space-time.

## 2.5 Universe as an algebraic hologram?

Quaternionic primes have a natural indentification as four-momenta. If the Minkowski norm for the quaternion is defined using the algebraic norm of the real extension of rationals involved with the state, mass squared is integer-valued as in super-conformal theories. The use of the algebraic norm means a loss of information carried by the units of the real algebraic extension $K(\theta)$ (see the appendix of this chapter). Hence one can say that besides ordinary elementary particle quantum numbers there are algebraic quantum numbers which presumably carry algebraic information. Very effective coding of information about quantum numbers becomes possible and these quantum numbers commute with ordinary quantum numbers. This information does not become manifest for matter-like regions where a real completion of rationals are used. In p-adic regions representing geometric correlates of mind the situation is different since p-adic number field in question is a finite algebraic extension of rationals.

Almost every calculation is approximation and completion to reals or p-adics makes possible to measure how good the approximation is. Real numbers are extremely practical in this respect but the failure of the real number based physics is that it reduces number to a mere quantity having a definite size but no number-theoretical properties. This is practical from the point of view of numerics but means huge loss of capacity for information storage and representation. In algebraic number theory number contains representation for its construction recipe. It seems that the correct manner to see numbers is as elements of the state space provided by the algebraic extension. p-Adic physics using p-adic versions of the algebraic extensions does not lead to a loss of this information unlike real physics. Thus the basic topology of the space-time sheet could code the quantum numbers associated with it.

Since the algebraic extension of rationals, and hence also of p-adics, depends on the number of particles present in the Fock state coded by the infinite prime, the only possible interpretation is that the additional quantum numbers code information about the many-particle state. Hence the idea about 'cognitive representation' of the fractal quantum numbers of particles of the external world suggests itself naturally. In particular, the degree of the minimal polynomial for the real extension $Q(\theta)$ is $n$, where $n$ is the number of particles in the Fock state in the casethe resulting state represents infinite prime. This means that there are $n-1$ quantum numbers represented by fractal scalings (see Appendix for Dirichlet's unit theorem). The interpretation as a representation for the fractal quantum numbers representing information about states of other particles in the system suggests itself. One cannot exclude the possibility that the fractal quantum numbers represent momenta or some other quantum numbers of other particles.

If this rather un-orthodox interpretation is correct, then cognitive representations are present already at the elementary particle level in p-adic regions associated with particles and are realized as algebraic holograms. Universe as a Computer consisting of sub-computers mimicking each other would be realized already at the elementary particle level. This view is consistent with the TGD inspired theory of consciousness. Algebraic physics would also make possible kind of a Gödelian loop by providing a representation for how the information about the structure of a physical system is coded into its properties.

This view has also immediate implications for complexity theory. The dimension of the minimal algebraic extension containing the algebraic number is a unique measure for its complexity. More concretely: the degree of the minimal polynomial measures the complexity. Everyone can solve second order polynomial but very few of us remembers formulas for the roots of fourth order polynomials. For higher orders quadratures do not even exist. Of course, numbers represent typically coordinates and this is consistent with the general coordinate invariance only if some preferred coordinates exist. In TGD based physics these coordinates exist: imbedding space allows (apart from isometries) unique coordinates in which the components of the metric tensor are rational functions of the coordinates.

Similar realization is fundamental in the second almost-proof of Riemann hypothesis described in the chapter "Riemann Hypothesis and Physics". In this case $\zeta$ is interpreted as an element in an infinite-dimensional algebraic extension of rationals allowing all roots of rationals. The vanishing of $\zeta$ requires that all components of this infinite-dimensional vector contain a common rational factor which vanishes. This is possible only if an infinite number of partition functions in the product representation of the modulus squared of $\zeta$ are rational and their product vanishes. This implies Riemann hypothesis. The assumption that only square roots of rationals are needed is very probably wrong and must be replaced with the assumption that $p^{iy}$ is algebraic numbers when $z = 1/2 + iy$ is zero of $\zeta$ for any prime $p$. It is quite possible that the almost-proof survives this generalization.

The notion of Platonia discussed already in the introduction adds cognition to this picture and allows to understand where all those mathematical structures continually invented by mathematicians but not realized physically in the conventional sense of the word reside. This notion takes also the notion of algebraic hologram to its extreme by making space-time points infinitely structured.

## 2.6 How to assign a p-adic prime to a given real space-time sheet?

p-Adic mass calculations force to assign p-adic prime also to the real space-time sheets and the longstanding problem is how this p-adic prime, or possibly many of them, are determined. Number theoretic view about information concept provides a possible solution of this long-standing problem.

### 2.6.1 Number theoretic information concept

The notion of information in TGD framework differs in some respects from the standard notion.

1. The definition of the entropy in p-adic context is based on the notion p-adic logarithm depending on the p-adic norm of the argument only ($Log_p(x) = Log_p(|x|_p) = n$) [H2]. For rational- and even algebraic number valued probabilities this entropy can be regarded as a real number. The entanglement entropy defined in this manner can be negative so that the entanglement can carry genuine positive information. Rationally/algebraically entangled p-adic system has a positive information content only if the number of the entangled state pairs is proportional to a positive power of the p-adic prime $p$.

2. This kind of definition of entropy works also in the real-rational/algebraic case and makes always sense for finite ensembles. This would have deep implications. For ordinary definition of the entropy NMP [H2] states that entanglement is minimized in the state preparation process. For the number theoretic definition of entropy entanglement could be generated during state preparation for both p-adic and real sub-systems, and NMP forces the emergence of p-adicity (say the number of entangled state is power of prime). The fragility of quantum coherence is the basic problem of quantum computations and the good news would be that Nature itself (according to TGD) tends to stabilize quantum coherence both in the real and p-adic contexts.

3. Quantum-classical correspondence suggests that the notion of information is well defined also at the space-time level. In the presence of the classical non-determinism of Kähler action and p-adic non-determinism one can indeed define ensembles, and therefore also probability distributions and entropies. For a given space-time sheet the natural ensemble consists of the deterministic pieces of the space-time sheet regarded as different states of the same system.

### 2.6.2 Life as islands of rational/algebraic numbers in the seas of real and p-adic continua?

The possibility to define entropy differently for rational/algebraic entangl3ment raises deep questions.

1. Is physics rational/algebraic at Hilbert space level or does the rational/algebraic entanglement represent only a special kind of entanglement for which the number theoretic definition of entropy makes sense? If rational/algebraic entanglement corresponds to a bound state entanglement then the second option seems more sensible and has quite dramatic implications. For instance, bound-unbound and living-dead dichotomies would correspond to rational/irrational or algebraic/transcendental dichotomy. Life would correspond to islands of rationality/algebraicity in the seas of real and p-adic continua.

2. Life would metaphorically reside at the rational/algebraic intersection of reals and p-adics/algebraic extensions of p-adics. Does this plus quantum-classical correspondence mean that life is a boundary phenomenon at the space-time level: real and p-adic space-time sheets, action and intention, meet along common rational/algebraic points at the boundaries of the real space-time sheets?

3. Does life corresponds to rational or algebraic entanglement? Algebraic option would maximize the size of the living sector of the state space. Rational numbers are common for reals and all p-adics: in algebraic case this holds true only if one introduces algebraic extensions of p-adics. This might make rationals preferred.

### 2.6.3 Does space-time sheet represent integer and its prime factorization?

A long-standing problem of quantum TGD is how to associate to a given real space-time sheet a (not necessarily) unique p-adic prime as required by the p-adic length scale hypothesis. One could achieve this by requiring that for this prime the negentropy associated with the ensemble is maximal. The simplest hypothesis is that a real space-time sheet consisting of $N$ deterministic pieces corresponds to p-adic prime defining the largest factor of $N$. One could also consider a more general possibility. If $N$ contains $p^n$ as a factor, then the real fractality above n-ary p-adic length scale $L_p(n) = p^{(n-1)/2}L_p$ corresponds to smoothness in the p-adic topology. This option is more attractive since it predicts that the fundamental p-adic length scale $L_p$ for a given $p$ can be effectively replaced by any integer multiple $NL_p$, such that $N$ is not divisible by $p$. There is indeed a considerable evidence for small $p$ p-adicity in long length scales. For instance, genetic code and the appearance of binary pairs like cell membrane consisting of liquid layers suggests 2-adicity in nano length scales. This view means that the fractal structure of a given real space-time sheet represents both an integer $N$ and its decomposition to prime factors physically. This obviously conforms with the physics as a generalized number theory vision.

Quantum-classical correspondence suggests that quantum computation processes might have counterparts at the level of space-time. An especially interesting process of this kind is the factorization of integers to prime factors. The classical cryptography relies on the fact that the factorization of large integers to prime factors is a very slow process using classical computation: the time needed to factor 100 digit number using modern computer would take more than the recent age of the universe. For quantum computers the factorization is achieved very rapidly using the famous Shor's algorithm. Does the factorization process indeed have a space-time counterpart?

Suppose that one can map the integer $N$ to be factored to a real space-time sheet with $N$ deterministic pieces. If one can measure the powers $p_i^{n_i}$ of primes $p_i$ for which the fractality above the appropriate p-adic length scale looks smoothness in the p-adic topology, it is possible to deduce the factorization of $N$ by direct physical measurements of the p-adic length scales characterizing the representative space-time sheet (say from the resonance frequencies of the radiation associated with the space-time sheet). If only the p-adic topology corresponding to the largest prime $p_1$ is realized in this manner, one can deduce first it, and repeat the process for $N/p_1^n$, and so on, until the full factorization is achieved. A possible test is to generate resonant radiation in a wave guide of having length which is an integer multiple of the fundamental p-adic length scale and to see whether frequencies which correspond to the factors of $N$ appear spontaneously.

## 2.7 Gaussian and Eistenstein primes and physics

Gaussian and Eisenstein primes could give rise to what might be called G- and E-adicities and also these -adicities might be of physical interest.

### 2.7.1 Gaussian and Eisenstein primes and elementary particle quantum numbers

The properties of Gaussian and Eisenstein primes have intriguing parallels with quantum TGD at the level of elementary particle quantum numbers.

1. The lengths of the complex vectors defined by the non-degenerate Gaussian and Eisenstein primes are square roots of primes as are also the preferred p-adic length scales $L_p$: this suggests a direct connection with quantum TGD.

2. Each non-degenerate (purely real or imaginary) Gaussian prime of given norm $p$ corresponds to 8 different complex numbers $G = \pm r \pm is$ and $G = \pm s \pm ir$. This is the number of different spin states for the imbedding space spinors and also for the color states of massless gluons (note that in TGD quark color is not spin like quantum number but is analogous to

orbital angular momentum). Complex conjugation might be interpreted as a representation of charge conjugation and multiplication by $\pm 1, \pm i$ could give rise to different spin states. The 4-fold degeneracy associated with the $p\ mod\ 4 = 3$ Gaussian primes could correspond to the quartet of massless electro-weak gauge bosons with a given helicity $[(\gamma, Z^0) \leftrightarrow \pm p)$ and $(W^+, W^-) \leftrightarrow \pm ip]$.

3. For Eisenstein prime $E_{p_1}$ the multiplication by $\pm i$ does not respect the rationality of the real part of $|Z_{p_1}|^2$ and the number of states is reduced to four. Eisenstein primes $r + isw$ and $s + irw$ have however the same norm squared so that also now the 8-fold degeneracy is present. When $p_1^{iy}$ is of the general form $r + i\sqrt{k}s$ this degeneracy is not present.

4. The basic character of the quark color is triality realized as phases $w$ which are third roots of unity. The fact that the phases are associated with the Eisenstein primes suggests that they might provide a representation of quark color. One can indeed multiply any Eisenstein prime in the product decomposition by factor 1, $w$ or $\overline{w}$ and the interpretation is that the three primes represent three color states of quark. The obvious interpretation is that each factor $Z_{p_1}$ with $p_1\ mod\ 4 = 1$ could represent 8 possible leptonic states. Each factor $Z_{p_1}$ satisfying $p_1\ mod\ 4 = 3$ and $p_1\ mod\ 3 = 1$ conditions simultaneously would correspond to a product of Eisenstein prime with Eisenstein phase and each prime $p_i$ associated with Eisenstein phase would correspond to one color state of quark. Even a number theoretical counterpart of color confinement could be imagined.

   There is also a further interesting analogy supporting the idea about number theoretical counterpart of the quark color. $\zeta$ decomposes into a product $\zeta_1 \times \zeta_3$, such that $\zeta_1$ is the product of $p\ mod\ 4 = 1$ partition functions and $\zeta_3$ the product of $p\ mod\ 4 = 3$ partition functions. This decomposition reminds of the leptonic color singlets and color triplet of quarks. Rather interestingly, leptons and quarks correspond to Ramond and Neveu-Schwartz type super Virasoro representations and the fields of N-S representation indeed contain square roots of complex variable existing p-adically for $p\ mod\ 4 = 3$.

5. What about the most general factors $r+is\sqrt{k}$? Can one assign some kind of color degeneracy also with these factors? It seems that this is the case. One can always find phase factors of type $U_\pm = (r \pm is\sqrt{k})/n$ with minimal values of $n$ $(r^2 + s^2k = n^2)$. The factors $1, U_\pm$ clearly give rise to a 3-fold degeneracy analogous to color degeneracy.

6. What about interpretation of the components of the complex integers? For Super Virasoro representations basic quantum numbers of this kind correspond to energy and longitudinal momentum. This suggests the interpretation of $r^2 + s^2k$ as energy, $r^2 - s^2k$ as mass, and $2rs\sqrt{k}$ as momentum. For the squares $r^2 - s^2 + (2rs - s^2)w$ of Eisenstein primes $r^2 - s^2/2 - rs$ corresponds to mass, $r^2 + s^2 - rs$ to energy, and $(2rs - s^2)\sqrt{3}/2$ to momentum. Note that the sign of mass changes for Gaussian primes in the interchange $r \leftrightarrow s$. The fact that the hexagonal lattice defined by Eisenstein integers correspond to the root lattice of $SU(3)$ group means that energy, momentum and mass corresponds to the sides of the triangles in the root lattice of color group.

### 2.7.2 G-adic, E-adic and even more general fractals?

Still one line of thoughts relates to the possibility to generalize the notion of p-adicity so that could speak about G-adic and E-adic number fields. The properties of the Gaussian and Einsenstein primes indeed strongly suggest a generalization for the notion of p-adic numbers to include what might be called G-adic or E-adic numbers. In fact, the argument generalizes to the case of all nine $\sqrt{-d}$ type extensions of rationals allowing a unique prime decomposition so that one might perhaps speak about D-adics.

1. Consider for definiteness Gaussian primes. The basic point is that the decomposition into a product of prime factors is unique. For a given Gaussian prime one could consider the representation of the algebraic extension involved (complex integers in the case of Gaussian primes) as a ring formed by the formal power series

$$G = \sum_n z_n G_p^n \quad . \tag{3}$$

Here $z_n$ is Gaussian integer with norm smaller than $|G_p|$, which equals to $p$ for $p\ mod\ 4 = 3$ and $\sqrt{p}$ for $p\ mod\ 4 = 1$.

2. If any Gaussian integer $z$ has a unique expansion in powers of $G_p$ such that coefficients have norm squared smaller than $p$, modulo $G$ arithmetics makes sense and one can construct the inverse of $G$ and number field results. This is the case if Gaussian integers behave with respect to modulo $G_p$ arithmetics like finite field $G(p,2)$. For $p\ mod\ 4 = 1$ the extension of the p-adic numbers by introducing $\sqrt{-1}$ as a unit is not possible since $\sqrt{-1}$ exists as a p-adic number: the proposed structure might perhaps provide the counterpart of the p-adic complex numbers in the case$p\ mod\ 4 = 1$. Thus the question is whether one could regard Gaussian p-adic numbers as a natural complexification of p-adics for $p\ mod\ 4 = 1$, perhaps some kind of square root of $R_p$, and if they indeed form a number field, do they reduce to some known algebraic extension of $R_p$?

3. In the case of Eisenstein numbers one can identify the coefficients $z_n$ in the formal power series $E = \sum z_n E_p^n$ as Eisenstein numbers having modulus square smaller than $p$ associated with $E_p$ and similar argument works also in this case.

4. As already noticed, in the case of complex extensions of form $r + \sqrt{-d}s$ a unique prime factorization is obtained only in nine cases corresponding to $d = 1, 2, 3, 7, 11, 19, 46, 67, 163$ [18]. The poor man's argument above does not distinguish between G- and E-adics ($d = 1, 3$) and these extensions.One might perhaps call this extensions generally D-adics. This suggests that generalized p-adics could exist also in this case. In fact, the generalization p-adics could make sense also for higher-dimensional algebraic extensions allowing unique prime decomposition. For $d = 2$ complex algebraic primes are of form $r + s\sqrt{-2}$ satisfying the condition $r^2 + 2s^2 = p$. For $d > 2$ complex algebraic primes are of form $(r + s\sqrt{-d})/2$ such that both $r$ and $s$ are even or odd. Quite generally, the condition $p\ mod\ d = k^2$ must be satisfied. $\sqrt{-d}$ corresponds to a root of unity only for $d = 1$ and $d = 3$ so that the powers of a complex primes in this case have a finite number of possible phase angles: this might make G- and E-adics physically special.

TGD suggests rather interesting physical applications of D-adics.

1. What is interesting from the physics point of view is that for $p\ mod\ 4 = 1$ the points $D_p^n$ are on the logarithmic spiral $z_n = p^{n/2} exp(in\phi_0/2)$, where $\phi$ is the phase associated with $D_p^2$. The logarithmic spiral can be written also as $\rho = exp(nlog(p)\phi/\phi_0)$. This reminds strongly of the logarithmic spirals, which are fractal structures frequently encountered in self-organizing systems: D-adics might provide the mathematics for the modelling of these structures.

2. p-Adic length scale hypothesis should hold true also for Gaussian primes, in particular, Gaussian Mersennes of form $(1 \pm i)^k - 1$ should be especially interesting from TGD point of view.

   i) The integers $k$ associated with the lowest Gaussian Mersennes are following: $2, 3, 5, 7, 11, 19, 29, 47, 73, 79, 113$. $k = 113$ corresponds to the p-adic length scale associated with the atomic nucleus and muon.

Thus all known charged leptons, rather than only $e$ and $\tau$, as well as nuclear physics length scale, correspond to Mersenne primes in the generalized sense.

ii) The primes $k = 151, 157, 163, 167$ define perhaps the most fundamental biological length scales: $k = 151$ corresponds to the thickness of the cell membrane of about ten nanometers and $k = 167$ to cell size about 2.56 $\mu m$. This strongly suggests that cellular organisms have evolved to their present form through four basic stages.

iii) $k = 239, 241, 283, 353, 367, 379, 457$ associated with the next Gaussian Mersennes define astronomical length scales. $k = 239$ and $k = 241$ correspond to the p-adic time scales .55 ms and 1.1 ms: basic time scales associated with nerve pulse transmission are in question. $k = 283$ corresponds to the time scale of 38.6 $min$. An interesting question is whether this period could define a fundamental biological rhythm. The length scale $L(353)$ corresponds to about $2.6 \times 10^6$ light years, roughly the size scale of galaxies. The length scale $L(367) \simeq \times 3.3 \times 10^8$ light years is of same order of magnitude as the size scale of the large voids containing galaxies on their boundaries (note the analogy with cells). $T(379) \simeq 2.1 \times 10^{10}$ years corresponds to the lower bound for the order of the age of the Universe. $T(457) \sim 10^{22}$ years defines a completely super-astronomical time and length scale.

3. Eisenstein integers form a hexagonal lattice equivalent with the root lattice of the color group $SU(3)$. Microtubular surface defines a hexagonal lattice on the surface of a cylinder which suggests an interpretation in terms of E-adicity. Also the patterns of neural activity form often hexagonal lattices.

### 2.7.3 Gaussian and Eisenstein versions of infinite primes

The vision about quantum TGD as a generalized number theory generates a further line of thoughts.

1. As has been found, the zeros of $\zeta$ code for the physical states of a super-symmetric arithmetic quantum field theory. As a matter fact, the arithmetic quantum field theory in question can be identified as arithmetic quantum field theory in which single particle states are labelled by Gaussian primes. The properties of the Gaussian primes imply that the single particle states of this theory have 8-fold degeneracy plus the four-fold degeneracy related to the $\pm i$ or $\pm 1$-factor which could be interpreted as a phase factor associated with any $p \ mod \ 4 = 3$ type Gaussian prime. Also Eisenstein primes could allow the construction of a similar arithmetic quantum field theory.

2. The construction of the infinite primes reduces to a repeated second quantization of an arithmetic quantum field theory. A straightforward generalization of the procedure of the previous chapter allows to define also the notion of infinite Gaussian and Eisenstein primes. Since each infinite prime is in a well-defined sense a composite of finite primes playing the role of elementary particles, this would mean that each composite prime in the expansion of an infinite prime has either four-fold degeneracy or eight-fold degeneracy. The interpretation of infinite primes could thus literally be as many-particle states of quantum TGD. In TGD the topology of space-time surfaces of infinite size is characterized by infinite-P p-adic topology and the possibility of infinite-P p-, G- and E-adic (and more generally, D-adic) topologies suggests the fascinating possibility that this infinite-P p-adic topology carries implicitly information about the discrete quantum numbers of all particles represented as space-time sheets glued to the larger space-time sheet.

## 2.8 p-Adic length scale hypothesis and quaternionic primality

p-Adic length scale hypothesis states that fundamental length scales correspond to the so called p-adic length scales proportional to $\sqrt{p}$, $p$ prime. Even more: the p-adic primes $p \simeq 2^k$, $k$ prime or possibly power of prime, are especially interesting physically. The so called elementary particle-blackhole analogy gives strong support for this hypothesis. Elementary particles correspond to the so called $CP_2$ type extremals in TGD. Elementary particle horizon can be defined as a surface at which the Euclidian signature of the metric of the space-time surface containing topologically condensed $CP_2$ type extremal, changes to Minkowskian signature. The generalization of the Hawking-Bekenstein formula relates the real counterpart of the p-adic entropy associated with the elementary particle to the area of the elementary particle horizon. If one requires that the radius of the elementary particle horizon corresponds to a p-adic length scale: $R = L(k)$ or $k^{n/2}L(k)$ where $k$ is prime, then $p$ is automatically near to $2^{k^n}$ and p-adic length scale hypothesis is reproduced! The proportionality of length scale to $\sqrt{p}$, rather than $p$, follows from p-adic thermodynamics for mass squared (!) operator and from Uncertainty Principle.

What Tony Smith [19] suggested, was a beautiful connection with number theory based on the generalization of the concept of a prime number. In the so called $D^4$ lattice regarded as consisting of integer quaternions, one can identify prime quaternions as the generators of the multiplicative algebra of the integer quaternions. From the basic properties of the quaternion norm it follows directly that prime quaternions correspond to the 3-dimensional spheres $R^2 = p$, $p$ prime. The crucial point from the TGD:eish point of view is the appearance of the *square* of the norm instead of the norm. One can even define the product of spheres $R^2 = n_1$ and $R^2 = n_2$ by defining the product sphere with norm squared $R^2 = n_1 n_2$ to consist of the quaternions, which are products of quaternions with norms squared $n_1$ and $n_2$ respectively. Prime spheres correspond to $n = p$. The powers of sphere $p$ correspond to a multiplicatively closed structure consisting of powers $p^n$ of the sphere $p$. It is also possible to speak about the multiplication of balls and prime balls in the case of integer quaternions.

p-Adic length scale hypothesis follows if one assumes that the Euclidian piece of the space-time surrounding the topologically condensed $CP_2$ type extremal can be approximated with a quaternion integer lattice with radius squared equal to $r^2 = k^n$, $k$ prime. One manner to understand the finiteness in the time direction is that topological sum contacts of $CP_2$ type extremal are not static 3-dimensional topological sum contacts but genuinely four-dimensional: 3-dimensional contact is created, expands to a maximum size and is gradually reduced to point. The Euclidian space-time volume containing the contact would correspond to an Euclidian region $R^2 = k^n$ of space-time. The distances of the lattice points would be measured using the induced metric. These contacts could have arbitrarily long duration from the point of view of external observer since classical gravitational fields give rise to strong time dilation effects (strongest on the boundary of the Euclidian region where the metric becomes degenerate with the emergence of a light like direction).

Lattice structure is essential for the argument. Lattice structures of type $D^4$ indeed emerge naturally in the p-adic QFT limit of TGD as also in the construction of the p-adic counterparts of the space-time surfaces as p-adically analytic surfaces. The essential idea is to construct the p-adic surface by first discretizing space-time surface using a p-adic cutoff in $k$:th pinary digit and mapping this surface to its p-adic counterpart and complete this to a unique smooth p-adically analytic surface. This leads to a fractal construction in which a given interval is decomposed to $p$ smaller intervals, when the resolution is increased. In the 4-dimensional case one naturally obtains a fractal hierarchy of nested $D^4$ lattices. The interior of the elementary particle horizon with Euclidian signature corresponds to some subset of the quaternionic integer lattice $D^4$: an attractive possibility is that the absolute minimization of the Kähler action and the maximization of the Kähler function force this set to be a ball $R^2 \leq k^n$, $k$ prime.

# 3 Scaling hierarchies and physics as a generalized number theory

The scaling hierarchies defined by powers of $\Phi$ and primes $p$ probably reflect something very profound. Mueller has proposed also a scaling law in powers of $e$ [40]. This scaling law can be however questioned since $\Phi^2 = 2.6180..$ is rather near to $e = 2.7183...$ Note that powers of $e$ define p-dimensional extension of $R_p$ since $e^p$ exists as a p-adic number in this case.

The interpretation of the p-adic as physics of cognition and the vision about reduction of physics to rational physics continuable algebraically to various extensions of rationals and p-adic number fields is an attractive general framework allowing to understand how p-adic fractality could emerge in real physics. In this section it will be found that this vision provides a concrete tool in principle allowing to construct global solutions of field equations by reducing long length scale real physics to short length scale p-adic physics. Also p-adic length scale hypothesis can be understood and the notion of multi-p p-fractality can be formulated in precise sense in this framework. This vision leads also to a concrete quantum model for how intentions are transformed to actions and the S-matrix for the process has the same general form as the ordinary S-matrix.

The fractal hierarchy associated with Golden mean cannot be understood in a manner analogous to p-adic fractal hierarchies. Rather, the understanding of Golden Mean and Fibonacci series could reduce to the hypothesis that space-time surfaces, and thus the geometry of physical systems, provide a representations for the hierarchy of Fibonacci numbers characterizing the Jones inclusions of infinite-dimensional Clifford sub-algebras of configuration space spinors identifiable as infinite-dimensional von Neumann algebras known as hyper-finite factors of type $II_1$ (not that configuration space corresponds here to the "world of classical worlds"). The emergence of powers of $e$ has been discussed in [E8] and will not be discussed here.

## 3.1 p-Adic physics and the construction of solutions of field equations

The number theoretic vision about physics relies on the idea that physics or, rather what we can know about it, is basically rational number based. One interpretation would be that space-time surfaces, the induced spinors at space-time surfaces, configuration space spinor fields, S-matrix, etc..., can be obtained by algebraically continuing their values in a discrete subset of rational variant of the geometric structure considered to appropriate completion of rationals (real or p-adic). The existence of the algebraic continuation poses very strong additional constraints on physics but has not provided any practical means to solve quantum TGD.

In the following it is however demonstrated that this view leads to a very powerful iterative method of constructing global solutions of classical field equations from local data and at the same time gives justification for the notion of p-adic fractality, which has provided very successful approach not only to elementary particle physics but also physics at longer scales. The basic idea is that mere p-adic continuity and smoothness imply fractal long range correlations between rational points which are very close p-adically but far from each other in the real sense and vice versa.

### 3.1.1 The emergence of a rational cutoff

For a given p-adic continuation only a subset of rational points is acceptable since the simultaneous requirements of real and p-adic continuity can be satisfied only if one introduces ultraviolet cutoff length scale. This means that the distances between subset of rational points fixing the dynamics of the quantities involved are above some cutoff length scale, which is expected to depend on the p-adic number field $R_p$ as well as a particular solution of field equations. The continued quantities coincide only in this subset of rationals but not in shorter length scales.

The presence of the rational cutoff implies that the dynamics at short scales becomes effectively discrete. Reality is however not discrete: discreteness and rationality only characterize the inherent limitations of our knowledge about reality. This conforms with the fact that our numerical calculations are always discrete and involve finite set of points.

The intersection points of various p-adic continuations with real space-time surface should code for all actual information that a particular p-adic physics can give about real physics in classical sense. There are reasons to believe that real space-time sheets are in the general case characterized by integers $n$ decomposing into products of powers of primes $p_i$. One can expect that for $p_i$-adic continuations the sets of intersection points are especially large and that these p-adic space-time surfaces can be said to provide a good discrete cognitive mimicry of the real space-time surface.

Adelic formula represents real number as product of inverse of its p-adic norms. This raises the hope that taken together these intersections could allow to determine the real surface and thus classical physics to a high degree. This idea generalizes to quantum context too.

The actual construction of the algebraic continuation from a subset of rational points is of course something which cannot be done in practice and this is not even necessary since much more elegant approach is possible.

#### 3.1.2 Hierarchy of algebraic physics

One of the basic hypothesis of quantum TGD is that it is possible to define exponent of Kähler action in terms of fermionic determinants associated with the modified Dirac operator derivable from a Dirac action related super-symmetrically to the Kähler action.

If this is true, a very elegant manner to define hierarchy of physics in various algebraic extensions of rational numbers and p-adic numbers becomes possible. The observation is that the continuation to various p-adic numbers fields and their extensions for the fermionic determinant can be simply done by allowing only the eigenvalues which belong to the extension of rationals involved and solve field equations for the resulting Kähler function. Hence a hierarchy of fermionic determinants results. The value of the dynamical Planck constant characterizes in this approach the scale factor of the $M^4$ metric in various number theoretical variants of the imbedding space $H = M^4 \times CP_2$ glued together along subsets of rational points of $H$. The values of $\hbar$ are determined from the requirement of quantum criticality [C6] meaning that Kähler coupling strength is analogous to critical temperature.

In this approach there is no need to restrict the imbedding space points to the algebraic extension of rationals and to try to formulate the counterparts of field equations in these discrete imbedding spaces.

#### 3.1.3 p-Adic short range physics codes for long range real physics and vice versa

One should be able to construct global solutions of field equations numerically or by engineering them from the large repertoire of known exact solutions [D1]. This challenge looks formidable since the field equations are extremely non-linear and the failure of the strict non-determinism seems to make even in principle the construction of global solutions impossible as a boundary value problem or initial value problem.

The hope is that short distance physics might somehow code for long distance physics. If this kind of coding is possible at all, p-adicity should be crucial for achieving it. This suggests that one must articulate the question more precisely by characterizing what we mean with the phrases "short distance" and "long distance". The notion of short distance in p-adic physics is completely different from that in real physics, where rationals very close to each other can be arbitrary far away in the real sense, and vice versa. Could it be that in the statement "Short length scale physics codes for long length scale physics" the attribute "short"/"long" could refer to p-adic/real norm, real/p-adic norm, or both depending on the situation?

The point is that rational imbedding space points very near to each other in the real sense are in general at arbitrarily large distances in p-adic sense and vice versa. This observation leads to an elegant method of constructing solutions of field equations.

1. Select a rational point of the imbedding space and solve field equations in the real sense in an arbitrary small neighborhood $U$ of this point. This can be done with an arbitrary accuracy by choosing $U$ to be sufficiently small. It is possible to solve the linearized field equations or use a piece of an exact solution going through the point in question.

2. Select a subset of rational points in $U$ and interpret them as points of p-adic imbedding space and space-time surface. In the p-adic sense these points are in general at arbitrary large distances from each and real continuity and smoothness alone imply p-adic long range correlations. Solve now p-adic field equations in p-adically small neighborhoods of these points. Again the accuracy can be arbitrarily high if the neighborhoods are choose small enough. The use of exact solutions of course allows to overcome the numerical restrictions.

3. Restrict the solutions in these small p-adic neighborhoods to rational points and interpret these points as real points having arbitrarily large distances. p-Adic smoothness and continuity alone imply fractal long range correlations between rational points which are arbitrary distant in the real sense. Return to 1) and continue the loop indefinitely.

In this manner one obtains even in numerical approach more and more small neighborhoods representing almost exact p-adic and real solutions and the process can be continued indefinitely.

Some comments about the construction are in order.

1. Essentially two different field equations are in question: real field equations fix the local behavior of the real solutions and p-adic field equations fix the long range behavior of real solutions. Real/p-adic global behavior is transformed to local p-adic/real behavior. This might be the deepest reason why for the hierarchy of p-adic physics.

2. The failure of the strict determinism for the dynamics dictated by Kähler action and p-adic non-determinism due to the existence of p-adic pseudo constants give good hopes that the construction indeed makes it possible to glue together the (not necessarily) small pieces of space-time surfaces inside which solutions are very precise or exact.

3. Although the full solution might be impossible to achieve, the predicted long range correlations implied by the p-adic fractality at the real space-time surface are a testable prediction for which p-adic mass calculations and applications of TGD to biology provide support.

4. It is also possible to generalize the procedure by changing the value of $p$ at some rational points and in this manner construct real space-time sheets characterized by different p-adic primes.

5. One can consider also the possibility that several p-adic solutions are constructed at given rational point and the rational points associated with p-adic space-time sheets labelled by $p_1, ...., p_n$ belong to the real surface. This would mean that real surface would be multi-p p-adic fractal.

I have earlier suggested that even elementary particles are indeed characterized by integers and that only particles for which the integers have common prime factors interact by exchanging particles characterized by common prime factors. In particular, the primes $p = 2, 3, ....., 23$ would be common to the known elementary particles and appear in the expression of the gravitational constant. Multi-p p-fractality leads also to an explanation for the weakness of the gravitational constant. The construction recipe for the solutions would give a concrete meaning for these heuristic proposals.

This approach is not restricted to space-time dynamics but is expected to apply also at the level of say S-matrix and all mathematical object having physical relevance. For instance, p-adic four-momenta appear as parameters of S-matrix elements. p-Adic four-momenta very near to each other p-adically restricted to rational momenta define real momenta which are not close to each other and the mere p-adic continuity and smoothness imply fractal long range correlations in the real momentum space and vice versa.

#### 3.1.4 p-Adic length scale hypothesis

Approximate $p_1$-adicity implies also approximate $p_2$-adicity of the space-time surface for primes $p \simeq p_1^k$. p-Adic length scale hypothesis indeed states that primes $p \simeq 2^k$ are favored and this might be due to simultaneous $p \simeq 2^k$- and 2-adicity. The long range fractal correlations in real space-time implied by 2-adicity would indeed resemble those implied by $p \simeq 2^k$ and both $p \simeq 2^k$-adic and 2-adic space-time sheets have larger number of common points with the real space-time sheet.

If the scaling factor $\lambda$ of $\hbar$ appearing in the dark matter hierarchy is in good approximation $\lambda = 2^{11}$ also dark matter hierarchy comes into play in a resonant manner and dark space-time sheets at various levels of the hierarchy tend to have many intersection points with each other.

There is however a problem involved with the understanding of the origin of the p-adic length scale hypothesis if the correspondence via common rationals is assumed.

1. The mass calculations based on p-adic thermodynamics for Virasoro generator $L_0$ predict that mass squared is proportional to $1/p$ and Uncertainty Principle implies that $L_p$ is proportional to $\sqrt{p}$ rather than $p$, which looks more natural if common rationals define the correspondence between real and p-adic physics.

2. It would seem that length $d_p \simeq pR$, $R$ or order CP$_2$ length, in the induced space-time metric must correspond to a length $L_p \simeq \sqrt{p}R$ in $M^4$. This could be understood if space-like geodesic lines at real space-time sheet obeying effective p-adic topology are like orbits of a particle performing Brownian motion so that the space-like geodesic connecting points with $M^4$ distance $r_{M^4}$ has a length $r_{X^4} \propto r_{M^4}^2$. Geodesic random walk with randomness associated with the motion in CP$_2$ degrees of freedom could be in question. The effective p-adic topology indeed induces a strong local wiggling in $CP_2$ degrees of freedom so that $r_{X^4}$ increases and can depend non-linearly on $r_{M^4}$.

3. If the size of the space-time sheet associated with the particle has size $d_p \sim pR$ in the induced metric, the corresponding $M^4$ size would be about $L_p \propto \sqrt{p}R$ and p-adic length scale hypothesis results.

4. The strongly non-perturbative and chaotic behavior $r_{X^4} \propto r_{M^4}^2$ is assumed to continue only up to $L_p$. At longer length scales the space-time distance $d_p$ associated with $L_p$ becomes the unit of space-time distance and geodesic distance $r_{X^4}$ is in a good approximation given by

$$r_{X^4} = \frac{r_{M^4}}{L_p} d_p \propto \sqrt{p} \times r_{M^4} \quad , \tag{4}$$

and is thus linear in $M^4$ distance $r_{M^4}$.

#### 3.1.5 Does cognition automatically solve real field equations in long length scales?

In TGD inspired theory of consciousness p-adic space-time sheets are identified as space-time correlates of cognition. Therefore our thoughts would have literally infinite size in the real topology if p-adics and reals correspond to each other via common rationals (also other correspondence based on the separate canonical identification of integers $m$ and $n$ in $q = m/n$ with p-adic numbers).

The cognitive solution of field equations in very small p-adic region would solve field equations in real sense in a discrete point set in very long real length scales. This would allow to understand why the notions of Universe and infinity are a natural part of our conscious experience although our sensory input is about an infinitesimally small region in the scale of universe.

The idea about Universe performing mimicry at all possible levels is one of the basic ideas of TGD inspired theory of consciousness. Universe could indeed understand and represent the long length scale real dynamics using local p-adic physics. The challenge would be to make quantum jumps generating p-adic surfaces having large number of common points with the real space-time surface. We are used to call this activity theorizing and the progress of science towards smaller real length scales means progress towards longer length scales in p-adic sense. Also real physics can represent p-adic physics: written language and computer represent examples of this mimicry.

## 3.2 A more detailed view about how local p-adic physics codes for p-adic fractal long range correlations of the real physics

The vision just described gives only a rough heuristic view about how the local p-adic physics could code for the p-adic fractality of long range real physics. There are highly non-trivial details related to the treatment of $M^4$ and $CP_2$ coordinates and to the mapping of p-adic $H$-coordinates to their real counterparts and vice versa.

### 3.2.1 How real and p-adic space-time regions are glued together?

The first task is to visualize how real and p-adic space-time regions relate to each other. It is convenient to start with the extension of real axis to contain also p-adic points. For finite rationals $q = m/n$, $m$ and $n$ have finite power expansions in powers of $p$ and one can always write $q = p^k \times r/s$ such that $r$ and $s$ are not divisible by $p$ and thus have pinary expansion of in powers of $p$ as $x = x_0 + \sum_1^N x_n p^n$, $x_i \in \{0, p\}$, $x_0 \neq 0$.

One can always express p-adic number as $x = p^n y$ where $y$ has p-adic norm 1 and has expansion in non-negative powers of $p$. When $x$ is rational but not integer the expansion contains infinite number of terms but is periodic. If the expansion is infinite and non-periodic, one can speak about *strictly p-adic* number having infinite value as a real number.

In the same manner real number $x$ can be written as $x = p^n y$, where $y$ is either rational or has infinite non-periodic expansion $y = r_0 + \sum_{n>0} r_n p^{-n}$ in negative powers of $p$. As a p-adic number $y$ is infinite. In this case one can speak about strictly real numbers.

This gives a visual idea about what the solution of field equations locally in various number fields could mean and how these solutions are glued together along common rationals. In the following I shall be somewhat sloppy and treat the rational points of the imbedding space as if they were points of real axis in order to avoid clumsy formulas.

1. The p-adic variants of field equations can be solved in the strictly p-adic realm and by p-adic smoothness these solutions are well defined also in as subset of rational points. The strictly p-adic points in a neighborhood of a given rational point correspond as real points to infinitely distant points of $M^4$. The possibility of p-adic pseudo constants means that for rational points of $M^4$ having sufficiently large p-adic norm, the values of $CP_2$ coordinates or induced spinor fields can be chosen more or less freely.

2. One can solve the p-adic field equations in any p-adic neighborhood $U_n(q) = \{x = q + p^n y\}$ of a rational point $q$ of $M^4$, where $y$ has a unit p-adic norm and select the values of fields at

different points $q_1$ and $q_2$ freely as long as the spheres $U_n(q_1)$ and $U_n(q_2)$ are disjoint (these spheres are either identical or disjoint by p-adic ultra-metricity).

The points in the p-adic continuum part of these solutions are at an infinite distance from $q$ in $M^4$. The points which are well-defined in real sense form a discrete subset of rational points of $M^4$. The p-adic space-time surface constructed in this manner defines a discrete fractal hierarchy of rational space-time points besides the original points inside the p-adic spheres. In real sense the rational points have finite distances and could belong to disjoint real space-time sheets. The failure of the strict non-determinism for the field equations in the real sense gives hopes for gluing these sheets partially together (say in particle reactions with particles represented as 3-surfaces).

3. All rational points $q$ of the p-adic space-time sheet can be interpreted as real rational points and one can solve the field equations in the real sense in the neighborhoods $U_n(q) = \{x = q+p^n y\}$ corresponding to real numbers in the the range $p^n \leq x \leq p^{n+1}$. Real smoothness and continuity fix the solutions at finite rational points inside $U_n(q)$ and by the phenomenon of p-adic pseudo constants these values can be consistent with p-adic field equations. Obviously one can can continue the construction process indefinitely.

#### 3.2.2 p-Adic scalings act only in $M^4$ degrees of freedom

p-Adic fractality suggests that finite real space-time sheets around points $x + p^n$, $x = 0$, are obtained as by just scaling of the $M^4$ coordinates having origin at $x = 0$ by $p^n$ of the solution defined in a neighborhood of $x$ and leaving $CP_2$ coordinates as such. The known extremals of Kähler action indeed allow $M^4$ scalings as dynamical symmetries.

One can understand why no scaling should appear in $CP_2$ degrees of freedom. $CP_2$ is complex projective space for which points can be regarded as complex planes and for these p-adic scalings act trivially. It is worth of emphasizing that here could lie a further deep number theoretic reason for why the space $S$ in $H = M^4 \times S$ must be a projective space.

#### 3.2.3 What p-adic fractality for real space-time surfaces really means?

The identification of p-adic and real $M^4$ coordinates of rational points as such is crucial for p-adic fractality. On the other hand, the identification rational real and p-adic $CP_2$ coordinates as such would not be consistent with the idea that p-adic smoothness and continuity imply p-adic fractality manifested as long range correlations for real space-time sheets

The point is that p-adic fractality is not stable against small p-adic deformations of $CP_2$ coordinates as function of $M^4$ coordinates for solutions representable as maps $M^4 \rightarrow CP_2$. Indeed, if the rational valued p-adic $CP_2$ coordinates are mapped as such to real coordinates, the addition of large power $p^n$ to $CP_2$ coordinate implies small modification in p-adic sense but large change in the real sense so that correlations of $CP_2$ at p-adically scaled $M^4$ points would be completely lost.

The situation changes if the map of p-adic $CP_2$ coordinates to real ones is continuous so that p-adically small deformations of the p-adic space-time points are mapped to small real deformations of the real space-time points.

1. Canonical identification $I : x = \sum x_n p^n \rightarrow \sum x_n p^{-n}$ satisfies continuity constraint but does not map rationals to rationals.

2. The modification of the canonical identification given by

$$I(q = p^k \times \frac{r}{s}) = p^k \times \frac{I(r)}{I(s)} \tag{5}$$

is uniquely defined for rational points, maps rationals to rationals, has a symmetry under exchange of target and domain. This map reduces to a direct identification of rationals for $0 \leq r < p$ and $0 \leq s < p$.

3. The form of this map is not general coordinate invariant nor invariant under color isometries. The natural requirement is that the map should respect the symmetries of $CP_2$ maximally. Therefore the complex coordinates transforming linearly under U(2) subgroup of SU(3) defining the projective coordinates of $CP_2$ are a natural choice. The map in question would map the real components of complex coordinates to their p-adic variants and vice versa. The residual $U(2)$ symmetries correspond to rational unitary $2\times 2$-matrices for which matrix elements are of form $U_{ij} = p^k r/s$, $r < p, s < p$. It would seem that these transformations must form a finite subgroup if they define a subgroup at all. In case of $U(1)$ Pythagorean phases define rational phases but sufficiently high powers fail to satisfy the conditions $r < p, s < p$. Also algebraic extensions of p-adic numbers can be considered.

4. The possibility of pseudo constant allows to modify canonical identification further so that it reduces to the direct identification of real and p-adic rationals if the highest powers of $p$ in $r$ and $s$ ($q = p^n r/s$) are not higher than $p^N$. Write $x = \sum_{n\geq 0} x_n p^n = x^{N)} + p^{N+1}y$ with $x^{N)} = \sum_{n=0}^{N} x_n p^n$, $x_0 \neq 0$, $y_0 \neq 0$, and define $I_N(x) = x^{N)} + p^{N+1}I(y)$. For $q = p^n r/s$ define $I_N(q) = p^n I_N(r)/I_N(s)$. This map reduces to the direct identification of real and p-adic rationals for $y = 0$.

5. There is no need to introduce the imaginary unit explicitly. In case of spinors imaginary unit can be represented by the antisymmetric $2\times 2$-matrix $\epsilon_{ij}$ satisfying $\epsilon_{12} = 1$. As a matter fact, the introduction of imaginary unit as number would lead to problems since for $p \ mod \ 4 = 3$ imaginary unit should be introduced as an algebraic extension and $CP_2$ in this sense would be an algebraic extension of $RP_2$. The fact that the algebraic extension of p-adic numbers by $\sqrt{-1}$ is equivalent with an extension introducing $\sqrt{p-1}$ supports the view that algebraic imaginary unit has nothing to do with the geometric imaginary unit defined by Kähler form of $CP_2$. For $p \ mod \ 4 = 1$ $\sqrt{-1}$ exists as a p-adic number but is infinite as a real number so that the notion of finite complex rational would not make sense.

#### 3.2.4 Preferred $CP_2$ coordinates as a space-time correlate for the selection of quantization axis

Complex $CP_2$ coordinates are fixed only apart from the choice of the quantization directions of color isospin and hyper charge axis in SU(3) Lie algebra. Hence the selection of quantization axes seems to emerge at the level of the generalized space-time geometry as quantum classical correspondence indeed requires.

In a well-defined sense the choice of the quantization axis and a special coordinate system implies the breaking of color symmetry and general coordinate invariance. This breaking is induced by the presence of p-adic space-time sheets identified as correlates for cognition and intentionality. One could perhaps say that the cognition affects real physics via the imbedding space points shared by real and p-adic space-time sheets and that these common points define discrete coordinatization of the real space-time surface analogous to discretization resulting in any numerical computation.

#### 3.2.5 Relationship between real and p-adic induced spinor fields

Besides imbedding space coordinates also induced spinor fields are fundamental variables in TGD. The free second quantized induced spinor fields define the fermionic oscillator operators in terms of which the gamma matrices giving rise to spinor structure of the "world of classical worlds" can be expressed.

p-Adic fractal long range correlations must hold true also for the induced spinor fields and they are in exactly the same role as $CP_2$ coordinates so that the variant of canonical identification mapping rationals to rationals should map the real and imaginary parts of real induced spinor fields to their p-adic counterparts and vice versa at the rational space-time points common to p-adic and real space-time sheets.

#### 3.2.6 Could quantum jumps transforming intentions to actions really occur?

The idea that intentional action corresponds to a quantum jump in which p-adic space-time sheet is transformed to a real one traversing through rational points common to p-adic and real space-time sheet is consistent with the conservation laws since the sign of the conserved inertial energy can be also negative in TGD framework and the density of inertial energy vanishes in cosmological length scales [D5]. Also the non-diagonal transitions $p_1 \rightarrow p_2$ are in principle possible and would correspond to intersections of p-adic space-time sheets having a common subset of rational points. Kind of phase transitions changing the character of intention or cognition would be in question.

*1. Realization of intention as a scattering process*

The first question concerns the interpretation of this process and possibility to find some familiar counterpart for it in quantum field theory framework. The general framework of quantum TGD suggests that the points common to real and p-adic space-time sheets could perhaps be regarded as arguments of an n-point function determining the transition amplitudes for p-adic to real transition or $p_1 \rightarrow p_2$-adic transitions. The scattering event transforming an p-adic surface (infinitely distant real surface in real $M^4$) to a real finite sized surface (infinitely distant p-adic surface in p-adic $M^4$) would be in question.

*2. Could S-matrix for realizations of intentions have the same general form as the ordinary S-matrix?*

One might hope that the realization of intention as a number theoretic scattering process could be characterized by an S-matrix, which one might hope of being unitary in some sense. These S-matrix elements could be interpreted at fundamental level as probability amplitudes between intentions to prepare a define initial state and the state resulting in the process.

Super-conformal invariance is a basic symmetry of quantum TGD which suggests that the S-matrix in question should be constructible in terms of n-point functions of a conformal field theory restricted to a subset of rational points shared by real and p-adic space-time surfaces or their causal determinants. According to the general vision discussed in [C1], the construction of n-point functions effectively reduces to that at 2-dimensional sections of light-like causal determinants of space-time surfaces identified as partonic space-time sheets.

The idea that physics in various number fields results by algebraic continuation of rational physics serves as a valuable guideline and suggests that the form of the S-matrices between different number fields (call them non-diagonal S-matrices) could be essentially the same as that of diagonal S-matrices. If this picture is correct then the basic differences to ordinary real S-matrix would be following.

1. Intentional action could transform p-adic space-time surface to a real one only if the exponent of Kähler function for both is rational valued (or belongs to algebraic extension of rationals).

2. The points appearing as arguments of n-point function associated with the non-diagonal S-matrix are a subset of rational points of imbedding space whereas in the real case, where the integration over these points is well defined, all values of arguments can be allowed. Thus the difference between ordinary S-matrix and more general S-matrices would be that a continuous Fourier transform of n-point function in space-time domain is not possible in

the latter case. The inherent nature of cognition would be that it favors localization in the position space.

*3. Objection and its resolution*

Exponent of Kähler function is the key piece of the configuration space spinor field. There is a strong counter argument against the existence of the Kähler function in the p-adic context. The basic problem is that the definite integral defining the Kähler action is not p-adically well-defined except in the special cases when it can be done algebraically. Algebraic integration is however very tricky and numerically completely unstable.

The definition of the exponent of Kähler function in terms of Dirac determinants or, perhaps equivalently, as a result of normal ordering of the modified Dirac action for second quantized induced spinors might however lead to an elegant resolution of this problem. This approach is discussed in detail in [B4, D1]. The idea is that Dirac determinant can be defined as a product of eigenvalues of the modified Dirac operator and one ends up to a hierarchy of theories based on the restriction of the eigenvalues to various algebraic extensions of rationals identified as a hierarchy associated with corresponding algebraic extensions of p-adic numbers. This hierarchy corresponds to a hierarchy of theories (and also physics!) based on varying values of Planck constant. The elegance of this approach is that no discretization at space-time level would be needed: everything reduces to the generalized eigenvalue spectrum of the modified Dirac operator.

*4. A more detailed view*

Consider the proposed approach in more detail.

1. Fermionic oscillator operators are assigned with the generalized eigenvectors of the modified Dirac operator defined at the light-like causal determinants:

$$
\begin{aligned}
\Psi &= \sum_n \Psi_n b_n \ , \\
D\Psi_n &= \Gamma^\alpha D_\alpha \Psi_n = \lambda_n O \Psi_n \ , \ \ O \equiv n_\alpha \Gamma^\alpha \ . \qquad (6)
\end{aligned}
$$

   Here $\Gamma^\alpha = T^{\alpha k}\Gamma_k$ denote so called modified gamma matrices expressible in terms of the energy momentum current $T^{\alpha k}$ assignable to Kähler action [B4]. The replacement of the ordinary gamma matrices with modified ones is forced by the requirement that the supersymmetries of the modified Dirac action are consistent with the property of being an extremal of Kähler action. $n_\alpha$ is a light like vector assignable to the light-like causal determinant and $O = n_\alpha \Gamma^\alpha$ must be rational and have the same value at real and p-adic side at rational points. The integer $n$ labels the eigenvalues $\lambda_n$ of the modified Dirac operator, and $b_n$ corresponds to the corresponding fermionic oscillator operator.

2. The condition that the p-adic and real variants $\Psi$ if the $\Psi$ are identical at common rational points of real and p-adic space-time surface (the same applies to 4-surfaces corresponding to different p-adic number fields) poses a strong constraint on the algebraic continuation from rationals to p-adics and gives hopes of deriving implications of this approach.

3. Ordinary fermionic anti-commutation relations do not refer specifically to any number field. Super Virasoro (anti-)commutation relations involve only rationals. This suggest that fermionic Fock space spanned by the oscillator operators $b_n$ is universal and same for reals and p-adic numbers and can be regarded as rational. Same would apply to Super Virasoro representations. Also the possibility to interpret configuration space spinor fields as quantum superpositions of Boolean statements supports this kind of universality. This gives good hopes

that the contribution of the inner produces between Fock states to the S-matrix elements are number field independent.

4. Dirac determinant can be defined as the product of the eigenvalues $\lambda_n$ restricted to a given algebraic extension of rationals. The solutions of the modified Dirac equation correspond to vanishing eigen values and define zero modes generating conformal super-symmetries and are not of course included.

5. Only those operators $b_n$ for which $\lambda_n$ belongs to the algebraic extension of rationals in question are used to construct physical states for a given algebraic extension of rationals. This might mean an enormous simplification of the formalism in accordance with the fact that configuration space Clifford algebra corresponds as a von Neumann algebra to a hyper-finite factor of type $II_1$ for which finite truncations by definition allow excellent approximations [C6]. One can even ask whether this hierarchy of algebraic extensions of rationals could in fact define a hierarchy of finite-dimensional Clifford algebras. If so then the general theory of hyper-finite factors of type $II_1$ would provide an extremely powerful tool.

## 3.3 Cognition, logic, and p-adicity

There seems to be a nice connection between logic aspects of cognition and p-adicity. In particular, p-valued logic for $p = 2^k - n$ has interpretation in terms of ordinary Boolean logic with $n$ "taboos" so that p-valued logic does not conflict with common sense in this case. Also an interpretation of projections of p-adic space-time sheets to an integer lattice of real Minkowski space $M^4$ in terms of generalized Boolean functions emerges naturally so that $M^4$ projections of p-adic space-time would represent Boolean functions for a logic with $n$ taboos.

### 3.3.1 2-adic valued functions of 2-adic variable and Boolean functions

The binary coefficients $f_{nk}$ in the 2-adic expansions of terms $f_n x^n$ in the 2-adic Taylor expansion $f(x) = \sum_{n=0} f_n x^n$, assign a sequence of truth values to a 2-adic integer valued argument $x \in \{0, 1, ..., 2^N\}$ defining a sequence of $N$ bits. Hence $f(x)$ assigns to each bit of this sequence a sequence of truth values which are ordered in the sense that the truth values corresponding to bits are not so important p-adically: much like higher decimals in decimal expansion. If a binary cutoff in $N$:th bit of $f(x)$ is introduced, $B^M$-valued function in $B^N$ results, where $B$ denotes Boolean algebra fo 2 elements. The formal generalization to p-adic case is trivial: 2 possible truth values are only replaced by $p$ truth values representable as $0, ..., p-1$.

### 3.3.2 p-Adic valued functions of p-adic variable as generalized Boolean functions

One can speak of a generalized Boolean function mapping finite sequences of p-valued Boolean arguments to finite sequences of p-valued Boolean arguments. The restriction to a subset $x = kp^n$, $k = 0, ..., p-1$ and the replacement of the function $f(x)$ with its lowest pinary digit gives a generalized Boolean function of a single p-valued argument. If $f(x)$ is invariant under the scalings by powers of $p^k$, one obtains a hologram like representation of the generalized Boolean function with same function represented in infinitely many length scales. This guarantees the robustness of the representation.

The special role of 2-adicity explaining p-adic length scale hypothesis $p \simeq 2^k$, $k$ integer, in terms of multi-p-acidic fractality would correlate with the special role of 2-valued logic in the world order. The fact that all generalizations of 2-valued logic ultimately involve 2-adic logic at the highest level, where the generalization is formulated would be analog of p-adic length scale hypothesis.

### 3.3.3 $p = 2^k - n$-adicity and Boolean functions with taboos

It is difficult to assign any reasonable interpretation to $p > 2$-valued logic. Also the generalization of logical connectives AND and OR is far from obvious. In the case $p = 2^k - n$ favored by the p-adic length scale hypothesis situation is however different. In this case one has interpretation in terms $B^k$ with $n$ Boolean statements dropped out so that one obtains what might be called $\hat{B^k}$. Since $n$ is odd this set is not invariant under Boolean conjugation so that there is at least one statement, which is identically true and could be called taboo, axiom, or dogma: depending on taste. The allowed Boolean functions would be constructed in this case using standard Boolean functions AND and OR with the constraint that taboos are respected: in other words, both the inputs and values of functions belong to $\hat{B^k}$.

A unique manner to define the logic with taboos is to require that the number of taboos is maximal so that if statement is dropped its negation remains in the logic. This implies $n > B^k/2$.

### 3.3.4 The projections of p-adic space-time sheets to real imbedding space as representations of Boolean functions

Quantum classical correspondence suggests that generalized Boolean functions should have space-time correlates. Since Boolean cognition involves free will, it should be possible to construct space-time representations of arbitrary Boolean functions with finite number of arguments freely. The non-determinism of p-adic differential equations guarantees this freedom.

p-Adic space-time sheets and p-adic non-determinism make possible to represent generalization of Boolean functions of four Boolean variables obtained by replacing both argument and function with p-valued pinary digit instead of bit. These representations result as discrete projections of p-adic space-time sheets to integer valued points of real Minkowski space $M^4$. The interpretation would be in terms of 4 sequences of truth values of p-valued logic associated with a finite 4-D integer lattice whose lattice points can be identified as sequences of truth values of a p-valued logic with a set of p-valued truth value at each point so that in the 2-adic case one has map $B^{4M} \rightarrow B^{4N}$. Here the number of lattice points in a given coordinate direction of $M^4$ is $M$ and $N$ is the number of bits allowed by binary cutoff for $CP_2$ coordinates. For $p = 2^k - n$ representing Boolean algebra with n taboos, the maps can be interpreted as maps $\hat{B}^{4M} \rightarrow \hat{B}^{4N}$.

These lattices can be seen as subsets of rational shadows of p-adic space-time sheets to Minkowski space. The condensed matter analog would be a lattice with a a sequence of p-valued dynamical variables (sequence of bits/spins for $p = 2$) at each lattice point. At a fixed spatial point of $M^4$ the lowest bits define a time evolution of a generalized Boolean function: $B \rightarrow B$.

These observations support the view that intentionality and logic related cognition could perhaps be regarded as 2-adic aspects of consciousness. The special role of primes $p = 2^k - n$ could also be understood as special role of Boolean logic among p-valued logics and $p = 2^k - n$ logic would correspond to $B^k$ with $n$ axioms representing logic respecting a belief system with n beliefs. Recall that multi-p p-adic fractality involving 2-adic fractality is possible for the solutions of field equations and explains p-adic length scale hypothesis.

Most points of the p-adic space-time sheets correspond to real points which are literally infinite as real points. Therefore cognition would be in quite literal sense outside the real cosmos. Perhaps this is a direct correlate for the basic experience that mind is looking the material world from outside.

### 3.3.5 Connection with the theory of computational complexity?

There are interesting questions concerning the interpretation of four generalized Boolean arguments. TGD explains the number $D = 4$ for space-time dimensions and also the dimension of imbedding space. Could one also find explanation why $d = 4$ defines special value for the number of generalized Boolean inputs and outputs?

1. Could the general theory of computational complexity allow to understand $d = 4$ as a maximum number of inputs and outputs allowing the computation of something related to these functions in polynomial time? For instance, complexity theorist could probably immediately answer following questions. Could the computation of the 2-adic values of $CP_2$ coordinates as a function of 2-adic $M^4$ coordinates expressed in terms of fundamental logical connectives take a time which is polynomial as a function of the number of $N^4$ pinary digits of $M^4$ coordinates and $N^4$ pinary digits of $CP_2$ coordinates? Is this time non-polynomial for $M^d$ and $S_d$, $S_d$ d-dimensional internal space, $d > 4$. Unfortunately I do not possess the needed complexity theoretic knowhow to answer these questions.

2. The same question could make sense also for $p > 2$ if the notion of the logical connectives and functions generalizes as it indeed does for $p = 2^k - n$. Therefore the question would be whether p-adic length scale hypothesis and dimensions of imbedding space and space-time are implied by a polynomial computation time? This could be the case since essentially a restriction of values and arguments of Boolean functions to a subset of $B^k$ is in question.

### 3.3.6 Some calculational details

In the following the details of p-adic non-determinism are described for a differential equation of single p-adic variable and some comments about the generalization to the realistic case are given.

*1. One-dimensional case*

To understand the essentials consider for simplicity a solution of a p-adic differential equation giving function $y = f(x)$ of one independent variable $x = \sum_{n\geq n_0} x_n p^n$.

1. p-Adic non-determinism means that the initial values $f(x)$ of the solution can be fixed arbitrarily up to $N+1$:th pinary digit. In other words, $f(x_N)$, where $x_N = \sum_{n_0\leq n\geq N} x_n p^n$ is a rational obtained by dropping all pinary digits higher than $N$ in $x = \sum_{n\geq n_0} x_n p^n$ can be chosen arbitrarily.

2. Consider the projection of $f(x)$ to the set of rationals assumed to be common to reals and p-adics.

   i) Genuinely p-adic numbers have infinite number of positive pinary digits in their non-periodic expansion (non-periodicity guarantees non-rationality) and are strictly infinite as real numbers. In this regime p-adic differential equation fixes completely the solution. This is the case also at rational points $q = m/n$ having infinite number of pinary digits in their pinary expansion.

   ii) The projection of p-adic x-axis to real axis consists of rationals. The set in which solution of p-adic differential equations is non-vanishing can be chosen rather freely. For instance, p-adic ball of radius $p^{-n}$ consisting of points $x = p^M y$, $y \neq 0$, $|y|_p \leq 1$, can be considered. Assume $N > M$. p-Adic nondeterminism implies that $f(q)$ for $q = \sum_{M\leq n\leq N} x_n p^n$, can be chosen arbitrarily. For $M \geq 0$ $q$ is always integer valued and the scaling of $x$ by a suitable power of $p$ always allows to get a finite integer lattice at $x$-axis.

   iii) The lowest pinary digit in the expansion of $f(q)$ in powers of $p$ in defines a pinary digit. These pinary digits would define a representation for a sequence of truth values of p-logic. $p = 2$ gives the ordinary Boolean logic. It is also interpret this pinary function as a function of pinary argument giving Boolean function of one variable in 2-adic case.

*2. Generalization to the space-time level*

This picture generalizes to space-time level in a rather straight forward manner. $y$ is replaced with $CP_2$ coordinates, $x$ is replaced with $M^4$ coordinates, and differential equation with field equations deducible from the Kähler action. The essential point is that p-adic space-time sheets have projection to real Minkowski space which consists of a discrete subset of integers when suitable scaling of $M^4$ coordinates is allowed. The restriction of 4 $CP_2$ coordinates to a finite integer lattice of $M^4$ defines 4 Boolean functions of four Boolean arguments or their generalizations for $p > 2$. Also the modes of the induce spinor field define a similar representation.

## 3.4 Fibonacci numbers, Golden Mean, and Jones inclusions

The picture discussed above does not apply in the case of Golden Mean since powers of $\Phi$ do not have any special role for the algebraic extension of rationals by $\sqrt{5}$. It is however possible to understand the emergence of Fibonacci numbers and Golden Mean using quantum classical correspondence and the fact that the Clifford algebra and its sub-algebras associated with configuration space spinors corresponds to the so called hyper-finite factor of type $II_1$ (configuration space refers to the "world of classical worlds").

### 3.4.1 Infinite braids as representations of Jones inclusions

The appearance of hyper-finite factor of type $II_1$ at the level of basic quantum tGD justifies the expectation that Jones inclusions $\mathcal{N} \subset \mathcal{M}$ of these factors play a key role in TGD Universe. For instance, subsystem system inclusions could induce Jones inclusions.

For the Jones inclusion $\mathcal{N} \subset \mathcal{M}$ $\mathcal{M}$ can be regarded as an $\mathcal{N}$-module with fractal dimension given by Beraha number $B_n = 4cos^2(\pi/n)$, $n \geq 3$ or equivalently by the quantum group phases $exp(i\pi/n)$. $B_5$ satisfies $B_5 = 4cos^2(\pi/5) = \Phi^2 = \Phi + 1$ so that the special role of $n = 5$ inclusion could explain the special role of Golden Mean in Nature.

Hecke algebras $H_n$, which are also characterized by quantum phase $q = exp(i\pi/n)$ or the corresponding Beraha number $B_n = 4cos^2(\pi/n)$, characterize the anyonic quantum statistics of n-braid system. Braids are understood as threads which can get linked and define in this manner braiding. Braid group describes these braidings. Like any algebra, Hecke algebra $H_n$ can be decomposed into a direct sum of matrix algebras. Fibonacci numbers characterize the dimensions of these matrix algebras for $n = 5$. Interestinglyt, topological quantum computation is based on the idea that computer programs can be coded into braidings. What is remarkable is that $n = 5$ characterizes the simplest universal quantum computer so that Golden Mean could indeed have very deep roots to quantum information processing.

The so called Bratteli diagrams characterize the inclusions of various direct summands of $H_k$ to direct summands $H_{k+1}$ in the sequence $H_3 \subset H_4 \subset ... \subset H_k \subset ...$ of Hecke algebras. Essentially the reduction of the representations of $H_{k+1}$ to those of $H_k$ is in question. The same Bratteli diagrams characterize also the Jones inclusions $\mathcal{N} \subset \mathcal{M}$ of hyper-finite factors of type $II_1$ with index $n$ as a limit of a finite-dimensional inclusion. Thus Jones inclusion can be visualized as a system consisting of infinite number of braids. In TGD framework the braids could be represented by magnetic flux lines or flux tubes.

### 3.4.2 Logarithmic spirals as representations of Jones inclusions

The inclusion sequence for Hecke algebras has a representations as a logarithmic spiral. The angle $\pi/5$ can be identified as a limit for angles $\phi_n$ with $cos(\phi_n) = F_{n+1}/2F_n$ assignable to orthogonal triangle with hypothenuse $2F_n$ and short side $F_{n+1}$ and $\sqrt{4F_n^2 - F_{n+1}^2}$. Fibonacci sequence defines via this prescription a logarithmic spiral as a symbolic representation of the $n = 5$ Jones inclusion representable also in terms of infinite number of braids.

### 3.4.3 DNA as a topological quantum computer?

Quantum classical correspondence encourages to think that space-time geometry could define a correlate for Jones inclusions of hyper-finite factors of Clifford sub-algebras associated with Clifford algebra of configuration space spinors. The appearance of Fibonacci series in living systems could represent one example of this correspondence. The angle $\pi/10$ closely related to Golden Mean characterizes the winding of DNA double strand. Could this mean that DNA allows to realize topological quantum computer programs as braidings? A possible realization would be based on the notion of super-genes [L2], which are like pages of a book identified as magnetic flux sheets containing genomes of sequences of cell nuclei as text lines. These text lines would represent line through which magnetic flux lines traverse.

The braiding of magnetic flux lines (or possibly flux sheets regarded as flattened tubes) would define the braiding and the particles involved would be anyons obeying dynamics having quantum group $SU(2)_q$, $q = exp(i\pi/5)$, as its symmetries. The anyons could be assigned with DNA nucleotides or triplets.

TGD predicts also different kind of new physics to DNA double strand. So called $H_N$-atoms consist of ordinary proton an $N$ dark electrons at space-time sheet which is $\lambda$-fold covering of space-time sheet of ordinary hydrogen atom. The effective charge of $H_N$-atom is $1 - N/\lambda$ since the fine structure constant for dark electrons is scaled down by $1/\lambda$. $H_\lambda$-atoms have full electron shell and are therefore exceptionally stable. The proposal is that $H_\lambda$-atoms could replace ordinary hydrogen atoms in hydrogen bonds [L2, L4]. Single base pair corresponds to 2 or 3 hydrogen bonds. The question is whether $\lambda$-hydrogen atom might somehow relate to the anyons involved with topological quantum computation.

Anyons could be dark protons resulting in the formation dark hydrogen bond in the fusion of $H_N$ atom and its conjugate $H_{N_c}$, $N_c = \lambda - N$. Neutron scattering and electron diffraction suggest that 1/4:th of protons of water are in dark phase in attosecond time scale [41], and the model explains this number.

# 4 Quantum criticality and how to express it algebraically?

There are two problems related to the precise formulation of quantum TGD. The first, rather longstanding, problem is how to precisely define quantum criticality concept. Second problem, is the normal ordering anomaly implying non-conservation of the isometry currents associated with the modified Dirac action for the induced spinor fields. The assumption that modified Dirac action is the fundamental variational principle could solve these problems to some degree. A further, technical problem is to calculate the value of the Kähler coupling strength determined by the requirement of quantum criticality or equivalently, to deduce the value of the Kähler coupling strength from the modified Dirac action. The notion of the coupling constant evolution defined in terms of infinite primes combined with the notion of the fermionic effective action allows to "deduce" the dependence of the Kähler coupling strength on infinite prime characterizing the sector of the configuration space.

The understanding of the coupling constant evolution of Kähler coupling strength does not of course yet provide real mathematical description of quantum criticality. The cancellation of loop corrections is very attractive formulation for the notion of quantum criticality, and one might hope that this notion would allow an elegant algebraic formulation. The work with Hopf algebras and related structures indeed led to a proposal for a formulation of this kind [C5]. The notion of Platonia inspired the idea that Feynman diagrams in some suitably generalized sense could be regarded as characterizing computations and that two computations containing loops would be equivalent to those without loops by basic axioms of the associated algebra. Certain basic axiom relating the product and co-product of Hopf algebras indeed has a graphical representation stating that a box diagram is equivalent with a tree diagram and it is easy to identify axioms implying that also the diagrams with loops associated with vertex corrections and self energy corrections are equivalent to tree diagrams. The connection with ordinary Feynman diagrams emerges from the fact that algebraic diagrams involve also vacuum lines identifiable in terms of identity element.

## 4.1 The value of Kähler coupling strength from quantum criticality

Kähler coupling strength is the only free parameter of the theory. The hypothesis that TGD corresponds to a quantum critical system means that $\alpha_K$ is mathematically analogous to critical temperature and provides a possible solution to the problem of fixing the value of $\alpha_K$. This argument with some additional physical inputs has led to a rather detailed information about Kähler coupling strength.

1. Already the reduction of the classical action to Kähler action implies criticality in a well defined sense as is clear from the emergence of the huge vacuum degeneracy. This criticality however occurs for all values of $\alpha_K$ and is actually the most characteristic feature of TGD. A good guess is that quantum criticality corresponds to a value of $\alpha_K$ such that Kähler magnetic/electric field configurations dominate and below/above criticality in the vacuum functional. The most obvious interpretation of the quantum criticality is that below criticality magnetic field configurations dominate whereas above criticality electric fields are dominant.

2. Classical non-determinism is the characteristic feature of the Kähler action. This means that the absolute minima of the Kähler action associated with a given 3-surface $Y^3$ at the light cone boundary are degenerate. Analogous degeneracy is encountered in the thermodynamics of strings: in Hagedorn temperature $T_H$ the degeneracy factor $g(E)$ of a given energy eigen state becomes roughly equal to the inverse of the Boltzmann weight at critical temperature: $g(E) \sim exp(E/T_H)$. This motivates the guess that the number $N_d(Y^3)$ of the degenerate space-time surfaces $X^4(Y^3)$ associated with $Y^3$ is in a good approximation given by $N_d(Y^3) \sim exp(-K_{cr}(Y^3))$, where Kähler function is evaluated for a critical temperature. The hypothesis is that Kähler coupling strength corresponds to this maximum value of the Kähler coupling strength above which theory is not well defined. At criticality 3-surfaces for which Kähler action has arbitrarily large negative value can appear in states with considerable probability and the structure of the theory becomes richest possible. The strongest motivation for this hypothesis comes from TGD inspired theory of consciousness, which allows to interpret $N_d$ as a number characterizing cognitive resources of a given 3-surface: quantum critical universe is the most intelligent and most interesting universe possible in TGD framework.

   This argument seems to allow only single value of $\alpha_K$. Situation is however not so simple since possible dependence of the number of quantum fluctuating degrees of freedom on zero modes allows dependence of $\alpha_K$ on zero modes too.

3. The fundamental variational principle defined by the modified Dirac action contains no free parameters since the anti-commutation relations of the induced spinor field are fixed by requiring that various currents and super-currents generate super-canonical algebra. If Kähler action indeed results from the normal ordering of the Dirac action, the possible values of the Kähler coupling strength are in principle expected to depend on the sector $D_P$ of configuration space labelled by infinite prime $P$ if the proposed interpretation of infinite primes is sensible [E3]. It is tempting to equate the presence of the super-symmetry with quantum

criticality since the addition of a separate bosonic term to the action in general spoils super-conformal symmetry by allowing solutions of the modified Dirac equation which are not analogous to massless solutions but contain a 'mass term' resulting from the non-vanishing divergence of the vector field defined by the modified gamma matrices. Clearly this term defines a length scale and means breaking of the scale invariance characteristic for both super-symmetry and criticality.

## 4.2 Is $G$ or $\alpha_K$ RG invariant?

The value of Kähler coupling strength is analogous to a critical temperature. The simplest option is that there is just single universal value. This however leads to problems with gravitational constant solved if one assumes that $G$ is RG invariant. Both options will be discussed in the sequel.

The quantization of Planck constant assignable to $M^4$ degrees of freedom as a multiple of the integer $n$ in quantum phase $q = exp(i\pi/n)$ characterizing Jones inclusions in $CP_2$ degrees of freedom [A9] implies that $\alpha_K$ is scaled down by $1/n$ in a phase with $\hbar = n\hbar_0 = n$. Notice that only $g_K^2/4\pi$ is visible in Kähler action and in the following $\alpha_K$ is identified as this parameter regarding $\hbar_0 = 1$ as the basic unit.

### 4.2.1 $\alpha_K$ when $G$ is RG invariant

The following argument allows to deduce the dependence of $\alpha_K$ on the p-adic prime $p$ assuming RG invariance of $G$.

1. The discrete p-adic evolution of the Kähler coupling strength follows from the requirement that gravitational coupling constant is renormalization group invariant. When combined with the requirement that the exponent of $CP_2$ action is a power of prime (this is essential for p-adicization by algebraic continuation), the argument would give

$$\frac{1}{\alpha_K(p)} = \frac{4}{\pi} log(K^2) \ \ , \ \ K^2 = \prod_{q=2,3,...23} q \times p$$

with $\alpha_K(p = M_{127}) \simeq 136.5585$ and $\alpha/\alpha_K \simeq .9965$. The deviation from fine structure constant at the same length scale is .35 per cent ($1/\alpha_{em} = 137.0360211$). Note that $M_{127}$ corresponds to electron length scale. If the action is a rational fraction of $CP_2$ action, and the extension of p-adic numbers is by an appropriate root of $p$ is enough to guarantee the existence of the Kähler function.

The value of the ratio $K = \frac{R}{\sqrt{G}}$ of $CP_2$ length scale to Planck length is $K = k \times 10^4$, $k = 1.375$ to be compared with the value $k = 1.376$ deduced from electron mass. The deviation is .3 per cent and the unknown second order corrections to the masses allow a deviation of this magnitude.

The predicted value of the inverse of the Kähler coupling strength at p-adic length scale $p = M_{127}$ (electron length scale) is

$$\frac{1}{\alpha_K(p = M_{127})} = 136.5585 \ \ ,$$

which deviates from fine structure constant at the same length scale by .35 per cent ($1/\alpha_{em} = 137.0360211$). Note however that the evolution of Kähler coupling strength is much faster than the evolution of the electromagnetic coupling constant strength and a sheer coincidence might be in question. The value of the Kähler coupling strength for $p = 29$ is given by

$$\frac{1}{\alpha_K(p=23)} = 28.7630 \ ,$$

$$\frac{\alpha_K(p=M_{127})}{\alpha_K(p=23)} = .2106 \ .$$

What is interesting is that the primes $p_i$ appearing in the formula for $K^2$ are the *nine* smallest primes and that the total number of primes is 10. The p-adic prime $p$ labelling the sector of the configuration space is analogous to a time like dimension whereas the nine primes $p_i$ are analogous to nine space-time like dimensions. Furthermore, the values of $p_i$ are smaller than 24. These observations suggest a number-theoretic connection with super string models.

2. One can consider also an alternative ansatz based on the requirement that Kähler function is a rational number rather than a logarithm of a power of integer $K^2$. This requires an extension of p-adic numbers involving some root of $e$ and a finite number of its powers. $S_R$ must be rational valued using Kähler action $S_K(CP_2) = 2\pi^2$ of $CP_2$ type extremal as a basic unit. In fact, not only rational values of Kähler function but all values which differ from a rational value by a perturbation with a p-adic norm smaller than one and rationally proportional to a power of $e$ or even its root exist p-adically in this case if they have small enough p-adic norm. The most general perturbation of the action is in the field defined by the extension of rationals defined by the root of $e$ and algebraic numbers.

Since $CP_2$ action is rationally proportional to $\pi^2$, the exponent is rational if $4\pi\alpha_K$ satisfies the same condition. If the conjecture $log(p) = q_1(p)exp\left[q(p_2)\right]/\pi$ holds , then the earlier ansatz $1/\alpha_K(p) = (4/\pi)log(K^2)$ does not guarantee this, and $4/\pi$ must be replaced with a rational number $Q \simeq 4/\pi$. The presence of $log(K^2)$, $K^2$ product of primes, is well motivated also in this case because it gives the desired $1/\pi$ factor.

This gives for the Kähler function the expression

$$K = Q\left[q_1(p)exp\left[q_2(p)\right] + \sum_i q_1(q_i)exp\left[q_2(q_i)\right]\right]\frac{S}{S_{CP_2}} \ . \tag{7}$$

$exp(K)$ exists p-adically only provided that $K$ has p-adic norm smaller than one. For given $p$ this poses strong conditions unless one assumes that the condition $S/S_{CP_2} = p^n r$, $r$ rational. In the case of many-particle state of $CP_2$ extremals this would mean that particle number is divisible by a power of $p$.

For single $CP_2$ extremal, the fact that $p$ cannot divide $q_1(p)$ means that either $Q$ contains a power of $p$ or the sum of terms is proportional to a power of $p$. Obviously this condition is extremely strong and allows only very few primes. One might wander whether this could provide the first principle explanation for p-adic length scale hypothesis selecting primes $p \simeq 2^k$, $k$ integer, and with prime power powers being preferred.

Since $k = 137$ (atomic length scale) and $k = 107$ (hadronic length scale) are the most important nearest p-adic neighbors of electron, one could make a free fall into number mysticism and try the replacement $4/\pi \rightarrow 137/107$. This would give $\alpha_K = 137.3237$ to be compared with $\alpha = 137.0360$: the deviation from $\alpha$ is .2 per cent (of course, $\alpha_K$ need not equal to $\alpha$ and the evolutions of these couplings are quite different). Thus it seems that $log(p) = q_1 exp(q_2)/\pi$ hypothesis is supported also by the properties of Kähler action and might lead to an improved understanding of the origin of the mystery prime $k = 137$. Of course, one must be extremely cautious with the numerics. For instance, one could replace 137/107 with the ratio of $137/log(M_{107}$ and in this case the $M_{107}$ would become an "easy" prime.

### 4.2.2 What if $\alpha_K$ is RG invariant?

Electro-weak and color coupling strenghs should be proportional to $\alpha_K$. The p-adic evolution of $\alpha_K$ is however rather fast and makes it difficult to understand the much slower electro-weak coupling constant evolution. If $\alpha_K$ is RG invariant the situation changes.

The previous argument goes through also when $\alpha_K$ is RG invariant by putting $p = M_{127}$ so that $K$ would we constant and the primes $p = 2, 3....23$ and $M_{127}$ would be exceptional ($M_{127}$ is certainly exceptional). This option predicts that gravitational constant is proportional to $L_p^2$. The observed value corresponds to $M_{127}$. This picture is sensible only if one assumes that gravitational interactions are mediated by space-time sheets characterized by Mersenne primes. Since $M_{127}$ is the largest Mersenne prime for which the p-adic length scale is not super-astronomical, the strongest observable-to-us gravitational interaction would correspond to $M_{127}$ so that $G$ would be effectively RG invariant.

The resulting strongly predictive model for electro-weak and color coupling evolution reduces the evolution of color coupling strength $\alpha_s$ to that of electro-weak U(1) coupling strength $\alpha_{U(1)}$ and its success supports this picture [A9]. The model favors the value $\alpha_K = \alpha_{U(1)}(M_{127}) = [\alpha_{em}/cos^2(\theta_W)](M_{127}) \simeq 104$. In the recent situation one must however keep mind open and allow the possibility that $\alpha_K$ can depend on $p$.

## 4.3 The bosonic action defining Kähler function as the effective action associated with the induced spinor fields

One could *define* the classical action defining Kähler function as the bosonic action giving rise to the divergences of the isometry currents. In this manner bosonic action, especially the value of the Kähler coupling strength, would come out as prediction of the theory containing no free parameters. Since infinite primes have space-time surfaces as their geometric correlates, the dependence could be through the infinite prime $P$ characterizing the space-time surfaces $X^4(X^3)$ in sector $D_P$ of the configuration space.

Thus it is assumed that the action $S_B$ defining Kähler function as its absolute minimum is *defined* by the functional integral over the Grassmann variables for the exponent of the massless Dirac action. Formally the functional integral is defined as

$$\begin{aligned} exp(S_B(X^4)) &= \int exp(S_F) D\Psi D\bar{\Psi} \ , \\ S_F &= \bar{\Psi}\left[\hat{\Gamma}^\alpha D_\alpha^{\rightarrow} - D_\alpha^{\leftarrow}\hat{\Gamma}^\alpha\right]\Psi\sqrt{g} \ . \end{aligned} \tag{8}$$

Formally the bosonic effective action is expressible as a logarithm of the fermionic functional determinant resulting from the functional integral over the Grassmann variables

$$\begin{aligned} S_B(X^4) &= log(det(D)) \ , \\ D &= \hat{\Gamma}^\alpha D_\alpha^{\rightarrow} \ . \end{aligned} \tag{9}$$

The rigorous definition of this determinant has been already discussed in [B4]. The sum over the logarithms of the eigen values in turn can be identified as the derivative of the logarithm of the generalized Zeta function

$$\zeta_F(s) \equiv \sum_n \lambda_n^{-s} \ ,$$

$$\begin{aligned} D\Psi_n &= \lambda_n o\Psi_n \ , \\ o &= n^\alpha\gamma_\alpha \ , \ [D,0]=0 \ . \end{aligned} \tag{10}$$

at $s=0$:

$$S_B(X^4) = log(det(D)) = \sum_n \log(\lambda_n) = -\frac{d}{ds}log(\zeta_F)(s,X^4) \ . \tag{11}$$

The vector $n_\alpha$ identified as the gradient of a coordinate $x^N$ normal to $X^3$. As shown in [B4], the hermiticity of the modified Dirac operator is guaranteed if $X^3$ is minimal hyper-surface or if Kähler action density $L_K$ vanishes at $X^3$.

The vanishing of the normal components $T^{nk}$ of the conserved currents associated with the isometries of $H$ is necessary in order to have effective 3-dimensionality in the sense that the modified Dirac equation contains only derivatives acting on $X^3$ coordinates. The reduction to the boundary and the dependence on the normal derivatives of the imbedding space coordinates realizes quantum gravitational holography.

The definition relying on the generalized Zeta function allows to circumvent the possible technical difficulties related to the precise definition of the Grassmannian functional integral and of the functional determinant since the possibly divergent sum over the logarithms of the eigenvalues can be identified as the derivative of Zeta function at $s=0$, which can be defined by analytically continuing the zeta function outside the domain where the definition in terms of the eigenvalues works.

The eigenvalues of the modified Dirac operator vanish for the vacuum extremals but the Dirac determinant equals to one since zero eigenvalues by definition do not contribute to it. In this case the determinant is well-defined without regularization, which suggests that Zeta function regularization might not be needed at all. The product of the eigenvalues must approach to unity for non-vacua at the limit $S_K(X^4)=0$.

#### 4.3.1 The regularization of the ordinary Dirac determinant as a guideline

There are good reasons to expect that time translations are replaced by scalings in TGD framework and the eigenvalues of the modified Dirac operator can be interpreted as scaling momenta. This does not however yet explain the required partially broken $\lambda \leftrightarrow 1/\lambda$ symmetry for the operator determining the Dirac determinant. The $E \leftrightarrow -E$ symmetry of the ordinary Dirac operator transforming positive energy solutions to their negative energy counterparts looks much more natural.

The partially broken $\lambda \leftrightarrow 1/\lambda$ symmetry does not seem plausible for the modified Dirac operator itself. The regularization of the ordinary Dirac determinant by dividing it by the Dirac determinant of the free Dirac operator however suggests how to achieve this symmetry. Causal determinants involve always pairs of maximal strictly deterministic space-time regions, and the natural hypothesis is that the ratio of the Dirac determinants of the two adjacent deterministic space-time regions contains information about their Kähler actions.

The assumption that the modified Dirac operators $D_+$ and $D_-$ of the adjacent deterministic regions commute at the causal determinant and their spectra coincide apart from a finite number of eigenvalues, is a strong statement about the character of the classical non-determinism. $\lambda \rightarrow 1/\lambda$ approximate symmetry would be realized for $D_+D_-^{-1}$ since most of its eigen values would equal to unity.

If the asymmetric eigenvalues deviate by the exponent of the Kähler action for the deterministic region, Dirac determinant gives the exponent for the difference of Kähler actions of the two regions, and one can identify the result as the ratio of exponents of Kähler actions for the two regions identifiable as exponents of Kähler function Nothing hinders from defining the regularized Dirac determinant for a given region as a corresponding finite exponent.

This approach has several nice features.

1. The construction brings in mind the difference bundle construction giving rise to a non-trivial gerbe in turn defining a regularized Dirac determinant [34] as the multiplicative analog of gauge potential for which curvature form corresponds formation of the ratio of determinants and gives the ratios of determinants correctly. One could see this approach as the deeper one and Dirac determinants as a mere calculational trick to deduce the vacuum functional and Kähler action.

2. The value of the Dirac determinant does not depend on the normalization of the modified Dirac operator if the non-vanishing eigenvalues are in one-one correspondence.

3. There is also a consistency with the number theoretical conjectures about the relationship between Kähler coupling strength and gravitational constant.

#### 4.3.2 Could generalized index theorems provide information about the spectrum?

The presence of only a finite number of "active" eigen values for a given causal determinant enhances the hopes that information about the exponent of Kähler function could be deduced by using a generalization of index theorems [33]. Ordinary index theorems typically give the number of solutions $D\Psi = 0$ of the modified Dirac operator not expressible in the form $\Psi = D\Psi_1$ (covariantly constant right handed neutrino spinor with two spin directions is one example) in terms of topological invariants of the manifold.

For the option c) the conservative view would be that the generalized index theorem expresses the number of the eigenvalues which become vanishing in the transition between the adjacent regions. A more radical interpretation is that index theorem expresses the number of those solutions of the modified Dirac operators in the adjacent deterministic regions which correspond to different eigen values in terms of some natural topological invariant associated with the causal determinant.

The Chern-Simons action associated with the induced Kähler form defines a 3-connection form of 2-gerbe [D1]. The more than obvious guess is that its value for a given causal determinant gives the number of "active" eigen values with sign telling the sign of asymmetry. Presumably the integer value of C-S action corresponds to a surface for which the projection of the causal determinant to $CP_2$ is many-to-one map.

The vanishing of the net C-S charge does not imply the vanishing of the Kähler function since only the value of the Kähler action for a particular maximal strictly non-deterministic region follows from the spectral asymmetry. If the value of entire Kähler function were in question Kähler function would be non-vanishing only when the dimension $D$ of $CP_2$ projection is $D = 4$ in the interior of the space-time sheet.

If the $CP_2$ projection of the causal determinant is 2-dimensional, the spectral asymmetry would be trivial. The Kähler function could be non-vanishing for the solutions for which space-time sheets have $D = 3$ everywhere. The regions of the space-time sheet with $D = 4$ (such as $CP_2$ extremals representing elementary particles and non-asymptotic regions of sheets having $D = 3$ asymptotically) would certainly contribute to the Kähler function.

In this approach the values of the Kähler coupling strength and gravitational coupling strength (and even quantum criticality itself equated with super-symmetry) are predicted directly, rather than being input parameters and the theory is essentially unique without any additional assumptions.

## 4.4 An attempt to evaluate the Kähler coupling strength from the fermionic determinant in terms of infinite primes

As already found, the notion of infinite primes provides estimate for the Kähler coupling strength and its is interesting to see whether the same estimate follows from the fermionic determinant under reasonable assumptions. The hypothesis that bosonic action is the logarithm of the action defined by the partition function associated with the induced Dirac action, makes it possible to identify it as the ordinary derivative of the Zeta function $\zeta_D(s) = \sum_\lambda \lambda^s$ of the modified Dirac operator at $s = 0$. That no zeta function regularization is needed would be the deep meaning of the conformal invariance at light like causal determinants.

By the previous arguments bosonic action $S_B$ should be Kähler action and by the broken $\lambda \rightarrow 1/\lambda$ symmetry only finite number of eigen values contribute to the determinant of the operator $D_+D_-^{-1}$. One ends up with the hypothesis that the eigenvalues contributing to the determinant correspond to a finite set of primes and thus to a fermionic infinite prime.

The number theory inspired idea which originally led to the formulas for the Kähler coupling constant and gravitational constant is that there is a connection with arithmetic quantum field theory. This means that the logarithm of the Dirac determinant can be interpreted as a "thermal energy" for an arithmetic quantum field theory at infinite temperature limit, that is as the logaritmic derivative $-dlog(Z_F(s,X^4))/ds$ of the partition function $Z_F(s,X^4)$ such that only a finite product of number theoretic partition functions for number theoretic fermions contributes at the limit $s \rightarrow 0$. The stronger requirement $\zeta_D(s) = -log(Z_F(s,X^4))$ can be considered but is not necessary. The infinite prime would characterize the non-thermalized modes. The broken $\lambda \rightarrow 1/\lambda$ symmetry would correspond to the broken $E \rightarrow -E$ symmetry in the arithmetic quantum field theory and makes sense only if one considers the TGD counterpart for the regularization of the Dirac determinant discussed already in the chapter "Configuration Space Spinor Structure".

### 4.4.1 A model for the Dirac Zeta function as a product of number theoretic partition functions

The decomposition of the configuration space to sectors $D_P$ labelled by infinite prime $P$ and the experience from the construction of infinite primes suggest strongly the hypothesis that the fermionic zeta function $\zeta_D(s = 0, X^4)$ is at least at the limit $s \rightarrow 0$, very closely related with the partition function $Z_F(s,X^4)$ of a generalized arithmetic quantum field theory involving both fermionic and bosonic labelled by primes and possessing super-symmetry. This means that $Z_F$ must have interpretation as a partition function constructible as a product of the bosonic and fermionic single particle partition functions $Z_p(B,s)$ and $Z_p(F,s)$ serving as the building blocks of the Riemann zeta function $\zeta(s)$ and its fermionic counterpart. The requirements that the coupling constant evolution of the Kähler coupling strength as a function of p-adic prime is the evolution forced by the renormalization group invariance of the gravitational constant and that $G$ has value consistent with the value implied by elementary particle mass calculations, fix the scenario completely.

The logarithm of the fermionic determinant allows a formal physical interpretation as a "thermal energy" at the infinite temperature limit for an 'arithmetic' system for which the ten modes $2 \leq p_i \leq 23$ and $p$ defining an infinite purely fermionic prime are thermalized. The requirement that the bosonic effective action is finite, forces to use fermionic partition functions $Z_F(p,s) = 1 + p^{-s}$ rather than bosonic partition functions $Z_p(B,u) = 1/(1-p^{-s})$ as one might expect. This motivates the guess that fermionic determinant is expressible as a logarithmic derivative of a number theoretic fermionic partition function at $s = 0$ for a system in which only the modes $p_i$ and $p$ are effectively present.

If infinite prime $P$ characterizes the ground state it should somehow determine the value of the fermionic determinant and thus the value of the Kähler coupling strength. Obviously $P$ should characterize the exceptional primes $p_i$ and $p$, and could thus be an infinite prime generated from $X+1$ or $X-1$ by adding number theoretic fermions in the modes $p$ and $p_i$. The simplest assumption is that all other fermionic and all bosonic modes are thermalized with identical oscillator frequencies so that the fermion modes of type $X+1$ and antifermion modes of type $X-1$ compensate each other in the fermionic determinant and only finite number of modes for which the difference of the theoretic frequencies is proportional to the difference of Kähler actions for the adjacent maximal deterministic regions, remain visible.

*1. The identification of the frequency spectrum*

The basic harmonic oscillator frequencies $\omega_p$ associated with the number theoretic fermionic and bosonic oscillators are given by

$$\begin{aligned} \omega(p) &= \omega_0 log(p) \ , \\ \omega_0 &= 2^{-\#\{p_i\}-1}\frac{S_K(X^4_+)-S_K(X^4_-)}{S_K(CP_2)} \ . \end{aligned} \tag{12}$$

The multiplication by the factor $2^{-\#\{p_i\}-1}$, where $\#\{p_i\}=9$ is the number of the small bosonic primes $p_i$ present in the infinite prime $P$, is necessary in order to obtain the correct normalization of the Kähler coupling strength. $\omega_0$ is proportional to the difference of Kähler actions for the maximal deterministic regions meeting at the causal determinant. That $CP_2$ action serves as a natural frequency scale is a guess motivated by the fundamental role of $CP_2$ type extremals.

The eigenvalues of the operator $D_+D_-^{-1}$ formed from the operators $D_+$ and $D_-$ assumed to commute, are exponentials of the thermalized single particle energies of the arithmetic system $\lambda_p = p^{\omega_0}$ and most of them are equal to one.

*2. Number theoretic partition function as a product of number theoretic fermionic partition functions*

The guess is that the fermionic determinant $DET(D_+D_-^{-1}$ corresponds to a logarithmic derivative at infinite temperature limit for the product of fermionic partition functions for the small 'background' primes $p_i$ and prime $p$:

$$\begin{aligned} Z_F(s,X^4) &= \prod_{2\le p_i\le p_{max}} Z_{p_i}(F,u)^{k(p_i)}\times Z_p(F,u)^{k(p)} \ , \\ Z_p(F,u) &= 1+p^{-u} \ , \\ u &= \omega_0 s \ . \end{aligned} \tag{13}$$

The parameter $s$ is analogous to the inverse of temperature: $s\leftrightarrow 1/T$. The values of the integers $k(p_i)$ and $k(p)$ turn out to be equal to unity.

This ansatz makes sense only in the sense that it gives fermionic determinant correctly so that the equality $\zeta_D(s) = -log(Z_F(s,X^4))$ need not hold true. The presence of an infinite number of eigenvalues compensating each other in the fermionic determinant is expected by general considerations. If the transition between maximal deterministic regions modifies only a finite number of eigenvalues in the spectrum of the modified Dirac operator, the spectrum of the operator $D=D_+D_-^{-1}$ constructed from the modified Dirac operators $D_+$ and $D_-$ for the adjacent maximal deterministic regions is invariant under the transformation $\lambda\rightarrow 1/\lambda$ apart from the eigen values labelled by $p_i\in\{2,..,23\}$ and $p$ for which one has $\lambda_+(q)/\lambda_-(q) = q^{\omega_+-\omega_-}$, $q\in\{p_i,p\}$. Other eigenvalues would be equal to one. If Chern-Simons action gives the dimension of the spectral asymmetry, the value of Chern-Simons charge for the causal determinant would be a multiple of 10.

*3. Expression for the bosonic action*

The bosonic action $S_B$ can be deduced by using the general formula for the fermionic determinant displayed already earlier:

$$S_B(X^4_+) - S_K(X^4_-) = \frac{d\zeta_d(s)}{ds}_{|s=0} = -\frac{d\left[log(Z_F(s,X^4)\right]}{ds}_{|s=0} \ . \tag{14}$$

The formula for the effective action is same as given by the direct renormalization group argument for infinite primes:

$$\begin{aligned} S_B(X^4) &= \frac{\pi}{8\alpha_K(p)}\frac{S_K(X^4)}{S_K(CP_2)} \ , \\ \frac{1}{\alpha_K(p)} &= \frac{4}{\pi}(log(p)+log(K^2)) \ , \\ K^2 &= \frac{R^2}{G} = \prod_{2\leq p_i\leq p_{max}} p_i \ . \end{aligned} \tag{15}$$

Depending on whether $\alpha_K$ or $G$ is RG invariant $p$ denotes $M_{127}$ or arbitrary prime. $p_{max} = 23$ and $k(p_i) = k(p) = 1$ predicts a correct value for gravitational constant as already found. $k(p_i) = k(p) = 1$ is very natural in the light of thermodynamical interpretation.

*4. About the expression for the Kähler coupling strength*

The expression of the Kähler coupling constant is deduced using the requirement that exponent of $CP_2$ Kähler action is power of integer. Number theoretical considerations of the chapter "Fusion of p-Adic and Real Variants of Quantum TGD to a More General Theory" support the expression for the Kähler coupling strength as

$$\frac{1}{\alpha_K(p)} = \frac{137}{107}(log(p)+log(K^2)) \tag{16}$$

obtained by the replacement $4/\pi \rightarrow 137/107$. The resulting expression is physically highly attractive since it would automatically raise the primes $k = 137$ and $k = 107$ corresponding to the length scales of atomic physics and hadron physics to a very special position concerning p-adicization. In this case one should replace the parameter $\omega_p$ by

$$\omega_p \rightarrow \hat{\omega}_0 log(p) \ , \quad \hat{\omega}_0 = \tfrac{\pi}{4}\tfrac{137}{107}\omega_0 \ , \quad \omega_0 = 2^{-\#\{p_i\}-1}\tfrac{S_K(X^4)}{S_K(CP_2)} \ . \tag{17}$$

A possible justification for this replacement comes from the rationality of $\omega_p$ becomes from the ansatz $log(p) = q_1 exp(q_2)/\pi$, $q_2(p_1) \neq q_2(p_2)$ for $p_1 \neq p_2$ ansatz discussed in the chapter "Fusion of p-Adic and Real Variants of Quantum TGD to a More General Theory".

This replacement gives hopes of generalizing the procedure defining the fermionic determinant also to the finitely extended p-adic case. $p^{-\omega s} = exp(-\omega_p s)$ exists in this case in a finite extension of p-adic numbers p-adically for all rational numbers $s$ which are divisible by $p$ and if finite-dimensional extensions of p-adics defined by $e^{1/p^n}$ are allowed they exists even more generally. Thus the limiting procedure $s \rightarrow 0$ could be carried in p-adic number field by keeping $s$ rational.

The real units defined by infinite primes could allow to define the exponent for almost all p-adic primes. By definition infinite primes are not divisible by any finite prime. The simplest infinite prime is of form $\Pi = 1 + X,\ X = \prod_i p_i$, where product is over all finite primes. More general infinite primes are of form $\Pi = mX/s + ns$, $s = \prod_i q_i$, $n = \prod_i q_i^{m_i}$ such that $m = \prod_i p_i^{n_i}$ and $s$ have no common factors. The interpretation could be as a counterpart for a state of a super-symmetric theory containing fermion and $m_i$ bosons in the mode labelled by $q_i$ and $n_i$ bosons in the mode labelled by $p_i$.

One can extend rational numbers by the multiplicative group of real units obtained from infinite primes and powers of $X$. Real number 1 could carry infinite information resources in its hidden structure! Note that the inclusion of the entire hierarchy of infinite primes gives even more information storage capacity. The idea generalizes to the level of rational octonions and generalize octonionic units could in principle code the quantum state of the universe to their structure. This group is generated as products of powers of $Y(n/m) = (n/m) \times [X/\Pi(m/n)]$, which is a unit in the real sense. Each $Y(n/m)$ would define a subgroup of units and the power of $Y(n/m)$ would code for the number of factors of a given integer with unit counted as a factor. This would give a hierarchy of integers with their p-adic norms coming as powers of $p$ with the prime factors of $m$ and $n$ forming an exception manifested in p-adic physics of cognition. Universe would "feel" its real or imagined state with its every point, be it a point of space-time sheet, of imbedding space, or of configuration space.

### 4.4.2 The space-time interpretation for the special primes

A possible space-time interpretation for the presence of the small primes $p_i$ and prime $p$ in the active spectrum could be following.

1. The index theorem in its standard form for the product $D_+D_-^{-1}$ of the modified Dirac operators would suggest that these primes label modes for which the conformal transformation inducing $\lambda \rightarrow 1/\lambda$ in symmetric case maps them to zero modes. The generalization of the index theorem would tell the number of the eigenvalues which change at the causal determinant and would be a highly non-trivial statement about the nature of the non-determinism. Physically this option looks more plausible.

2. The simplest causal determinant corresponds to an elementary particle condensed at a space-time sheet characterized by $p$, whereas more complex many particle causal determinants would give in a good approximation integer power of the vacuum functional.

3. The simplest possibility is that topological condensation occurs in 10 steps $CP_2$ type extremal $\rightarrow 2 \rightarrow 3 \cdots 23 \rightarrow p$ so that a 10-level hierarchy of space-time sheets is involved. Various symmetry breaking spinor zero modes would reside at these space-time sheets.

4. Without a proper normalization of the modified Dirac action one would have $\lambda_+(n) = \sqrt{1/k} \times n^{\omega}$ and $\lambda_-(n) = \sqrt{k} \times n^{-\omega}$ so that there product would give unity if the eigenvalues are in 1-1 correspondence.

If some eigenvalues go to zero some power of $k$ results in Dirac determinant by the presence of "lonely" eigen values. In this case the requirement that the eigenvalues $p^{\omega}$ reduce to unit for a vanishing Kähler action fixes the normalization. The simplest guess is that the un-renormalized eigenvalues $\hat{\lambda}$ and normalized eigenvalues $\lambda$ are related by

$$\begin{aligned} \hat{\lambda} &= \frac{R^2}{G}\lambda \quad , \\ \frac{R^2}{G} &= \prod_{2 \leq p_i \leq 23} p_i \quad . \end{aligned} \tag{18}$$

### 4.4.3 The spectral asymmetry, infinite primes, negative energies, and electric-magnetic duality

The interpretation of the ansatz for the fermionic determinant in terms of infinite primes would be following.

1. The spectra of $D_+$ and $D_-$ could correspond to the presence of the infinite primes generated from infinite vacuum primes $X \pm 1$, $X = \prod_k p_k$ (product over all finite primes). The asymmetric modes labelled by $p_i$ and $p$ would be coded by a fermionic infinite prime of type $X - 1$ or $X + 1$. In the arithmetic quantum field theory the asymmetric modes correspond to a difference of frequency scales proportional to $\omega_0 = \omega_+ - \omega_-$.

2. The two families of infinite primes correspond to the two possible signs of scaling momentum and "arrow of scaling" in a complete analogy with the corresponding interpretation based on time orientation in the case of ordinary Dirac operator.

3. The "arrow of scaling" would mean that for the 10 exceptional primes generated from $X+1$ or $X-1$ the asymmetric modes for number theoretical fermions would have different frequency scales. The "arrow of scaling" would naturally correlate with whether the sub-cosmology is expanding or contracting and thus with the arrow of the geometric and experienced time.

4. The two vacua correspond to the two possible quantizations in which creation and annihilation operators change their role and would thus characterize positive and negative energy matter predicted by quantum TGD. Phase conjugate photons behaving like negative energy photons would indeed be negative energy matter.

5. If the configuration space is identified as the union of configuration spaces $CH(a)$ ("sub-cosmologies") associated with all possible unions of future and past light cones $M^4_\pm(a)$ with the dip any point $a$ of $M^4$ then $M^4_\pm$ naturally corresponds to $X \pm 1$. Why the ontology of TGD makes this picture natural is discussed in [E7].

6. An important implication is that Kähler coupling strength $\alpha_K$ changes sign in the transformation $\lambda \rightarrow 1/\lambda$ induced by the transformation $X - 1 \leftrightarrow X + 1$ of vacuum infinite prime and changing the direction of the "scaling arrow". The change of the sign would be a scaling version of the $g \rightarrow 1/g$ duality relating weak and strong coupling regimes in gauge theories. The absolute minimization of Kähler action would require that the extremals would be dominated by Kähler magnetic *resp.* electric fields in the two phases so that the transformation would actually correspond to the electric-magnetic duality suggested to be a basic symmetry of quantum TGD. The two possible manners to define configuration space metric using induced Kähler electric and magnetic fields would correspond to the two dually related phases [B2, B3]. The earlier belief that duality acts trivially does not seem to be true. Note that the requirement that the value of the Kähler function is finite for the entire universe requires the vanishing of the amount of Kähler function per volume. This density must be vanishing in both sectors.

7. It is possible to understand how the sub-cosmologies are created from vacuum at the moments of "big bangs" for sub-cosmologies in the fractal cosmology. Negative energy magnetic flux tubes stable in the reverse time direction are time-reflected as positive energy magnetic flux tubes which are unstable and decay to thicker magnetic flux tubes with a weaker field strength. This picture justifies the TGD inspired cosmology in which positive and negative energy sectors dissipate in different directions of geometric time. Both sectors give positive contribution to the gravitational mass and negative energy sector would give the dominating contribution to the dark gravitational energy.

## 4.5 Equivalence of loop diagrams with tree diagrams from the axioms of generalized ribbon category

Concerning the algebraic description of quantum criticality, the key idea is that generalized Feynman diagrams are analogous to knot and link diagrams in the sense that they allow also "moves" allowing to identify classes of diagrams and that the diagrams containing loops are equivalent with tree diagrams, so that there would be no summation over diagrams. This would be a generalization of duality symmetry of string models.

TGD itself provides general arguments supporting same idea. The identification of absolute minimum of Kähler action as a four-dimensional Feynman diagram characterizing particle reaction means that there is only single Feynman diagram instead of functional integral over 4-surfaces: this diagram is expected to be minimal one. At quantum level S-matrix element can be seen as a representation of a path defining continuation of configuration space ($CH$) spinor field between different sectors of $CH$ corresponding to different 3-topologies. All continuations and corresponding Feynman diagrams are equivalent. The idea about Universe as a computer and algebraic hologram allows a concrete realization based on the notion of infinite primes, and space-time points become infinitely structured monads [E10]. The generalized Feynman diagrams differing only by loops are equivalent since they characterize equivalent computations.

Hopf algebra related structures and appropriately generalized ribbon categories [35, 36] could provide a concrete realization of this picture. Generalized Feynman diagrams which are identified as braid diagrams with strands running in both directions of time and containing besides braid operations also boxes representing algebra morphisms with more than one incoming and outgoing strands. 3-particle vertex should be enough, and the fusion of 2-particles and $1 \rightarrow 2$ particle decay would correspond to generalizations of the algebra product $\mu$ and co-product $\Delta$ to morphisms of the category defined by the super-canonical algebras associated with 3-surfaces with various topologies and conformal structures. The basic axioms for this structure generalizing ribbon algebra axioms [35] would state that diagrams with self energy loops, vertex corrections, and box diagrams are equivalent with tree diagrams.

Tensor categories might provide a deeper understanding of p-adic length scale hypothesis. Tensor primes can be identified as vector/Hilbert spaces, whose real or complex dimension is prime. They serve as "elementary particles" of tensor category since they do not allow a decomposition to a tensor product of lower-dimensional vector spaces. The unit $I$ of the tensor category would have an interpretation as a one-dimensional Hilbert space or as the number field associated with the Hilbert space and would act like identity with respect to tensor product. Quantum jump cannot decompose tensor prime system to an unentangled product of sub-systems. This elementary particle like aspect of tensor primes might directly relate to the origin of p-adicity. Also infinite primes are possible and could distinguish between different infinite-dimensional state spaces.

Quantum criticality means that renormalization group acts like isometry group at a fixed point rather than acting like a gauge symmetry as in the standard quantum field theory context. Despite this difference it is possible to understand how Feynman graph expansion with vanishing loop corrections relates to generalized Feynman graphs and a nice connection with the Hopf- and Lie algebra structures assigned by Connes and Kreimer to Feynman graphs emerges [37]. For instance, it is possible to deduce an explicit representation for the universal momentum and p-adic length scale dependence of propagators in this picture. The renormalization parameter $Z$ is expressible solely in terms of the zeros of Riemann Zeta. The condition that loop diagrams are equivalent with tree diagrams gives explicit equations which might fix completely also the p-adic length scale evolution of vertices. Quantum criticality in principle fixes completely the values of the masses and coupling constants as a function of p-adic length scale.

# 5 The quantum dynamics of topological condensation and connection with string models

Despite its central role in the applications of quantum TGD, the mathematical description of the topological condensation has remained a longstanding challenge for quantum TGD. Even at the classical level the problem has stubbornly resisted all attacks to solve it. The situation looks even more formidable at quantum level: how one could understand topological sum contacts at quantum level when even the proper classical understanding is lacking?

A related open problem is whether Equivalence Principle holds true in TGD framework or not. For instance, the physical interpretation of vacuum extremals, in particular the fact that Robertson Walker cosmologies are vacuum extremals, have been a deep puzzle from the first days of TGD. The new view about energy forcing to reformulate Equivalence Principle resolved the paradoxes and allowed to understand what happens in topological condensation.

A further longstanding but much more technical problem has been to understand precisely how gravitational constant, which is not fundamental constant in TGD, emerges. The success of string models suggests that the analog of the string model action defined as area normalized by gravitational constant or $CP_2$ length squared should somehow pop up from the theory and the question is how this could occur.

Even these steps of progress do not tell anything precise about the interaction of the levels of the topological condensate at quantum level. The newest step of progress relates to the identification of the hierarchical structure of topological condensates with the hierarchical structure of infinite primes to be discussed in the later sections.

## 5.1 Questions related to topological condensation

One group of important questions relates to the relationship between inertial and gravitational masses. Are they identical or does the proportionality constant depend on external conditions, such as external electromagnetic field in which particle resides? Does the very attractive notion of gravitational flux generalize from the Newtonian context to TGD framework? That this might occur is suggested since one expects that the throats of wormhole contact behave like objects possessing some gravitational mass. This in turn raises the question how to generalize the idea about geodesic motion of a point like singularity so that it applies in the case of the wormhole throats.

The stumbling block has been the same as in general relativity, where point like test particles are assumed to move along geodesic lines of the background metric but actually correspond to the singularities of the metric. This kind 'as if'-description is highly un-satisfying. In the case of TGD singularity means that the induced metric is degenerate at the inner and outer elementary particle horizons accompanying the wormhole contact with Euclidian metric and connecting two Minkowskian space-time sheets. For $CP_2$ type extremals representing elementary particles and having Euclidian metric only single elementary particle horizon is present. It seems extremely difficult to say anything about the physics in such extreme conditions when only the intuition based on the Minkowskian physics is available.

## 5.2 Super-conformal invariance and new view about energy as solution of the problems

At algebraic level TGD and string models are characterized by various super-conformal symmetries. The scaling operator $L_0$ is essentially mass squared operator and string model action generalizes the action for the geodesic line. One might therefore ask whether these extremely powerful symmetries might allow to generalize the idea about the motion of point like particle along geodesic line so that one would have model for the motions of wormhole throats along space-time sheets.

The properties of the elementary particle horizons, which are light like 3-surfaces surrounding wormhole contacts connecting two parallel space-time sheets, might help to understand what happens in the topological condensation. It is good to describe briefly the evolution of ideas to more clearly understand the general role of these symmetries.

1. The first discovery was that the metric 2-dimensionality of the future light cone boundary (moment of big bang) implies super-canonical and related conformal invariance. These symmetries reduce the construction of the configuration space geometry and spinor structure to symmetry considerations. If Kähler action were deterministic, the construction of the quantum theory would reduce to the light cone boundary and one would lose time as in the quantized general relativity. This would be mathematically very nice but physically a catastrophe. Fortunately, this is not the case and the problem was how to take into account the non-determinism of the Kähler action.

2. Ironically, the gigantic super-canonical symmetries did not relate directly to the elementary particle mass spectrum: in particular, electro-weak symmetries are a problem. The next discovery came with the number theoretical formulation of quantum TGD: there is also a second conformal invariance involved. This symmetry might be called quaternion conformal invariance and is due to the local quaternion structure of the 4-D space-time surface. This symmetry leads to the gauging (or Kac-Moodying) of Poincare, color and electro-weak symmetries which have nothing to do with the configuration space metric (allowing zero modes). Also a beautiful understanding of the fermionic degrees of freedom emerges. It is the quaternion conformal invariance which determines elementary particle masses. Contrary to the first belief, super-canonical invariance makes itself visible also at the level of elementary particle propagators and vertices and brings in non-stringy effects. It also involves new physics appearing in all length scales and being especially relevant for the understanding of consciousness.

3. The next discovery was that the light likeness for the boundaries of the space-time sheets provides a very general manner to satisfy boundary conditions of the Kähler action identically. Light likeness in turn implies metric two-dimensionality and conformal invariance. A second example of light like 3-surface $X^3_l \subset X^4$ is provided by elementary particle horizons at which the Minkowskian signature of the induced metric is changed to an Euclidian one. There is a strong temptation to identify this symmetry as quaternion conformal invariance but it is not quite obvious whether this identification is correct.

4. The new view about energy (it took 25 years to end up with it!) implies the net quantum numbers and classical conserved quantities of the Universe vanish. By the crossing symmetry the vanishing of the net energy of is consistent with elementary particle physics. Also a consistency with macroscopic physics is achieved. Gravitational energy can be identified as the difference of Poincare (inertial energies) of positive and negative energy matter and is thus non-vanishing even for vacuum extremals and non-conserved as it should be. This provides elegant interpretation for inertial vacuum property of the Robertson-Walker cosmologies.

   The notions of inertial and gravitational masses generalize to the notions of inertial and gravitational quantum numbers in general. An interesting question is whether the gravitational version of color Kac Moody algebra could have interpretation in terms of electro-weak Kac Moody algebra and generators having the quantum numbers of Higgs field. The identification of Higgs field as associated with the complement of $U(2)$ sub-algebra of color algebra have been already proposed but this proposal was based on the observation that for the known absolute minima of the Kähler action classical color currents decompose to two separately conserved currents.

5. Taking into account the non-determinism of Kähler action, the new view about energy forces to say a final good-bye to the idea about the Universe as a deterministic clockwork which started to run at the moment big bang. Pairs of space-time sheets of opposite energy can be created from vacuum anytime and this occurs most naturally at light like 7-surfaces $X_l^3 \times CP_2$ taking the role of causal determinants so that the TGD counterparts of big bangs are possible everywhere in $M^4$. The notion of deterministic cosmology can make sense only in a statistical sense.

   This means generalized super-canonical invariance at the level of the configuration space of 3-surfaces, and one must generalize the construction of the configuration space metric and spinor structure. If not anything else, this at least allows to understand something about the enormous fractal complexity of the configuration space geometry.

   Configuration space decomposes into sectors such that the line element of the metric contains the contributions from the light like 7-surfaces $X_l^3 \times CP_2$, $X_l^3$ light like surface. These sectors are related by Poincare transformations to each other so that Poincare invariance acts in a very natural manner. In particular, it becomes possible to assign definite value of geometric time to each sector of this kind: without the non-determinism of Kähler action this kind of assignment would not be possible and the assignment of definite value of geometric time to subjectively experience time would be lost as it is lost in general relativity. These light like surfaces have the structure of category with respect to the formation of set theoretic intersections and unions and category theory might help to disentangle the complexities of the configuration space geometry.

6. The assignment of Poincare momentum and other conserved quantum numbers with quaternion conformal invariance suggests that also gravitational momentum and gravitational quantum numbers should be assignable to some conformal invariance. Perhaps the quaternion conformal deformations of light like boundaries and elementary particle horizons restricted to deformations of the space-time surface in a direction normal to the light like surface could define this conformal symmetry.

## 5.3 Connection with string models and how gravitational constant appears

In the proposed framework the description of what happens in the topological condensation reduces to gravitational super-conformal invariance. The scaling operator $L_0$ has an interpretation as a mass squared operator at configuration space level and the interpretation of the $L_0$ associated with elementary particle horizons as gravitational mass squared suggests strongly itself. If so, then gravitational *resp.* inertial mass would correspond to the gravitational *resp.* quaternion conformal super-conformal invariance.

Equivalence Principle, if true in strong sense, would state that particles have same quantum numbers with respect to these two conformal symmetry algebras. Some kind of mapping of the states of these representations to each other would be obviously required if Equivalence Principle is true in strong sense. Quaternion conformal and gravitational conformal invariance might be even one and the same thing. In fact, quaternion conformal invariance implies that space-time region in a well-defined sense reduces to a two-dimensional representative inside which quaternionic coordinates are commutative and I have proposed that this representative corresponds to a 2-surface associated with the elementary particle horizon. If so, then one would a strong symmetry based argument supporting Equivalence Principle.

This finding forces to ask whether the counterpart of the string model action emerges somehow from the theory. Consider first canonical conformal invariance. In the case of the light cone boundary the area of the light like coordinate constant sphere is proportional to the distance squared from the dip of the light cone so that no unique surface area can be associated to the light like boundary. Same is true for the light like like boundaries having light like $M^4$ projections. Note however that conformal invariance states that the area as such does not matter.

For the elementary particle horizons associated with gravitational conformal invariance, which in many respects resemble the black hole horizons (recall only the elementary particle black hole analogy), situation might be different. If the situation is stationary with respect to some time coordinate $t$, which means that the area $A(t)$ of $t = constant$ section is invariant under time translations, the exponentials $exp(-A/A_0)$ of the time=constant section, where $A_0$ is some normalization area, can (but need not!) appear as additional multiplicative terms in the vacuum functional. Hence the two-dimensional string orbits would be replaced by the metrically 2-dimensional elementary particle horizons in TGD framework.

Note that 'strings' have now Euclidian signature of metric whereas in superstring models the signature is Minkowskian. However, the dynamical degrees of freedom for strings are always Euclidian which means that an automatic elimination of the non-physical degrees of freedom occurs in TGD framework. If this picture is correct, gravitational interaction could be modelled using a generalization of string models diagrams in which incoming and outgoing strings are replaced with wormhole contacts.

Consider next the identification of the constant $A_0$ assuming that the exponents $exp(-A/A_0)$ indeed are present in the state functionals. In TGD framework gravitational constant is not a fundamental constant but at the level of quaternion conformal representations its role is take by $CP_2$ length squared $R^2 \sim 10^8 G$ (only in this manner p-adic mass calculations give sensible results and cosmic strings have a sensible string tension).

1. The first geometrically motivated possibility is that one has $A_0 = xR^2$, where $x$ is of order unity. This option is unavoidable if gravitational and quaternion conformal symmetries are actually one and the same thing. One should be able to show that the interaction energy between wormhole contacts is $GM_1M_2/r$ with $G \sim 10^{-8}R^2$. The task would be to understand why the naive expectation for the value of the gravitational constant is scaled down by a factor of order $10^{-8}$. I have developed some arguments for this kind of reduction in the chapter devoted to the construction of S-matrix and the arguments are based on the value of $CP_2$ action exponentials associated with the virtual gravitons. Also p-adic physics based arguments giving explicit formula for $G$ in terms or $R^2$ exist.

2. The option suggested by naive generalization of string models is that $A_0$ is of the same order of magnitude as gravitational constant:

$$A_0 = kG \equiv 4\pi\alpha' \ . \tag{19}$$

   where $k$ is a numerical constant and $T = 1/4\pi\alpha'$ corresponds to string tension. If this option is correct, the area of the elementary particle horizon in the induced metric should be of order Planck length squared and much smaller than $CP_2$ length squared. Now one should develop a good argument for why gravitational constant, presumably constrained by internal consistency conditions, differs by 8 orders of magnitude from $CP_2$ length squared. Also one should show that the different scale factors for gravitational and quaternion conformal masses do not lead to contradiction. It must be emphasized that one should not draw too hasty conclusions since the functional integral defined by the exponential $exp(-A/A_0)$ and exponential of the K̈hler action is quite different thing from the stringy functional integral.

## 5.4 Elementary particle vacuum functionals and gravitational conformal invariance

Also a connection with elementary particle vacuum functionals emerges. Elementary particle vacuum functions [F1] are associated with the boundary components of particle like surfaces and conformal and modular invariance are key elements in their construction. Also this part of the theory has remained relatively disjoint from quantum TGD proper. If the 3-dimensional boundary components of $CP_2$ type extremals have degenerate metric, then the assumptions involved with the construction of the elementary particle vacuum functionals find a justification.

The construction of the elementary particle functionals might make sense also at the level elementary particle horizons and the question whether the successful genus-generation correspondence is associated with genuine boundary components of the $CP_2$ type extremal or with the elementary particle horizon remains still open!

Quaternion conformal invariance requires that $CP_2$ type extremals have holes. If one assigns elementary particle vacuum functional with the boundary component of $CP_2$ type extremal then one must assume that there is strong deformation of the $CP_2$ type extremal to the direction of $M^4$ implying that the boundary has degenerate metric. Therefore if the notion of free $CP_2$ type extremal makes sense at all, its boundary must be equivalent with elementary particle horizon. Could it be that in the topological condensation the hole of a free $CP_2$ type extremal becomes elementary particle horizon and is effectively replaced with the boundary of the elementary particle space-time sheet.

## 5.5 Questions about topological condensation

One can imagine two manners for how $CP_2$ type extremals topologically condense.

1. The first possibility is to glue $CP_2$ type extremals along the boundary of the hole to the corresponding hole of the elementary particle space-time sheet. This option is very natural since the boundary of $CP_2$ type extremal must have a degenerate metric in any case. The number of the boundary components is conserved in this process. If gauge fluxes are conserved at the Minkowskian space-time sheets (they need *not* be conserved and even less so for Euclidian space-time regions), the boundary of the elementary particle space-time sheet having a size of order Compton length would carry the gauge charges associated with the free $CP_2$ type extremal with a hole. The classical counterpart for charge normalization would perhaps be the renormalization of the gauge charges in this process.

2. Alternatively, a $CP_2$ type extremal representing elementary particle and having a boundary component with a given genus could be glued to the elementary particle space-time sheet so that boundary component does not disappear. Does the elementary particle horizon carry the negatives of the original elementary particle quantum numbers in this case? Does also the boundary of the elementary particle space-time sheet carry the negatives of the original elementary particle vacuum numbers as gauge flux interpretation suggests? If so, then one ends up with a rather paradoxal situation since the total quantum numbers of the structure vanish. Thus it would seem that the first option is more plausible for the topologically condensed elementary particles.

Also the topological condensation of a Minkowskian space-time sheet at a larger similar space-time sheet is possible. Now both inner and outer elementary particle horizons appear. If the electric gauge fluxes and gravitational flux are conserved as they flow through the wormhole contacts, the wormhole throats of the wormhole contact connecting two space-time sheets with a Minkowskian signature are carriers of opposite classical quantum numbers. The question is whether this generalizes to the level of the super-conformal representations as an excellent approximation or exactly.

# 6 Algebraic physics at the level of configuration space

This section is not a distilled final answer to the challenges involved with the p-adicization of the configuration space geometry and spinor structure. The reasons are simple: it is not even clear what the basic challenges are! Is finite-p p-adicization enough/possible or should one try to imagine what infinite-P p-adicization could mean? This serves as an excuse for why the subsections represent only a loose collection of essays written during a time period about one decade reflecting the evolution of ideas and the drift of focus.

The last subsection represents the most recent approach to the p-adicization of the configuration space geometry. The reduced configuration space identified as the space of the maxima of Kähler function is the key notion. This space is the direct counterpart of the spin glass landscape known to obey ultrametric topology naturally. This approach is reasonably concrete and relies heavily on the most recent, admittedly speculative, view about quantum TGD.

## 6.1 A possible view about basic problems

Algebraic approach provides new insights to the construction of configuration space geometry and spinor structure [B2, B3, B4]. In particular, one can provide answers to the puzzling questions about to the precise role of the p-adic numbers.

1. Are finite-p p-adics needed at all or should one allow only real and infinite-P p-adic topologies, which seem to be more easily realizable? Could the infinite prime associated with the space-time surface characterize the infinite-P p-adic topology of the region of configuration space where it belongs? What about the topology of the configuration space sector representing space-time surfaces representing given infinite integer $N$? Is the topology $N$-adic or infinite-P p-adic for some prime factor of $N$? Is infinite-P p-adic topology associated with any factor $P$ possible and does it characterize the view of the observer characterized by that particular $P$? By previous arguments infinite-P p-adicization looks surprisingly simple procedure and there seems no reason to exclude this possibility.

2. Do the finite-p p-adic regions of the space-time surface contribute to the configuration space geometry or not? Could it be that they give no contribution in accordance with the identification of p-adic regions as the geometric correlate for 'mind-stuff'? Or do both real, infinite-P, and finite-p p-adic configuration spaces make sense and represent configuration space as 'seen' by a particular observer? Is the number field associated with the configuration space metric and other structures same as that defining the local topology of the configuration space as it seems natural to assume?

Both the maximal algebraic and topological democracy and the generalized notion of number field suggest that one should allow maximal generality, which somewhat paradoxically should pay itself back as enormously tight constraints on the real physics. Algebraic approach raises also further questions.

1. Do quaternions and octonions emerge also at the level of the configuration space geometry as useful auxiliary tools? Could configuration space geometry inherit quaternion-holomorphic character of the space-time geometries? Quaternion-conformal structures certainly decompose configuration space into subsets corresponding to given quaternion-conformal structure which suggests that quaternions might serve as an important auxiliary tool. The hypothesis that configuration space has Hyper-Kähler structure fits nicely with the idea of quaternion structure and implies that configuration space is Ricci flat: this is necessary in order to have a configuration space integration measure which is free of infinities. The choice of a particular Kähler structure of configuration space (there is a full sphere $S^2$ of such structures) would

correspond at the level of conformal weights a selection of a commutative complex plane of quaternions.

2. Quaternionic super-conformal transformations act only inside the sets of quaternion conformally equivalent surfaces whereas the super-canonical invariance acts as a symmetry of the light like 7-surfaces $X_l^3 \times CP_2$, where $X_l^3$ is light like 3-surface of $M^4$ and transforms quaternion conformally equivalent surfaces to each other. How does quaternionic super-conformal invariance relate to the super-canonical conformal symmetries?

   The progress in the understanding of the super-conformal symmetries of the configuration space geometry [B2, B3, C5] allows to answer this question quite satisfactorily. Super-canonical conformal weights can be regarded as punctures in complex plane in which quaternion conformal symmetries act as conformal transformations. The generators of super-canonical algebra can be seen as quaternion conformal fields in the space of super-canonical conformal weights labelled by quaternion conformal weights. This view leads to beautiful connections between braid and quantum quantum groups, type $II_1$ factors of von Neumann algebras, and Clifford algebra defined by configuration space gamma matrices. Super-canonical conformal weights for the generators of super-canonical algebra correspond zeros of Riemann Zeta and this leads to a connection with Riemann Zeta and Beraha numbers and to a realization that the imaginary parts of the zeros can be regarded as primes in additive sense. Hence number theory would become a key element of the configuration space geometry.

   The existence of two separately conserved fermion numbers is essential part of the picture. Lepton-quark dichotomy corresponds to Ramond-NS and SUSY-kappa dichotomies and configuration space dynamics reduces to both Ramond and NS Super Virasoro conditions satisfied separately by both super-canonical and quaternion conformal algebras.

   A still unsettled question is whether quaternion conformal degrees of freedom do contribute to the configuration space geometry or not. One could define the contribution to metric in terms of anticommutators of the quark-like super-generators but it is unclear whether the symmetries have representations in terms of configuration space Hamiltonians as required.

3. Further questions are related to the possibility (necessity is perhaps too strong word) to generalize the configuration space to include configuration spaces for $4n$-dimensional surfaces in $8n$-dimensional imbedding spaces. As already explained this generalization has a very nice interpretation in terms of the notion of topological condensate and provides a powerful mathematical tool for the mathematical modelling of topological condensate and of many-particle states at configuration space level. The description is analogous to the description of $n$-particle state as a point in $E^{3n}$ and conforms nicely with the somewhat mysterious more or less identity $CH = CH^2 = ....$ making sense only in infinite-dimensional context.

## 6.2 Algebraic physics and configuration space geometry

The reader not familiar with the basic ideas related to the construction of the configuration space geometry and spinor structure is warmly encouraged to read the chapters [B2, B3, B4].

### 6.2.1 Configuration space as a union of symmetric spaces

The construction of the configuration space geometry and spinor structure is very algebraic process since configuration space is a union of infinite-dimensional symmetric spaces of form $G/H$ and the construction of the metric reduces to its construction in single point of the symmetric space and is dictated by super-canonical symmetry. The crucial Cartan decomposition of the tangent space of $G$ is defined by the decomposition of the super-canonical algebra $g$ to a direct sum $g = h + t$ of subspaces satisfying the conditions $[h, h] \subset h$, $[h, t] \subset t$, and $[t, t] \subset h$.

After some trials, the structure of this decomposition has become clear. The algebra has as generating elements generators labelled by both the trivial and non-trivial zeros of Riemann Zeta and their commutators give naturally rise to $g = t + h$ decomposition.

1. Only the generators with conformal weights $h = n - 1/2 - i\sum n_i y_i$, $\sum_i n_i =$ odd/even for $n$ even/odd and $h = 2n$, $n > 0$ contribute to $t$.

2. The generators with conformal weights with conformal weights $h = n - 1/2 - i\sum_i n_i y_i$, $\sum_i n_i =$ odd/even for for $n$ odd/even and $h = 2n - 1$, $n > 0$ contribute to $h$.

   The absence of the non-zero norm physical states with $h > n - 1/2 - i\sum_i n_i y_i$, $n > 1$, can be understood in terms both quaternion conformal or super-canonical conformal invariance. If these generators would not represent zero norm states, the basis would not satisfy orthogonality requirements. The action of quaternion conformal algebra on super-canonical conformal weights is non-trivial but by gauge invariance generators remain effectively invariant in the infinitesimal action whereas global action can give rise to a braiding operation. The present of complex conformal weights relates directly to the complex values of Casimir operator of Lorentz group which replaces mass squared operator in the stringy mass formula in the case of super-canonical Virosoro algebra.

The contribution of the quark-like quaternion conformal generators to the metric cannot be excluded. In this case the decomposition $g = t + h$ would correspond to half odd integer conformal weights for $t$ and integer conformal weights for $h$: this was the original guess for the Cartan decomposition in the case of super-canonical algebra.

What one should prove is that the properties of Kähler action indeed imply this decomposition.

### 6.2.2 Zero modes

There must be an infinite number of different mutually nonequivalent fundamental quaternionic coordinatizations $(q, p)$ for the imbedding space such that for each coordinatization the effective space-time metric defined by the Kähler action can be regarded as an induced effective metric of imbedding space. The symplectic transformations associated with the octonionic symplectic structure act as conformal transformations and cannot relate inequivalent coordinatizations. A good guess is that inequivalent coordinatizations (quaternion-conformal structures) of the imbedding space are related by the canonical transformations of $H$ and possibly other transformations of the imbedding space inducing changes of zero mode coordinates. These canonical transformations can be indeed extended from $\delta M^4_+ \times CP_2$ to entire $H$ uniquely since absolute minimization assigns to a deformation of a 3-surface $Y^3$ on light cone boundary a unique deformation of $X^4(Y^3)$.

What about quaternion-analytic deformations of the space-time surface: do they represent zero modes? It might quite well be that this the case. They would however correspond to genuine physical degrees of freedom since the ground states of the super-canonical representations would be different for them.

Space-time surfaces in a given quaternion conformal equivalence class have as their 'hard core' the 2-dimensional surface at which the quaternionic coordinates reduce to complex numbers. Every disjoint component of this surface has some genus probably related to the genus characterizing particle family. One might hope that one could understand at least the quaternion-conformal symmetry as an analytic continuation from this 2-dimensional surface. One might even hope that this allows to reduce the understanding of this aspect of theory to the existing wisdom provided by super-conformal theories. Complex subfields of octonions correspond to points of six-sphere and this brings in additional local degeneracy of conformal structures.

These observations suggest following working hypothesis.

1. Zero modes decompose to canonical covariants and invariants. The canonical transformations generated by the function basis of $M^4_+ \times CP_2$ labelled by even integer valued conformal weights correspond to the zero modes labelled by $H$-coordinates in the decomposition of the configuration space $CH = \cup_{zeromodes} G/H$ (do not confuse that $'H'$ with imbedding space!). There are also other zero modes labelled by canonical invariants described in chapter "Construction of the configuration space geometry". The size and shape of the 3-surface and classical Kähler field correspond to these zero modes. These invariants are useful for practical purposes if it is a good approximation to shift the dip of the light cone to the laboratory, otherwise these invariants have only cosmological significance.

2. Kähler action densities are same for the space-time surfaces related by quaternion-conformal transformations of $H$. Quaternion-conformal transformations in general correspond to non-trivial deformations of the space-time surface identifiable as zero modes.

3. The Hamiltonians labelled by odd integer valued conformal weights transform to each other non-equivalent quaternion-conformal structures of space-time surfaces which correspond to the fiber degrees of freedom of $G/H$ representing quantum fluctuations. The Hamiltonians labelled by even integer conformal weights correspond to zero modes and to the summand $h$ in the Cartan decomposition $g = t + h$ of the super-canonical algebra. Why this should be the case, is an open question and as long as a real justification for the assumption is lacking, one must take it with a grain of salt.

### 6.2.3 How to construct super-canonical algebra?

The configuration space of 3-surfaces $Y^3$ as a union of infinite-dimensional symmetric spaces labelled by zero modes obeying real or infinite-P p-adic topology associated with the infinite prime determining space-time surface $X^(Y^3)$ and having metric and spinor structure determined solely by super-symmetry, is the basic intuitive picture about configuration space geometry.

Algebraic physics vision suggests that the representation of the generators of the canonical transformations of the lightlike 7-surface $X^3_l \times CP_2$ must be expressible in terms of rational functions. In the case that Hamiltonians correspond to irreducible representations of $SU(3)$, they are products of rational functions of $CP_2$ coordinates with functions depending on coordinates of $X^3_l$. If the Hamiltonians transform according to an irreducible representation of the rotation group leaving $r_M = constant$ sphere $S^2$ invariant, they are rational functions of the complex coordinates of $S^2$. If the hypothesis that the phases $q^{iy}$ for all primes $q$ and all non-trivial zeros $z = 1/2 + iy$ of Riemann Zeta belong to a finite algebraic extension of $R_p$ for any prime $p$, the p-adicization of the Hamiltonians at imbedding space level is possible and the remaining problems relate to the 3-integrals appearing in the definition of configuration space Hamiltonians.

The modified Dirac action allows to deduce explicit expressions for the super generators. This allows to extend the formulas for the configuration space Hamiltonians in terms of the classical canonical charges associated with the Kähler action to the formulas for super-canonical charges. Configuration space metric, being numerically equal to the Kähler form in complex coordinates, in turn relates directly to the canonical charges. A natural expectation is that gamma matrices can be related by an analogous formula to the expressions for the super-canonical charges. The super algebra of quaternion-analytic transformations acts in zero modes.

### 6.2.4 Some questions

Configuration space spinors [B4] correspond to the representations of the super-canonical algebra and the basic implication of the infinite prime concept is that one can assign to a given space-time surface a unique ground state of a super-conformal representation. There are several questions to be answered.

1. Why the ground states in higher dimensions should be constructible as Fock states with many-particle states of the previous level taking the role of elementary particles? The interpretation of $4n$-dimensional surfaces as $n$ space-time sheets of topological condensate provides answer to this question. The lower level many-particle states serve as elementary particle at the next level of the topological condensate. Super-symmetry allows also to make given many-boson state fermionic and vice versa.

2. A further problem is the interpretation of the infinite primes represented by higher order irreducible polymials of infinite primes which are in one-one-correspondence with purely bosonic states at higher level. This problem is encountered already at the first level. The physical interpretation of these states is as bound states with the degree of the polynomial having interpretation as a particle number in consistency with the interpretation of the infinite integers as representations for interacting but not bound many particle states. The irreducibily of the polynomial would be number theoretical analog of bound state entanglement having join along boundaries bonds as space-time correlate. The particle number in question is clearly topological and naturally corresponds to the number of space-time sheets. This picture suggests a mapping of n:th order polynomial of generating infinite primes to n-fold product of infinite primes associated with an algebraic extension of quaternions. At the level of configuration space spinor fields higher level infinite primes correspond to definite ground states of super-canonical representations so that in principle no problems are encountered.

## 6.3 Generalizing the construction of the configuration space geometry to the p-adic context

A problematics analogous to that related with the entanglement between real and p-adic number fields is encountered also in the construction of the configuration space geometry. The original construction was performed in the real context. What is needed are Kähler geometry and spinor structure for the configuration space of three-surfaces, and a construction of the configuration space spinor fields. What (as I believe) solves these immense architectural challenges are the equally immense symmetries of the configuration space and algebraic continuation as the method of p-adicization. What I hope that everything of physical interest reduces to the level of algebra (rational or algebraic numbers) and that topology (be it real or p-adic) disappears totally at the level of the matrix elements of the metric and of S-matrix.

### 6.3.1 Generalizing the construction for configuration space metric

It is not enough to generalize this construction to the p-adic context or infinite-P p-adic context. 3-surfaces contain both real and p-adic regions and should be able to perform the construction for this kind of objects.

1. Very naively, one could start from the Riemannian construction of the line element which tells the length squared between infinitesimally close points at each point of the Riemann manifold. The notion of line element involves the notion of nearness and one obviously cannot do without topology here. The line element makes formally sense sense for real and p-adic contexts but not for the situation in which 3-surface contains both real and p-adic regions: it does not make sense to sum real and p-adic line-elements together. One can however construct a collection of real and p-adic line-elements coming from various regions of the 3-surface.

2. The notion of line-element is not actually needed in the quantum theory. Only the matrix elements of the configuration space metric matter and one could consider the possibility that configuration space metric is a collection of these matrix elements for real and p-adic regions

with the deformations of 3-surface selected so that they vanish only in real or p-adic regions with fixed value of p. If the rational boundaries between the regions of the 3-surface belonging to different number fields correspond to zero modes (and behave effectively classically), there is no need to construct the metric associated with these modes.

3. With these assumptions metric tensor would reduce to a direct sum of tensors belonging to different number fields. One cannot exclude the possibility that the values of the matrix elements are rational or complex rational (or algebraic) so that everything would algebraize and topology would disappear at the level of matrix elements completely.

4. The explicit construction of the matrix elements of the metric in the real context involves canonical symmetries, and thus also configuration space Hamiltonians, whose definition involves integrals over 3-surfaces. Definite integrals are problematic in the p-adic context, as is clear from the fact that in-numerable number of definitions of definite integral have been proposed. One might however hope that one could reduce the construction in the real case to that for the representations of super-conformal and canonical symmetries, and analytically continue the construction from the real context to the p-adic contexts by *defining* the matrix elements of the metric to be what the symmetry respecting analytical continuation gives.

5. Configuration space integration should be also continued algebraically to the finite-p p-adic context. Quantum criticality realized as the vanishing of loop corrections, in particular those associated with the configuration space integral, would reduce configuration space integration to purely algebraic process much like in free field theory and this would give could hopes about p-adicization. Matrix elements would be proportional to the exponent of Kähler function at its maximum plus matrix elements expressible as correlation functions of conformal field theory with arguments of the correlations functions identifiable as super-canonical conformal weights. The idea about the vanishing of loop corrections can be expressed as equivalence of loop diagrams with tree diagrams in the algebraic formulation for the quantum criticality discussed in [C5]. This encourages further the hopes about complete algebraization of the theory so that the independence of the basic formulation on number field could be raised to a principle analogous to general coordinate invariance.

#### 6.3.2 Generalizing the notion of configuration space spinor field

One must also construct spinor structure. Also this construction relies crucially quaternion conformal and super-canonical symmetries. Spinors at a given point of the configuration space correspond to the Fock space spanned by fermionic oscillator operators and again one might hope that supersymmetries would allow algebraization of the whole procedure. Configuration space spinor fields depend on the point of the configuration space and here the hopes are based on the construction of an orthonormal basis, whose elements are normalized to unity with respect to an inner product involving the integral over the configuration space. p-Adic configuration space integral poses deep technical problems but again analytical continuation from the real context using super-symmetries might save the situation.

Assume that all these difficulties can be overcome using super-symmetry based analytical continuation and that everything is algebraic at the S-matrix level. How to generalize of the notion of configuration space spinor field?

1. For a moment restrict the consideration to the space of 3-surfaces with fixed decomposition to real and p-adic regions such that the boundaries between regions belonging to different number fields consist of fixed sets of rational numbers. The whole configuration space can be regarded as a union over all these sectors. If the rational boundaries can be regarded as zero modes (classical degrees of freedom in which localization occurs in each quantum jump),

there is no need to integrate over these boundaries in the inner product for configuration space spinors fields. Assume that this is the case.

2. Assume that one has found an orthonormal basis for the spaces of real and p-adic regions of the 3-surfaces with a fixed rational boundary.

3. Armed with these assumptions (plus many other about which I am not yet conscious of) one can construct formal products of the configuration space spinor fields belonging to different regions. One can also construct formal sums of the products with rational or algebraic entanglement coefficients. Although these expressions do not belong to any definite number field, their inner products are complex rational numbers and this is all that is needed for doing physics.

This construction, if it really works, would mean that it is possible to construct a quantum theory which is able to describe also the interaction between cognitive representations and matter as well as cognitive representations characterized by different values of p-adic prime p.

#### 6.3.3 Could the trivial solution be the only one?

One must consider also the possibility that the trivial solution to these challenges is the only possible one. The line element would vanish in all finite-p p-adic sectors. p-Adic space-time regions would correspond to zero modes in which a complete localization (or at least localization to discrete set of points) would occur in each quantum jump so that there would be no need to define p-adic configuration space integral. The p-adic counterparts of the field equations defined by the Kähler action would be satisfied. This option would save from the trouble of trying to define the four-dimensional integral defining the Kähler action. One would also avoid the challenge of giving p-adic meaning to the exponent of the Kähler action defining the vacuum functional.

Since configuration space metric would not pose any conditions on the fermionic oscillators operators, they could but need not anti-commute to zero.

*1. Do fermionic oscillator operators anti-commute to zero...*

Consistency would suggest that fermionic oscillator operators anti-commute to zero in p-adic sector. This would mean that p-adic fermionic oscillator algebra reduces to Grassmann numbers and thus also fermionic degrees of freedom represent classical zero modes. The fact that classical theory is an essential part of all quantum theories could be interpreted as reflecting the fact that cognition obeys purely classical physics. The objection is that fermionic anticommutation relations are universal and should be more or less independent of the number field involved.

*2. ... or do they generate quaternion conformal algebra at least?*

On physical grounds it seems that fermionic oscillator operators cannot generate a mere Grassmann algebra. p-Adic mass calculations are based on p-adic thermodynamics which assumed p-adic conformal invariance and p-adic Super Kac Moody algebra based on fermionic oscillator operators. Certainly fermionic oscillator operators cannot anti-commute to zero in this case and in cognitive degrees of freedom one would have quaternion conformal fermionic field theory in a fixed p-adic background space-time. The non-anti-commutativity of fermions is consistent with the vanishing of the p-adic configuration space metric if quaternion conformal algebra does not contribute to the configuration space metric. Mathematician would however question the asymmetric treatment of super-canonical and quaternion conformal degrees of freedom. After all, there is tight interaction between them.

One might hope that this theory allows to define various symmetry generators purely representation theoretically without any reference to the problematic integrals over p-adic 3-surface. This seems to be the case. For Kac Moody representations momentum and mass squared operators do not correspond to conserved charges defined by integrals. Quaternion conformal invariance implies effective two-dimensionality in the sense that the quantization of the theory reduces to that in a commutive sub-manifold of the quaternionic space-time. At each point of a commutative sub-manifold quaternionic tangent space reduces to some subspace of quaternions isomorphic to the field of complex numbers. The integrals defining conserved charges are integrals over circles in the real context and integration can be done by residue calculus without any resort to a numerical definition of the integral. The residue calculus would provide a natural p-adic generalization for these integrals. Clearly this option could solve all technical problems related to the construction of the configuration space spinors and configuration space spinor fields and be consistent with p-adic mass calculations.

From the point of view of cognition the situation would look like follows (assuming the option allowing quaternion conformal representations).

1. Quaternion conformal fermionic algebra would have interpretation as an infinite-dimensional Boolean algebra and as a physical correlate of logical thinking.

2. Cognitive representations would be completely classical in space-time degrees of freedom and induced by the common boundaries of the real and p-adic regions.

3. The unitary time development operator $U$ could generate rational or algebraic entanglement between different number fields in both bosonic and fermionic degrees of freedom. Thus the mapping of the real to cognitive by quantum entanglement would occur in the same manner as in quantum measurement. Thus even the trivial solution gives all what can dream of and would predict that the world of cognitive experience is classical.

This picture suggests that physical theories do not only reflect the structure of the physical world but also the structure and limitations of our cognitive consciousness.

1. Fermionic quaternion conformal symmetries would be realized as p-adic cognitive representations unlike the super-canonical symmetries and super-conformal symmetries of light like manifolds of 4-dimensional Minkowski space related to the configuration space metric. Only the standard model symmetries (Poincare, color, and electro-weak super Kac Moody symmetries) would be realized at the level of cognitive representations in terms of effectively two-dimensional quaternion conformal invariance. One can say that cognition would represent space-time as effectively two-dimensional and makes it to look like the world of superstring models.

2. The cognitive non-representability of super-canonical and conformal symmetries of light like manifolds of 4-dimensional Minkowski space could provide answer to several intriguing questions. Why these symmetries, which are realized at the level of sensory qualia but not at the level of cognition, have not been invented although the massless extremals possessing super-canonical symmetries seem to be everywhere and have fantastic explanatory power in the physics of living matter? Why the conformal symmetries of the light like manifolds of 4-dimensional Minkowski space, which is extremely natural and obvious mathematically, have still not found their way to the mathematical physics literature although they could have been invented already at the times of Einstein and quantum gravitational holography almost forces to discover these symmetries? Could the effective two-dimensionality of cognition and heavy left-brainy character of the recent day theoretical physics explain the otherwise mysterious looking success of super string models despite the fact that the world of super strings is so far from the experiental reality: did the cognitive representation of the world mask the world behind it?

Despite these arguments, my own vision when I am writing this is that finite-p p-adicity is realized also at the level of the configuration space and the difficult challenge is to add details to this vision.

## 6.4 p-Adicization of quantum TGD by algebraic continuation

Assuming that S-matrix exists simultaneously in all number fields (allowing finite-dimensional extensions of p-adics), the immediate question is whether also the construction procedure of the real S-matrix could have a p-adic counterpart for each $p$, and whether the mere requirement that this is the case could provide non-trivial intuitions about the general structure of the theory. Not only the configuration space but also Kähler function and its exponent, Kähler metric, and configuration space functional integral should have p-adic variants. In the following this challenge is discussed in a rather optimistic number theoretic mood using the ideas stimulated by the connections between number theory and cognition.

### 6.4.1 The p-adic variants of configuration space geometry and spinor structure

The basic structure is the rational configuration space whose points have rational valued coordinates. This space is common to both real and p-adic variants of the configuration space. Therefore the construction of the generalized configuration space as such might not be a problem.

The assumption that configuration space decomposes into a union of symmetric spaces labelled by zero modes means that the left invariant metric for each space in the union is dictated by isometries. It should be possible to interpret the matrix elements of the configuration space metric in the basis of properly normalized isometry currents as p-adic numbers in some finite extension of p-adic numbers allowing perhaps also some transcendentals. Note that the Kähler function is proportional to the inverse of Kähler coupling strength $\alpha_K$ which depends on p-adic prime $p$, and does seem to be a rational number if one takes seriously various arguments leading to the hypothesis $\alpha_K = \pi/4K^2$, $K^2 = p \times 2 \times 3 \times 5.. \times 23$. If so then $e$, $\pi$ and logarithms of primes seem to be the minimum generating set of transcendentals required in the extensions used.

The continuation of the exponent of Kähler function and of configuration space spinor fields to p-adic sectors would require some selection of a subset of points of the rational configuration space. On the other hand, the minimum requirement is that it is possible to define configuration space integration in the p-adic context. The only manner to achieve this is by defining configuration space integration purely algebraically by perturbative expansion. For free field theory Gaussian integrals are in question and one can calculate them trivially. The Gaussian can be regarded as a Kähler function of a flat Kähler manifold having maximal translational and rotational symmetries. Physically infinite number of harmonic oscillators are in question. The origin of the symmetric space is preferred point as far as Kähler function is considered: metric itself is invariant under isometries.

### 6.4.2 Algebraization of the configuration space functional integral

Configuration space is a union of infinite-dimensional symmetric spaces labelled by zero modes. One can hope that the functional integral could be performed perturbatively around the maxima of the Kähler function. In the case of $CP_2$ Kähler function has only single maximum and is a monotonically decreasing function of the radial variable $r$ of $CP_2$ and thus defines a Morse function. This suggests that a similar situation is true for all symmetric spaces and this might indeed be the case. The point is that the presence of several maxima implies also saddle points at which the matrix defined by the second derivatives of the Kähler function is not positive definite. If the derivatives of type $\partial_K \partial_L K$ and $\partial_{\overline{K}} \partial_{\overline{L}} K$ vanish at the saddle point (this is the crucial assumption) in some complex coordinates holomorphically related to those in which the same holds true at maximum, the Kähler metric is not positive definite at this point. On the other hand, by symmetric space property the metric should be isometric with the positive define metric at maxima so that a contradiction results.

If this argument holds true, for given values of zero modes Kähler function has only one maximum, whose value depends on the values zero modes. Staying in the optimistic mood, one could go on to guess that the Duistermaat-Heckman theorem generalizes and the functional integral is simply the exponent of the Kähler function at the maximum (due to the compensation of Gaussian and metric determinants). Even more, one could bravely guess that for configuration space spinor fields belonging to the representations of symmetries the inner products reduces to the generalization of correlation functions of Gaussian free field theory. Each configuration space spinor field would define a vertex from which lines representing the propagators defined by the contravariant configuration space metric in isometry basis emanate.

If this optimistic line of reasoning makes sense, the definition of the p-adic configuration space integral reduces to a purely algebraic one. What is needed is that the contravariant Kähler metric fixed by the symmetric space-property exists and that the exponent of the maximum of the Kähler function exists for rational values of zero modes or subset of them if finite-dimensional algebraic extension is allowed. This would give could hopes that the $U$-matrix elements resulting from the configuration space integrals would exist also in the p-adic sense.

### 6.4.3 Are the exponential of the Kähler function and reduced Kähler action rational functions?

The simplest possibilities one can imagine are that the exponent $e^{2K}$ of Kähler function appearing in the configuration space inner products is a rational or at most a simple algebraic function existing in a finite-dimensional algebraic extension of p-adic numbers. One could also require that the reduced Kähler action without the $1/4\pi\alpha_K$ factor, which affects in no manner the dynamics of the absolute minimization, is a rational function.

*1. Is $e^{2K}$ a rational function?*

The exponent of the $CP_2$ Kähler function is a rational function of the standard complex coordinates and thus rational-valued for all rational values of complex $CP_2$ coordinates. Therefore one is lead to ask whether this property holds true quite generally for symmetric spaces and even in the infinite-dimensional context. If so, then the continuation of the vacuum functional to the p-adic sectors of the configuration space would be possible in the entire configuration space. Also the spherical harmonics of $CP_2$ are rational functions containing square roots in normalization constants. That also configuration space spinor fields could use rational functions containing square roots as normalization constant as basic building blocks would conform with general number theoretical ideas as well as with the general features of harmonic oscillator wave functions.

This idea is supported by the earlier work. Various arguments for the p-adic evolution of the Kähler coupling strength imply that the exponent of the Kähler function for $CP_2$ type extremal is a rational number being the product $K^2 = p \times 2 \times 3 \times 5... \times 23$. In this case the Kähler coupling strength is $1/\alpha_K = (4/\pi) \times log(K^2)$. $\alpha_K$. The general number theoretical conjectures implied by p-adic physics and physics of cognition and intention state that this is the case. Although one must take these arguments with a big grain of salt, the general idea might be correct. Also the elements of the configuration space metric would be rational functions as is clear from the fact that one can express the second derivatives of the Kähler function in terms of $F = exp(K)$ as

$$\partial_K \partial_{\overline{L}} K = \frac{\partial_K \partial_{\overline{L}} F}{F} - \frac{\partial_K F \partial_{\overline{L}} F}{F^2} \quad .$$

*2. Is reduced Kähler action a rational function?*

Kähler coupling strength does not appear at all in the field equations for the extremals of the Kähler action. Therefore one could argue that also the reduced Kähler action $S_R(X^4(X^3))$ defined as $S_R = \int J^{\mu\nu} J_{\mu\nu} \sqrt{g} d^4x$ should be rational valued, when $X^3$ corresponds to a rational point of the configuration space (including zero modes). On the other hand, also the exponent $e^{2K} = e^{S_R/8\pi\alpha_K}$ appearing in configuration space inner products should be rational valued for rational points of the configuration space.

If one takes seriously the conjecture $1/\alpha_K = (4/\pi)\times log(K^2)$, $K^2 = p\times\prod_{q=2,3,...23} q$, one one can write the exponent of Kähler function as

$$e^K = \left[p\times\prod_{q=2,3,...23} q\right]^{S_R/2\pi^2} .$$

This corresponds numerically to $1\alpha_K \simeq 136.5585$ giving $\alpha/\alpha_K \simeq .9965$.

The rational-valuedness for the $e^{2K}$ appearing in the configuration space inner products would require a quantization of the absolute minimum of the Kähler action as integer multiples of $CP_2$ action

$$S_R = n\times S_R(CP_2) = n\times 2\pi^2 , \quad (20)$$

where $n$ is integer. Needless to say, the quantization of the absolute minimum of Kähler action as multiples of $CP_2$ Kähler action would be quite a dramatic implication and would correspond to the basic idea of the first days of the quantum theory about quantization of action. Probably something like this is too much to hope for and would probably have un-physical consequences.

The milder assumption that the exponent of Kähler function has values in a finite-dimensional algebraic extension requires that $S_R$ is rational multiple of $CP_2$ reduced Kähler action $S_R(CP_2) = 2\pi^2$: this would require an extension generated by a finite root of $p$. That $CP_2$ action would serve as a universal unit for the absolute minima associated with the rational valued zero modes, sounds reasonable.

An alternative option is based on the fact that $e^p$ is always an ordinary p-adic number in $R_p$ so that powers of $e^{m/n}$ for a given value of $n$ exist always in a finite-dimensional transcendental extension of p-adic numbers. In this case not only rational values of Kähler function but all values which are rationally proportional to an $n$:th root of $e$ would be possible. The proportionality of the Kähler function to a power of $e$ requires much less than the proportionality to a rational number.

One can consider also an alternative ansatz based on the requirement that Kähler function is rational number rather than a logarithm of a power of integer $K^2$. This requires an extension of p-adic numbers involving some root of $e$ and a finite number of its powers. The reduced action $S_R$ must be rational valued using Kähler action $S_K(CP_2) = 2\pi^2$ of $CP_2$ type extremal as a basic unit. In fact, not only rational values of Kähler function but all values which differ from a rational value by a perturbation with a p-adic norm smaller than one and rationally proportional to a power of $e$ or even its root exist p-adically in this case if they have small enough p-adic norm. The most general perturbation of the action is in the field defined by the extension of rationals defined by the root of $e$ and algebraic numbers.

Since $CP_2$ action is rationally proportional to $\pi^2$, the exponent is rational if $g_K^2 = 4\pi\alpha_K$ is also proportional to $\pi^2$. If the $log(p) = q_1 exp(q_2)/\pi$ ansatz holds true for every prime, then the earlier ansatz $1/\alpha_K(p) = (4/\pi)log(K^2)$ does not guarantee this, and $4/\pi$ must be replaced with a rational number $q \simeq 4/\pi$. The presence of $log(K^2)$, $K^2$ product of primes, is well motivated also in this case because it gives the desired $1/\pi$ factor. The replacement is also supported by the proposal that Kähler action can be defined as a fermionic effective action using $\zeta$ function regularization [B4].

Since $k = 137$ (atomic length scale) and $k = 107$ (hadronic length scale) are the most important nearest p-adic neighbors of electron, one could make a free fall into number mysticism and try the replacement $4/\pi \rightarrow 137/107$. This would give $\alpha_K = 137.3237$ to be compared with $\alpha = 137.0360$: the deviation from $\alpha$ is .2 per cent (of course, $\alpha_K$ need not equal to $\alpha$ and the evolutions of these couplings are quite different). Thus it seems that $log(p) = q_1 exp(q_2)/\pi$ hypothesis is supported also by the properties of Kähler action and leads to an improved understanding of the origin of the mystery prime $k = 137$.

*3 .Could infinite primes appear in the p-adicization of the exponent of Kähler action?*

The difficulties related to the p-adic continuation of Kähler function to an arbitrary p-adic number field and the fact that infinities are every day life in quantum field theory bring in mind infinite primes discussed in [E3].

Infinite primes are not divisible by any finite prime. The simplest infinite prime is of form $\Pi = 1 + X$, $X = \prod_i p_i$, where product is over all finite primes. The factor $Y = X/(1 + X)$ is in the real sense equivalent with 1. In p-adic sense it has norm $1/p$ for every prime. Thus one could multiply Kähler function by $Y$ or its positive power in order to guarantee that the continuation to p-adic number fields exists for all primes. Of course, these states might differ physically in p-adic sense from the states having $Y = 1$. Thus it would seem that the physics of cognition could differentiate between states which are in real sense equivalent.

More general infinite primes are of form $\Pi = nX/m + n$, such that $m = \prod_i q_i$ and $n = \prod_i p_i^{n_i}$ have no common factors. The interpretation could be as a counterpart for a state of a super-symmetric theory containing fermion in each mode labelled by $q_i$ and $n_i$ bosons labelled in modes labelled by $p_i$. Also positive powers of the ratio $Y = X/\Pi$, $\Pi$ some infinite prime, are possible as a multiplier of the Kähler function. In the real sense this ratio would correspond to the ratio $m/n$.

If this picture is correct, infinite primes would emerge naturally in the p-adicization of the theory. Since octonionic infinite primes could correspond to the states of a super-symmetric quantum field theory more or less equivalent with TGD, the presence of infinite primes could make it possible to code the quantum physical state to the vacuum functional via coupling constant renormalization.

One could also consider the possibility of defining functions like $exp(x)$ and $log(1+x)$ p-adically by replacing $x$ with $Yx$ without introducing the algebraic extension. The series would converge for all values of $x$ also p-adically and would be in real sense equivalent with the function. This trick would apply to a very general class of Taylor series having rational coefficients. One could also say that p-adic physics allowing infinite primes would be very similar to real physics.

The fascination of infinite primes is that the ratios of infinite primes which are ordinary rational numbers in the real sense could code the particle number content of a super-symmetric arithmetic quantum field theory. For the octonic version of the theory natural in the TGD framework these states could represent the states of a real Universe. Universe would be an algebraic hologram in the sense that space-time points, something devoid of any structure in the standard view, could code for the quantum states of possible Universes!

The simplest manner to realize this scenario is to consider an extension of rational numbers by the multiplicative group of real units obtained from infinite primes and powers of $X$. Real number 1 would code everything in its structure! This group is generated as products of powers of $Y(m/n) = (m/n) \times [X/\Pi(m/n)]$ which is a unit in the real sense. Each $Y(m/n)$ would define a subgroup of units and the power of $Y(m/n)$ would code for the number of factors of a given integer with unit counted as a factor. This would give a hierarchy of integers with their p-adic norms coming as powers of $p$ with the prime factors of $m$ and $n$ forming an exception and being reflected in p-adic physics of cognition, Universe would "feel" its real or imagined state with its every point, be it a point of space-time surface, of imbedding space, or of configuration space.

## 6.5 Minimal approach: p-adicize only the reduced configuration space

The most recent view about what p-adicizationmight be characterized as minimalism and would involve geometrization of only the reduced configuration space consisting of the maxima of Kähler function.

### 6.5.1 p-Adicization at the level of space-time

The minimum amount of p-adicization correspond to the p-adicization for the maxima of the Kähler function. The basic question is whether the equations characterizing real space-time sheet make sense also p-adically. Suppose that TGD indeed reduces to almost topological field theory defined by Chern-Simons action for the light-like 3-surfaces interpreted as orbits of partonic 2-surfaces [B4, C1, C2]. If this is the case, then the starting point here would be the algebraic equations defining light-like partonic 3-surfaces via the condition that the determinant of the induced metric vanishes. If the coordinate functions appearing in the determinant are algebraic functions with algebraic coefficients, p-adicization should make sense. This of course, means the assumption of some preferred coordinates and the construction of solutions of field equations leads naturally to such coordinates [D1].

If the corresponding 4-dimensional real space-time sheet is expressible as a hyper-quaternionic surface of hyper-octonionic variant of the imbedding space as number-theoretic vision suggests [E2], it might be possible to construct also the p-adic variant of the space-time sheet by algebraic continuation in the case that the functions appearing in the definition of the space-time sheet are algebraic.

### 6.5.2 p-Adicization of second quantized induced spinor fields

Induction procedure makes it possible to geometrize the concept of a classical gauge field and also of the spinor field with internal quantum numbers. In the case of the electro-weak gauge fields induction means the projection of the $H$-spinor connection to a spinor connection on the space-time surface.

In the most recent formulation induced spinor fields appear only at the 3-dimensional light-like partonic 3-surfaces and the solutions of the modified Dirac equation can be written explicitly [C1, C2] as simple algebraic functions involving powers of the preferred coordinate variables very much like various operators in conformal field theory can be expressed as Laurent series in powers of a complex variable $z$ with operator valued coefficients. This means that the continuation of the second quantized induced spinor fields to various p-adic number fields is a straightforward procedure. The second quantization of these induced spinor fields as free fields is needed to construct configuration space geometry and anti-commutation relation between spinor fields are fixed from the requirement that configuration space gamma matrices correspond to super-canonical generators.

The idea about rational physics as the intersection of the physics associated with various number fields inspires the hypothesis that induced spinor fields have only modes labelled by rational valued quantum numbers. Quaternion conformal invariance indeed implies that zero modes are characterized by integers. This means that same oscillator operators can define oscillator operators are universal. Powers of the quaternionic coordinate are indeed well-define in any number field provided the components of quaternion are rational numbers since p-adic quaternions have in this case always inverse.

### 6.5.3 Should one p-adicize at the level of configuration space?

If Duistermaat-Heckman theorem [38] holds true in TGD context, one could express configuration space functional integral in terms of exactly calculable Gaussian integrals around the maxima of the Kähler function defining what might be called reduced configuration space $CH_{red}$. The huge super-conformal symmetries raise the hope that the rest of S-matrix elements could be deduced using group theoretical considerations so that everything would become algebraic. If this optimistic scenario is realized, the p-adicization of $CH_{red}$ might be enough to p-adicize all operations needed to construct the p-adic variant of S-matrix.

The optimal situation would be that S-matrix elements reduce to algebraic numbers for rational valued incoming momenta and that p-adicization trivializes in the sense that it corresponds only to different interpretations for the imbedding space coordinates (interpretation as real or p-adic numbers) appearing in the equations defining the 4-surfaces. For instance, space-time coordinates would correspond to preferred imbedding space coordinates and the remaining imbedding space coordinates could be rational functions of the latter with algebraic coefficients. Algebraic points in a given extension of rationals would thus be common to real and p-adic surfaces. It could also happen that there are no or very few common algebraic points. For instance, Fermat's theorem says that the surface $x^n + y^n = z^n$ has no rational points for $n > 2$.

This picture is probably too simple. The intuitive expectation is that ordinary S-matrix elements are proportional to a factor which in the real case involves an integration over the arguments of an n-point function of a conformal field theory defined at a partonic 2-surface. For p-adic-real transitions the integration should reduce to a sum over the common rational or algebraic points of the p-adic and real surface. Same applies to $p_1 \rightarrow p_2$ type transitions.

If this picture is correct, the p-adicization of the configuration space would mean p-adicization of $CH_{red}$ consisting of the maxima of the Kähler function with respect to both fiber degrees of freedom and zero modes acting effectively as control parameters of the quantum dynamics. If $CH_{red}$ is a discrete subset of $CH$ ultrametric topology induced from finite-p p-adic norm is indeed natural for it. 'Discrete set in $CH$' need not mean a discrete set in the usual sense and the reduced configuration space could be even finite-dimensional continuum. Finite-p p-adicization as a cognitive model would suggest that p-adicization in given point of $CH_{red}$ is possible for all p-adic primes associated with the corresponding space-time surface (maximum of Kähler function) and represents a particular cognitive representation about $CH_{red}$.

A basic technical problem is, whether the integral defining the Kähler action appearing in the exponent of Kähler function exists p-adically. Here the hypothesis that the exponent of the Kähler function is identifiable as a Dirac determinant of the modified Dirac operator defined at the light-like partonic 3-surfaces [B4] suggests a solution to the problem. By restricting the generalized eigen values of the modified Dirac operator to an appropriate algebraic extension of rationals one could obtain an algebraic number existing both in the real and p-adic sense if the number of the contributing eigenvalues is finite. The resulting hierarchy of algebraic extensions of $R_p$ would have interpretation as a cognitive hierarchy. If the maxima of Kähler function assignable to the functional integral are such that the number of eigenvalues in a given algebraic extension is finite this hypothesis works.

If Duistermaat-Heckman theorem generalizes, the p-adicization of the entire configuration space would be un-necessary and it certainly does not look a good idea in the light of preceding considerations.

1. For a generic 3-surface the number of the eigenvalues in a given algebraic extension of rationals need not be finite so that their product can fail to be an algebraic number.

2. The algebraic continuation of the exponent of the Kähler function from $CH_{red}$ to the entire $CH$ would be analogous to a continuation of a rational valued function from a discrete set to a real or p-adic valued function in a continuous set. It is difficult to see how the continuation could be unique in the p-adic case.

## 6.6 The most recent vision about zero energy ontology and p-adicization

The generalization of the number concept obtained by fusing real and p-adics along rationals and common algbraics is the basic philosophy behind p-adicization. This however requires that it is possible to speak about rational points of the imbedding space and the basic objection against the notion of rational points of imbedding space common to real and various p-adic variants of the imbedding space is the necessity to fix some special coordinates in turn implying the loss of a manifest general coordinate invariance. The isometries of the imbedding space could save the situation provided one can identify some special coordinate system in which isometry group reduces to its discrete subgroup. The loss of the full isometry group could be compensated by assuming that WCW is union over sub-WCW:s obtained by applying isometries on basic sub-WCW with discrete subgroup of isometries.

The combination of zero energy ontology realized in terms of a hierarchy causal diamonds and hierarchy of Planck constants providing a description of dark matter and leading to a generalization of the notion of imbedding space suggests that it is possible to realize this dream. The article [16] provides a brief summary about recent state of quantum TGD helping to understand the big picture behind the following considerations.

### 6.6.1 Zero energy ontology briefly

1. The basic construct in the zero energy ontology is the space $CD \times CP_2$, where the causal diamond $CD$ is defined as an intersection of future and past directed light-cones with time-like separation between their tips regarded as points of the underlying universal Minkowski space $M^4$. In zero energy ontology physical states correspond to pairs of positive and negative energy states located at the boundaries of the future and past directed light-cones of a particular $CD$. $CD$:s form a fractal hierarchy and one can glue smaller $CD$:s within larger $CD$ along the upper light-cone boundary along a radial light-like ray: this construction recipe allows to understand the asymmetry between positive and negative energies and why the arrow of experienced time corresponds to the arrow of geometric time and also why the contents of sensory experience is located to so narrow interval of geometric time. One can imagine evolution to occur as quantum leaps in which the size of the largest $CD$ in the hierarchy of personal $CD$:s increases in such a manner that it becomes sub-$CD$ of a larger $CD$. p-Adic length scale hypothesis follows if the values of temporal distance $T$ between tips of $CD$ come in powers of $2^n$. All conserved quantum numbers for zero energy states have vanishing net values. The interpretation of zero energy states in the framework of positive energy ontology is as physical events, say scattering events with positive and negative energy parts of the state interpreted as initial and final states of the event.

2. In the realization of the hierarchy of Planck constants $CD \times CP_2$ is replaced with a Cartesian product of book like structures formed by almost copies of $CD$:s and $CP_2$:s defined by singular coverings and factors spaces of $CD$ and $CP_2$ with singularities corresponding to intersection $M^2 \cap CD$ and homologically trivial geodesic sphere $S^2$ of $CP_2$ for which the induced Kähler form vanishes. The coverings and factor spaces of $CD$:s are glued together along common $M^2 \cap CD$. The coverings and factors spaces of $CP_2$ are glued together along common homologically non-trivial geodesic sphere $S^2$. The choice of preferred $M^2$ as subspace of tangent space of $X^4$ at all its points and having interpretation as space of non-physical polarizations, brings $M^2$ into the theory also in different manner. $S^2$ in turn defines a subspace of the much larger space of vacuum extremals as surfaces inside $M^4 \times S^2$.

3. Configuration space (the world of classical worlds, WCW) decomposes into a union of sub-WCW:s corresponding to different choices of $M^2$ and $S^2$ and also to different choices of the quantization axes of spin and energy and and color isospin and hyper-charge for each choice of this kind. This means breaking down of the isometries to a subgroup. This can be compensated by the fact that the union can be taken over the different choices of this subgroup.

4. p-Adicization requires a further breakdown to discrete subgroups of the resulting sub-groups of the isometry groups but again a union over sub-WCW:s corresponding to different choices

of the discrete subgroup can be assumed. Discretization relates also naturally to the notion of number theoretic braid.

Consider now the critical questions.

1. Very naively one could think that center of mass wave functions in the union of sectors could give rise to representations of Poincare group. This does not conform with zero energy ontology, where energy-momentum should be assignable to say positive energy part of the state and where these degrees of freedom are expected to be pure gauge degrees of freedom. If zero energy ontology makes sense, then the states in the union over the various copies corresponding to different choices of $M^2$ and $S^2$ would give rise to wave functions having no dynamical meaning. This would bring in nothing new so that one could fix the gauge by choosing preferred $M^2$ and $S^2$ without losing anything. This picture is favored by the interpretation of $M^2$ as the space of longitudinal polarizations.

2. The crucial question is whether it is really possible to speak about zero energy states for a given sector defined by generalized imbedding space with fixed $M^2$ and $S^2$. Classically this is possible and conserved quantities are well defined. In quantal situation the presence of the light-cone boundaries breaks full Poincare invariance although the infinitesimal version of this invariance is preserved. Note that the basic dynamical objects are 3-D light-like "legs" of the generalized Feynman diagrams.

### 6.6.2 Definition of energy in zero energy ontology

Can one then define the notion of energy for positive and negative energy parts of the state? There are two alternative approaches depending on whether one allows or does not allow wave-functions for the positions of tips of light-cones.

Consider first the naive option for which four momenta are assigned to the wave functions assigned to the tips of $CD$:s.

1. The condition that the tips are at time-like distance does not allow separation to a product but only following kind of wave functions

$$\Psi = exp[ip\cdot(m_+ - m_-)]\Theta(T^2)\Theta(m_+^0 - m_-^0)\Phi(p) \ , \ T^2 = (m_+ - m_-)^2 \ . \qquad (21)$$

Here $m_+$ and $m_-$ denote the positions of the light-cones and $\Theta$ denotes step function. $\Phi$ denotes configuration space spinor field in internal degrees of freedom of 3-surface. One can introduce also the decomposition into particles by introducing sub-$CD$:s glued to the upper light-cone boundary of $CD$.

2. The first criticism is that only a local eigen state of 4-momentum operators $p_\pm = \hbar\nabla/i$ is in question everywhere except at boundaries and at the tips of the $CD$ with exact translational invariance broken by the two step functions having a natural classical interpretation. The second criticism is that the quantization of the temporal distance between the tips to $T = 2^kT_0$ is in conflict with translational invariance and reduces it to a discrete scaling invariance.

The less naive approach relying of super conformal structures of quantum TGD assumes fixed value of $T$ and therefore allows the crucial quantization condition $T = 2^kT_0$.

1. Since light-like 3-surfaces assignable to incoming and outgoing legs of the generalized Feynman diagrams are the basic objects, can hope of having enough translational invariance to

define the notion of energy. If translations are restricted to time-like translations acting in the direction of the future (past) then one has local translation invariance of dynamics for classical field equations inside $\delta M^4_\pm$ as a kind of semigroup. Also the $M^4$ translations leading to interior of $X^4$ from the light-like 2-surfaces surfaces act as translations. Classically these restrictions correspond to non-tachyonic momenta defining the allowed directions of translations realizable as particle motions. These two kinds of translations have been assigned to super-canonical conformal symmetries at $\delta M^4_\pm \times CP_2$ and and super Kac-Moody type conformal symmetries at light-like 3-surfaces. Equivalence Principle in TGD framework states that these two conformal symmetries define a structure completely analogous to a coset representation of conformal algebras so that the four-momenta associated with the two representations are identical [C1].

2. The condition selecting preferred extremals of Kähler action is induced by a global selection of $M^2$ as a plane belonging to the tangent space of $X^4$ at all its points [C1]. The $M^4$ translations of $X^4$ as a whole in general respect the form of this condition in the interior. Furthermore, if $M^4$ translations are restricted to $M^2$, also the condition itself - rather than only its general form - is respected. This observation, the earlier experience with the p-adic mass calculations, and also the treatment of quarks and gluons in QCD encourage to consider the possibility that translational invariance should be restricted to $M^2$ translations so that mass squared, longitudinal momentum and transversal mass squared would be well defined quantum numbers. This would be enough to realize zero energy ontology. Encouragingly, $M^2$ appears also in the generalization of the causal diamond to a book-like structure forced by the realization of the hierarchy of Planck constant at the level of the imbedding space.

3. That the cm degrees of freedom for $CD$ would be gauge like degrees of freedom sounds strange. The paradoxical feeling disappears as one realizes that this is not the case for sub-$CD$:s, which indeed can have non-trivial correlation functions with either upper or lower tip of the $CD$ playing a role analogous to that of an argument of n-point function in QFT description. One can also say that largest $CD$ in the hierarchy defines infrared cutoff.

### 6.6.3 p-Adic variants of the imbedding space

Consider now the construction of p-adic variants of the imbedding space.

1. Rational values of p-adic coordinates are non-negative so that light-cone proper time $a_{4,+} = \sqrt{t^2 - z^2 - x^2 - y^2}$ is the unique Lorentz invariant choice for the p-adic time coordinate near the lower tip of $CD$. For the upper tip the identification of $a_4$ would be $a_{4,-} = \sqrt{(t-T)^2 - z^2 - x^2 - y^2}$. In the p-adic context the simultaneous existence of both square roots would pose additional conditions on $T$. For 2-adic numbers $T = 2^n T_0$, $n \geq 0$ (or more generally $T = \sum_{k \geq n_0} b_k 2^k$), would allow to satisfy these conditions and this would be one additional reason for $T = 2^n T_0$ implying p-adic length scale hypothesis. The remaining coordinates of $CD$ are naturally hyperbolic cosines and sines of the hyperbolic angle $\eta_{\pm,4}$ and cosines and sines of the spherical coordinates $\theta$ and $\phi$.

2. The existence of the preferred plane $M^2$ of un-physical polarizations would suggest that the 2-D light-cone proper times $a_{2,+} = \sqrt{t^2 - z^2}$ $a_{2,-} = \sqrt{(t-T)^2 - z^2}$ can be also considered. The remaining coordinates would be naturally $\eta_{\pm,2}$ and cylindrical coordinates $(\rho, \phi)$.

3. The transcendental values of $a_4$ and $a_2$ are literally infinite as real numbers and could be visualized as points in infinitely distant geometric future so that the arrow of time might be said to emerge number theoretically. For $M^2$ option p-adic transcendental values of $\rho$ are infinite as real numbers so that also spatial infinity could be said to emerge p-adically.

4. The selection of the preferred quantization axes of energy and angular momentum unique apart from a Lorentz transformation of $M^2$ would have purely number theoretic meaning in both cases. One must allow a union over sub-$WCW$s labeled by points of $SO(1,1)$. This suggests a deep connection between number theory, quantum theory, quantum measurement theory, and even quantum theory of mathematical consciousness.

5. In the case of $CP_2$ there are three real coordinate patches involved [Appendix]. The compactness of $CP_2$ allows to use cosines and sines of the preferred angle variable for a given coordinate patch.

$$\begin{aligned} \xi^1 &= tan(u)exp(i\frac{(\Psi+\Phi)}{2})cos(\frac{\Theta}{2}) \ , \\ \xi^2 &= tan(u)exp(i\frac{(\Psi-\Phi)}{2})sin(\frac{\Theta}{2}) \ . \end{aligned} \tag{22}$$

   The ranges of the variables $u, \Theta, \Phi, \Psi$ are $[0, \pi/2], [0, \pi], [0, 4\pi], [0, 2\pi]$ respectively. Note that $u$ has naturally only the positive values in the allowed range. $S^2$ corresponds to the values $\Phi = \Psi = 0$ of the angle coordinates.

6. The rational values of the (hyperbolic) cosine and sine correspond to Pythagorean triangles having sides of integer length and thus satisfying $m^2 = n^2 + r^2$ ($m^2 = n^2 - r^2$). These conditions are equivalent and allow the well-known explicit solution [39]. One can construct a p-adic completion for the set of Pythagorean triangles by allowing p-adic integers which are infinite as real integers as solutions of the conditions $m^2 = r^2 \pm s^2$. These angles correspond to genuinely p-adic directions having no real counterpart. Hence one obtains p-adic continuum also in the angle degrees of freedom. Algebraic extensions of the p-adic numbers bringing in cosines and sines of the angles $\pi/n$ lead to a hierarchy increasingly refined algebraic extensions of the generalized imbedding space. Since the different sectors of $WCW$ directly correspond to correlates of selves this means direct correlation with the evolution of the mathematical consciousness. Trigonometric identities allow to construct points which in the real context correspond to sums and differences of angles.

7. Negative rational values of the cosines and sines correspond as p-adic integers to infinite real numbers and it seems that one use several coordinate patches obtained as copies of the octant ($x \geq 0, y \geq 0, z \geq 0,$). An analogous picture applies in $CP_2$ degrees of freedom.

8. The expression of the metric tensor and spinor connection of the imbedding in the proposed coordinates makes sense as a p-adic numbers in the algebraic extension considered. The induction of the metric and spinor connection and curvature makes sense provided that the gradients of coordinates with respect to the internal coordinates of the space-time surface belong to the extensions. The most natural choice of the space-time coordinates is as subset of imbedding space-coordinates in a given coordinate patch. If the remaining imbedding space coordinates can be chosen to be rational functions of these preferred coordinates with coefficients in the algebraic extension of p-adic numbers considered for the preferred extremals of Kähler action, then also the gradients satisfy this condition. This is highly non-trivial condition on the extremals and if it works might fix completely the space of exact solutions of field equations. Space-time surfaces are also conjectured to be hyper-quaternionic [E2], this condition might relate to the simultaneous hyper-quaternionicity and Kähler extremal property. Note also that this picture would provide a partial explanation for the decomposition of the imbedding space to sectors dictated also by quantum measurement theory and hierarchy of Planck constants.

### 6.6.4 p-Adic variants for the sectors of WCW

One can also wonder about the most general definition of the p-adic variants of the sectors of the world of classical worlds.

1. The restriction of the surfaces in question to be expressible in terms of rational functions with coefficients which are rational numbers of belong to algebraic extension of rationals means that the world of classical worlds can be regarded as a a discrete set and there would be no difference between real and p-adic worlds of classical worlds: a rather unexpected conclusion.

2. One can of course whether one should perform completion also for WCWs. In real context this would mean completion of the rational number valued coefficients of a rational function to arbitrary real coefficients and perhaps also allowance of Taylor and Laurent series as limits of rational functions. In the p-adic case the integers defining rational could be allowed to become p-adic transcendentals infinite as real numbers. Also now also Laurent series could be considered.

3. In this picture there would be close analogy between the structure of generalized imbedding space and $WCW$. Different WCW:s could be said to intersect in the space formed by rational functions with coefficients in algebraic extension of rationals just real and p-adic variants of the imbedding space intersect along rational points. In the spirit of algebraic completion one might hope that the expressions for the various physical quantities, say the value of Kähler action, Kähler function, or at least the exponent of Kähler function (at least for the maxima of Kähler function) could be defined by analytic continuation of their values from these sub-WCW to various number fields. The matrix elements for p-adic-to-real phase transitions of zero energy states interpreted as intentional actions could be calculated in the intersection of real and p-adic WCW:s by interpreting everything as real.

## 6.7 Zero energy ontology, self hierarchy, and the notion of time

One manner to test the internal consistency of the picture about zero energy ontology and p-adication is by formulating the basic notions and problems of TGD inspired quantum theory of consciousness and quantum biology in terms of zero energy ontology.

In consciousness theory the basic challenges are to understand the asymmetry between positive and negative energies and between two directions of geometric time at the level of conscious experience, the correspondence between experienced and geometric time, and the emergence of the arrow of time. One should also explain why human sensory experience is about a rather narrow time interval of about .1 seconds and why memories are about the interior of much larger $CD$ with time scale of order life time. One should also have a vision about the evolution of consciousness takes place: how quantum leaps leading to an expansion of consciousness take place.

Negative energy signals to geometric past - about which phase conjugate laser light represents an example - provide an attractive tool to realize intentional action as a signal inducing neural activities in the geometric past (this would explain Libet's classical findings), a mechanism of remote metabolism, and the mechanism of declarative memory as communications with the geometric past. One should understand how these signals are realized in zero energy ontology and why their occurrence is so rare.

In the following my intention is to demonstrate that TGD inspired theory of consciousness and quantum TGD proper indeed seem to be in tune and that this process of comparison helps considerably in the attempt to develop the TGD based ontology at the level of details.

## 6.8 Causal diamonds as correlates for selves

Quantum jump as a moment of consciousness, self as a sequence of quantum jumps integrating to self, and self hierarchy with sub-selves experienced as mental images, are the basic notions of TGD inspired theory of consciousness. In the most ambitious vision self hierarchy reduces to a fractal hierarchy of quantum jumps within quantum jumps.

It is natural to interpret $CD$s as correlates of selves. $CD$s can be interpreted either as subsets of the generalized imbedding space or as sectors of WCW. Accordingly, selves correspond to $CD$s of the generalized imbedding space or sectors of WCW, literally separate interacting quantum Universes. The spiritually oriented reader might speak of Gods. Sub-selves correspond to sub-$CD$s geometrically. The contents of consciousness of self is about the interior of the corresponding $CD$ at the level of imbedding space. For sub-selves the wave function for the position of tip of $CD$ brings in the delocalization of sub-WCW.

The fractal hierarchy of $CD$s within $CD$s is the geometric counterpart for the hierarchy of selves: the quantization of the time scale of planned action and memory as $T(k) = 2^k T_0$ suggest an interpretation for the fact that we experience octaves as equivalent in music experience.

## 6.9 Why sensory experience is about so short time interval?

CD picture implies automatically the 4-D character of conscious experience and memories form part of conscious experience even at elementary particle level. Amazingly, the secondary p-adic time scale of electron is $T = 0.1$ seconds defining a fundamental time scale in living matter. The problem is to understand why the sensory experience is about a short time interval of geometric time rather than about the entire personal $CD$ with temporal size of order life-time. The explanation would be that sensory input corresponds to subselves (mental images) with $T \simeq .1$ s at the upper light-like boundary of $CD$ in question. This requires a strong asymmetry between upper and lower light-like boundaries of $CD$s.

The localization of the contents of the sensory experience to the upper light-cone boundary and local arrow of time could emerge as a consequence of self-organization process involving conscious intentional action. Sub-$CD$s would be in the interior of $CD$ and self-organization process would lead to a distribution of $CD$s concentrated near the upper or lower boundary of $CD$. The local arrow of geometric time would depend on $CD$ and even differ for $CD$ and sub-$CD$s.

1. The localization of contents of sensory experience to a narrow time interval would be due to the concentration of sub-$CD$s representing mental images near the either boundary of $CD$ representing self.

2. Phase conjugate signals identifiable as negative energy signals to geometric past are important when the arrow of time differs from the standard one in some time scale. If the arrow of time establishes itself as a phase transition, this kind of situations are rare. Negative energy signals as a basic mechanism of intentional action and transfer of metabolic energy would explain why living matter is so special.

3. Geometric memories would correspond to the regions near "lower" boundaries of $CD$. Since the density of sub-$CD$s is small there geometric memories would be rare and not sharp. A temporal sequence of mental images, say the sequence of digits of a phone number, would correspond to a temporal sequence of sub-$CD$s.

4. Sharing of mental images corresponds to a fusion of sub-selves/mental images to single sub-self by quantum entanglement: the space-time correlate could be flux tubes connecting space-time sheets associated with sub-selves represented also by space-time sheets inside their $CD$s.

## 6.10 Arrow of time

TGD forces a new view about the relationship between experienced and geometric time. Although the basic paradox of quantum measurement theory disappears the question about the arrow of geometric time remains.

1. Selves correspond to $CD$s. The $CD$s and their projections to the imbedding space do not move anywhere. Therefore the standard explanation for the arrow of geometric time cannot work.

2. The only plausible interpretation at classical level relies on quantum classical correspondence and the fact that space-times are 4-surfaces of the imbedding space. If quantum jump corresponds to a shift for a quantum superposition of space-time sheets towards geometric past in the first approximation (as quantum classical correspondence suggests), one can understand the arrow of time. Space-time surfaces simply shift backwards with respect to the geometric time of the imbedding space and therefore to the 8-D perceptive field defined by the $CD$. This creates in the materialistic mind a temporal variant of train illusion. Space-time as 4-surface and macroscopic and macro-temporal quantum coherence are absolutely essential for this interpretation to make sense.

Why this shifting should always take place to the direction of geometric past of the imbedding space? Does it so always? The proposed mechanism for the localization of sensory experience to a short time interval suggests an explanation in terms of intentional action.

1. $CD$ defines the perceptive field for self. Selves are curious about the space-time sheets outside their perceptive field and perform quantum jumps tending to shift the superposition of the space-time sheets so that unknown regions of space-time sheets emerge to the perceptive field. Either the upper or lower boundary of $CD$ wins in the competition and the arrow of time results as a spontaneous symmetry breaking. The arrow of time can depend on $CD$ but tends to be the same for $CD$ and its sub-$CD$s. Global arrow of time could establish itself by a phase transitions establishing the same arrow of time globally by a mechanism analogous to percolation phase transition.

2. Since the news come from the upper boundary of $CD$, self concentrates its attention to this region and improves the resolution of sensory experience. The sub-$CD$s generated in this manner correspond to mental images with contents about this region. Hence the contents of conscious experience, in particular sensory experience, tends to be about the region near the upper boundary.

## 6.11 Can selves interact and evolve?

Interesting questions relate to how dynamical selves are.

1. Is self doomed to live inside the same sub-WCW eternally as a lonely god? This question has been already answered: there are interactions between sub-$CD$s of given $CD$, and one can think of selves as quantum superposition of states in $CD$s with wave function having as its argument the tips of $CD$, or rather only the second one since $T$ is assumed to be quantized.

2. Is there largest $CD$ in the personal $CD$ hierarchy of self in an absolute sense? Or is the largest $CD$ present only in the sense that the contribution to the contents of consciousness coming from very large $CD$s is negligible? Long time scales $T$ correspond to low frequencies and thermal noise might mask these contributions. Here however the hierarchy of Planck constants and generalization of the imbedding space could come in rescue by allowing dark EEG photons to have energies above thermal energy.

3. Can selves evolve in the sense that the size of $CD$ increases in quantum leaps so that the corresponding time scale $T = 2^k T_0$ of memory and planned action increases? Geometrically this kind of leap would mean that $CD$ becomes a sub-$CD$ of a larger $CD$ - either at the level of conscious experience or in absolute sense. The leap can occur in two senses: as an increase of the largest p-adic time scale in the personal hierarchy of space-time sheets or as increase of the largest value of Planck constants in the personal dark matter hierarchy. At the level of individual organism this would mean emergence of new lower frequencies of generalized EEG and levels of personal dark matter hierarchy with larger value of Planck constant.

# 7 Appendix: Basic facts about algebraic numbers, quaternions and octonions

To understand the detailed connection between infinite primes, polynomial primes and Fock states, some basic concepts of algebraic number theory related to the generalization of prime and prime factorization [18, 25, 23] (the first reference is warmly recommended for a physicist because it teaches the basic facts through exercises; also second book is highly enjoyable reading because of its non-Bourbakian style of representation).

## 7.1 Generalizing the notion of prime

Algebraic numbers are defined as roots of polynomial equations with rational coefficients. Algebraic integers are identified as roots of monic polynomials (highest coefficient equals to one) with integer coefficients. Algebraic number fields correspond to algebraic extensions of rationals and can have any dimension as linear spaces over rationals. The notion of prime is extremely general and involves rather actract mathematics in general case.

Quite generally, commutative ring $R$ called integral domain, if the product $ab$ vanishes only if $a$ or $b$ vanishes. To a given integral domain one can assign a number field by essentially the same construction by which one assigns the field of rationals to ordinary integers. The integer valued function $a \rightarrow N(a)$ in $R$ is called norm if it has the properties $N(ab) = N(a)N(b)$ and $N(1) = 1$. For instance, for the algebraic extension $Q(\sqrt{-D})$ of rationals consisting of points $z = r + \sqrt{-D}s$, the function $N(z) = r^2 + Ds^2$ defines norm. More generally, the determinant of the linear map defined by the action of $z$ in algebraic number field defines norm function. This determinant reduces to the product of all conjugates of $z$ in K and is n:th order polynomial with respect to the components of $z$ when $K$ is n-dimensional.

Irreducible elements (almost the counterparts of primes) can be defined as elements $P$ of integral domain having the property that if one has $P = bc$, then either $b$ or $c$ has unit norm. Elements with unit norm are called units and elements differing by a multiplication with unit are called associates. Note that in the case of p-adics all p-adic numbers with unit norm are units.

## 7.2 UFDs, PIDs and EDs

If the elements of $R$ allow a unique factorization to irreducible elements, $R$ is said to be unique factorization domain (UFD). Ordinary integers are obviously UFD. The field $Z(\sqrt{-5})$ is not UFD: for instance, one has $6 = 2 \times 3 = (1 + \sqrt{-5})(1 - \sqrt{-5})$. The fact that prime factorization is not unique forces to generalize the notion of primeness such that ideals in the ring of algebraic integers take the role of integers. The counterparts of primes can be identified as irreducible elements, which generate prime ideals containing one and only one rational prime. Irreducible elements, such as $1 \pm \sqrt{-5}$ in $Z(\sqrt{-5})$, are not primes in this sense.

Principal ideal domain (PID) is defined as an integral domain for which all ideals are principal, that is are generated as powers of single element. In the case of ordinary integers powers of integers define PID.

Euclidian domain (ED) is integral domain with the property that for any pair $a$ and $b$ one can find pair $(q,r)$ such that $a = bq + r$ with $N(r) < N(a)$. This guarantees that the Euclidian algorithm used in the division of rationals converges. Integers form an Euclidian domain but polynomials with integer coefficients do not (elements 2 and $x$ do not allow decomposition $2 = q(x)x + r$). It can be shown that EDs are PIDs in turn are UFDs. For instance, for complex quadratic extensions of integers $Z(\sqrt{-d})$ there are only 9 UFDs and they correspond to $d = 1, 2, 3, 7, 11, 19, 43, 67, 163$. For extensions of type $Z(\sqrt{d})$ the number of UFD:s is infinite. There are not too many quadratic extensions which are ED:s and the possible values of $d$ are $d = -1, \pm 2, \pm 3, 5, 6, \pm 7, \pm 11, 13, 17, 19, 21, 29, 33, 37, 41, 57, 73$.

Any algebraic number field $K$ is representable always as a polynomial ring $Q[\theta]$ obtained from the polynomial ring $Q[x]$ by replacing $x$ with an algebraic number $\theta$, which is a root of an irreducible polynomial with rational coefficients. This field has dimension $n$ over rationals, where $n$ is the degree of the polynomial in question.

## 7.3 The notion of prime ideal

As already noticed, a general algebraic number field $K$ does not allow a unique factorization into irreducibles and one must generalize the notion of prime number and integer in order to achieve a unique factorization. The ideals of the ring $O_K$ of algebraic integers in $K$ take the role of integers whereas prime ideals take the role of primes. The factorization of an ideal to a product of prime ideals is unique and each prime ideal contains single rational prime characterizing it. One can assign to an ideal norm which orders the ideals: $N(a) < N(b) \leftrightarrow b \subset a$. The smaller the integer generating ideal, the larger the ideal is and the ideals generated by primes are maximal ones in PID. The equivalence classes of the ideals of $O_K$ under equivalence defined by integer multiplication form a group. The number of classes is a characteristic of an algebraic number field. For class-one algebraic number fields prime factorization of ideals is equivalent with the factorization to irreducibles in $K$. $Z(\sqrt{-5})$, which is not UFD, allows two classes of prime ideals. Cyclotomic number fields $Q(\zeta_m)$, where $\zeta_m$ is $m$:th root of unity have class number one for $3 \leq m \leq 10$. In particular, the four-dimensional algebraic number fields $Q(\zeta_8)$ and $Q(\zeta_5) = Q(\zeta_{10})$ are ED and thus UFD.

### 7.3.1 Basic facts about primality for polynomial rings

The notion of primality can be abstracted to the level of polynomial algebras in field $K$ and these polynomial algebras seem to be more or less identical with the algebra formed by infinite integers. The following two results are crucial for the argument demonstrating that this is indeed the case.

### 7.3.2 Polynomial ring associated with any number field is UFD

The elements in the ring $K[x_1, ..., x_n]$ formed by the polynomials having coefficients in *any* field $K$ and $x_i$ having values in $K$, allow a unique decomposition into prime factors. This means that things are much simpler at the next abstraction level, since there is no need for refined class theories needed in the case of algebraic number fields.

The number field $K$ appearing as a coefficient field of polynomials could correspond to finite fields (Galois fields), rationals, any algebraic number field obtained as an extension of rational, p-adic numbers, reals or complex numbers. For $Q[x]$, where $Q$ denotes rationals, the simplest prime factors are monomials of form $x - q$, $q$ rational number. More complicated prime factors correspond to minimal polynomials having algebraic number $\alpha$ and its conjugates as their roots.

In the case of complex number field only monomomials $x - z$, $z$ complex number are the only prime polynomials. Clearly, the primes at the higher level of abstraction are generalized rationals of previous level plus numbers which are algebraic with respect to the generalized rationals.

### 7.3.3 The polynomial rings associated with any UFD are UFD

If $R$ is a unique factorization domain (UFD), then also $R[x]$ is UFD: this holds also for $R[x_1, ..., x_n]$. Hence one obtains an infinite hierarchy of UFDs by a repeated abstraction process by starting from a given algebraic number field $K$. At the first step one obtains the ring $K[x]$ of polynomials in $K$. At the next step one obtains the ring of polynomials $K^{2)}[y]$ having as coefficient ring the ring $K[x] \equiv K^{1)}[x]$ of polynomials. At the next step one obtains $K^{2)}[z]$, etc.. Note that $O_K[x]$ is not ED in general and need not be UFD neither unless $O_K$ is UFD. $O_K[x]$ is not however interesting from the viewpoint of TGD.

An element of $K^{2)}(y)$ corresponds to a polynomial $P(y, x)$ of $y$ such that its coefficients are K-rational functions of $x$. A polynomial in $K^{3)}(z)$ corresponds to a polynomial of $P(z, y, z)$ such that the coefficients of $z$ are K-rational functions of functions of $y$ with coefficients which are K-rational functions of $z$. Note that as a special case, polynomials of all $n$ variables result. Note also the hierarchical ordering of the variables. Thus the hierarchy of polynomials gives rise to a hierarchy of functions having increasingly number of independent variables.

## 7.4 Examples of two-dimensional algebraic number fields

The general two-dimensional (in algebraic sense) algebraic extension of rationals corresponds to $K(\theta)$, where $\theta = (-b \pm \sqrt{b^2 - 4c})/2$ is root of second order irreducible polynomial $x^2 + bx + c$. Depending on whether the discriminant $D = b^2 - 4c$ is positive or negative, one obtains real and complex extensions. $\theta$ and its conjugate generate equivalent extensions and all extensions can be obtained as extensions of form $Q(\sqrt{\pm d})$.

For $Q(\sqrt{d})$, $d$ square-free integer, units correspond to powers of $x = \pm(p_{n-1} + q_{n-1}\sqrt{d})$, where $n$ defines the period of the continued fraction expansion of $\sqrt{d}$ and $p_k/q_k$ defines $k$:th convergent in the continued fraction expansion. For $Q(\sqrt{-d})$, $d > 1$ units form group $Z_2$. For $d = 1$ the group is $Z_2^2$ and for $Q(w)$ where $w = -1/2 + \sqrt{3}/2$ is the third root of unity ($w^3 = 1$), this group is $Z_2 \times Z^3$ (note that in this case the minimal polynomial is $(x^3 - 1)/(x - 1)$.

$Z(w)$ and $Z(i)$ are exceptional in the sense that the group of the roots of unity is exceptionally large. $Z(i)$ and $Z(w)$ allow a unique factorization of their elements into products of irreducibles. The primes $\pi$ of $Z(w)$ consist of rational primes $p$, $p\ mod\ 4 = 3$ and complex Gaussian primes satisfying $N(\pi) = \pi\overline{\pi} = p$, $p\ mod\ 4 = 1$. Squares of the Gaussian primes generate as their product complex numbers giving rise to Pythagorean phases. The primes $\pi$ of $Z(w)$ consist of rational primes $p$, $p\ mod\ 3 = 2$ and complex Eisenstein primes satisfing $N(\pi) = \pi\overline{\pi} = p$, $p\ mod\ 3 = 1$.

## 7.5 Cyclotomic number fields as examples of four-dimensional algebraic number fields

By the 'theorem of primitive element' all algebraic number fields are obtained by replacing the polynomial algebra $Q[x]$, by $Q[\theta]$, where $\theta$ is a root of an irreducible minimal polynomial which is of fourth order. One can readily calculate the extensions associated with a given irreducible polynomial by using quadratures for 4:th order polynomials. These polynomials are of general form $P_4(x) = x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$ and by a substitution $x = y - a_3/4$) which does not change the nature of algbebraic number field, they can be reduced to a canonical form $P_4(x) = x^4 + a_2x^2 + a_1x + a_0$. Thus a very rough view is that three rationals parametrize the 4-dimensional algebraic number fields.

A second manner to represent extensions is in form $K(\theta_1, \theta, ..)$ such that the units $\theta_i$ have no common factors different from one. In this case the dimension of the extension is $2^n$, where $n$ is the number of units. Examples of four-dimensional extensions are the algebraic extensions $Q(\sqrt{\pm d_1}, \sqrt{\pm d_2})$ of rationals, where $d_i$ are square-free integers, reduce to form $Q(\theta)$. The cyclic extension of rationals by the powers of the m:th root of unity with $m = 5, 8, 12$ are four-dimensional extensions called cyclotomic number fields. Also the extensions $Q((\pm)d)^{1/4})$ are simple four-dimensional extensions. These extensions allow completion to a corresponding p-adic algebraic extension for some p-adic primes.

Quite generally, cyclotomic number fields $Q(\zeta_m)$ are obtained from polynomial algebra $Q[x]$ by replacing $x$ with the $m$:th primitive root of unity denoted by $\zeta_m$ and thus satisfying $\zeta_m^m = 1$. There are three cyclic extensions of dimension 4 and they correspond to $Q(\zeta_5) = Q(\zeta_{10})$, $Q(\zeta_8)$ and $Q(\zeta_{12})$. Cyclotomic extensions are higly symmetric since the roots of unity act as symmetries of the norm.

The units of cyclotomic field $Q(\zeta_m)$ form group $Z_2 \times Z_m \times Z$. $Z$ corresponds to the powers of units for $Q(\zeta_m + 1/\zeta_m)$. These powers have unit norm only with respect to the norm of $Q(\zeta_m)$ whereas with respect to the ordinary complex norm they correspond to fractal scalings. What looks fractal obtained by repeated scalings of the same structure with respect to the real norm looks like a lattice when algebraic norm is used.

*1. $Q(\zeta_8)$*

The cyclotomic number field $Q(\zeta_8)$, $\zeta_8 = exp(i\pi/4)$ satisfying $\zeta_8^8 = 1$, consists of numbers of form $k = m + in + \sqrt{i}(r + is)$. All roots ($\pm i^{1/2}$ and $\pm i^{3/2}$) are complex. The group of units is $Z_2^4 \times Z$. $Z$ corresponds in real topology to the fractal scalings generated by $L = 1 + \sqrt{2}$. The integer multiples of $log(L)$ could be interpreted as a quantized momentum. $Q(\zeta_8)$ can be generated by $\pm\zeta_8$ and $\pm i\zeta_8$. This means additional $Z_2^2$ Galois symmetry which does not define multiplicative quantum number.

*2. $Q(\zeta_{12})$*

The extension $Q(\sqrt{-1}, w)$, $w = \zeta_3$, can be regarded as a cyclic extension $Q(iw) = Q(\zeta_{12})$ as is clear from the fact that the six lowest powers of $iw$ come as $iw, -w^2, -i, w = -1 - w^2, iw^2 = -iw - i, -1$. $Z(iw)$ is especially interesting because it contains $Q(i)$ and $Q(w)$ for which primes correspond to Gaussian and Eisenstein primes. A unique factorization to a product of irreducibles is possible only for $Q(\zeta_m)$ $m \leq 10$: thus the algebraic integers in $Z(iw)$ do not always allow a unique decomposition into irreducibles. The most obvious candidates for primes not allowing unique factorization are primes satisfying simultaneously the conditions $p \, mod \, 4 = 3 = 1$ implying decomposition into a product of Gaussian prime and its conjugate and $p \, mod \, 3 = 1$ guaranteing the decomposition into a product of Eisestein prime and its conjugate.

The group of units reduces to $Z_2^2 \times Z_3 \times Z$ might have something to do with the group of discrete quantum numbers C,P and $SU(3)$ triality telling the number of quarks modulo 3 in the state. For the extensions $Q(\sqrt{-1}, \sqrt{d})$ the roots of unity form the group $Z_2^2$: these extensions could correspond to gauge bosons and the quantum numbers would correspond to $C$ and $P$. For real extensions the group of the roots of unity reduces to $Z_2$: in this case the interpretation inters of parity suggests itself.

The lattice defined by $Z$ corresponds to the scalings by powers of $\sqrt{3} + 2$. It could be also interpreted also as the lattice of longitudinal momenta for hadronic quarks which move collinearly inside space-time sheet which might be identified as a massless extremal (ME) for which longitudinal direction is a preferred spatial direction.

$Q(\zeta_{12})$ can be generated by $\pm iw, \pm iw^2$ and the replacement of $iw$ with these alternatives generates $Z_2^2$ symmetry not realizable as a multiplication with units.

*3. $Q(\zeta_5)$ and biology*

$Q(\zeta_5)$ indeed gives 4-dimensional extension of rationals since one has $1+\zeta_5+...\zeta_5^4=0$ implying that $\zeta_5^4 = 1/\zeta_5$ is expressible as rational combination of other units. Both $Q(\zeta_5)$ and $Q(\zeta_8)$ allows a unique decomposition of rational integers into prime factors. The primes in $Q(\zeta_5)$ allow decomposition to a product of $r = 1, 2$ or 4 primes of $Q(\zeta_5)$ [25]. The value of $r$ for a given $p$ is fixed by the requirement that $f = 4/r$ is the smallest natural number for which $p^f - 1\ mod\ p = 0$ holds true. For instance, $p = 2, 3$ correspond to $f = 4$ and are primes of $Q(\zeta_5)$, $p = 11$ has decomposition into a product of four primes of $Q(\zeta_5)$, and $p = 19$ has decomposition into two primes of $Q(\zeta_5)$).

What makes this extension interesting is that the phase angle associated with $\zeta_5$ corresponds to the angle of 72 degrees closely related with Golden Mean $\tau = (1+\sqrt{5})/2$ satisfying the equation $\tau^2 - \tau - 1 = 0$. The phase of the fifth root is given by $\zeta_5 = (\tau - 1 + i\sqrt{2+\tau})2$. The group of units is $Z_2 \times Z_5 \times Z$. $Z$ corresponds to the factal scalings by $\tau = (1 + \sqrt{5})/2$. The conjugations $\zeta_5 \rightarrow \zeta_5^k$, $k = 1, 2, 3, 4$ leave the norm invariant and generate group $Z_2^5$.

Fractal scalings by Golden Mean and the closely related Fibonacci numbers are closely related with the fractal structures associated with living systems (botany is full of logarithmic spirals involving Golden Mean and the phase angle 36 is involved even with DNA). It has been suggested that Golden Mean might be even a fundamental constant of physics [29]. Of course, the very fact that Golden Mean emerges in biological length scales provides strongest evidence for its dynamical origin in algebraic framework.

$Q(\zeta_5)$ cannot be realized as an algebraic extension $K(\theta, i)$ naturally associated with the transversal part of quaternionic primes but can appear only as a subfield of the 8-dimensional extension $K(i, cos(2\pi/5), sin(2\pi/5)$ containing also 20:th root of unity as $\zeta_{20} = i\zeta_5$. In [E9] it is indeed found that Golden Mean plays a fundamental role in topological quantum computation and is indeed a fundamental constant in TGD Universe.

### 7.5.1 Fractal scalings

By Dirichlet's unit theorem the group of units quite generally reduces to $Z_m \times Z^r$, where $Z_m$ is cyclic group of roots of unity and $Z^r$ can be regarded as an $r$-dimensional lattice with latticed units determined by the extension. For real extensions $Z_m$ reduces to $Z_2$ since the only real roots of unity are $\{\pm 1\}$. All components of four-momentum represented by a quaternionic prime can be multiplied by separate real units of $Q(\theta)$. For a given quaternionic prime, one can always factor out the common factor of the units of $Q(\theta)$ or $Q(\theta, i)$.

The units generate nontrivial transformations at the level of single quaternionic prime. If the dimension of the real extension is $n$, the transformations form an $n-1$-dimensional lattice of scalings. Alternative but less plausible interpretation is that the logarithms of the scalings represent $n-1$-dimensional momentum lattice. Particle would be like a part of an algebraic hologram carrying information about external world in accordance with the ideas about fractality. Of course, units represent fractal scalings only with respect to ordinary real norm, with respect to number theoretical norm they act like phase factors.

For instance, in the case of $Q(\sqrt{5})$ the units correspond to scalings by powers of Golden Mean $\tau = (1+\sqrt{5})/2$ having number theoretic norm equal to one. Bio-systems are indeed full of fractals with scaling symmetry. For $K = Q(\sqrt{3})$ the scalings correspond to powers of $L = 2 + \sqrt{3}$. An interesting possibility is that hadron physics might reveal fractality in powers of $L$. More generally, for $Q(\sqrt{d})$, $d$ square-free integer, the basic fractal scaling is $L = p_{n-1} + q_{n-1}\sqrt{d}$, where $n$ defines the period of the continued fraction expansion of $\sqrt{d}$ and $p_k/q_k$ defines $k$:th convergent in the continued fraction expansion.

Four-dimensional algebraic extensions are very interesting for several reasons. First, algebraic dimension four is a borderline in complexity in the sense that for higher-dimensional irreducible algebraic extensions there is no general quadratures analogous to the formulas associated with second order polynomials giving the roots of the polynomial. Secondly, in transversal degrees of freedom the minimal dimension for $K(\theta, i)$ is four. The units of $K$ which are algebraic integers having a unit norm in $K$. Quite generally, the group of units is a product $Z_{2k} \times Z_r$ of two groups. $Z_{2k} = Z_2 \times Z_k$ is the cyclic group generated by k:th root of unity. For real extensions one has $k = 1$. In transversal degrees of freedom one can have $k > 1$ since extension is $Q(\theta, i)$. The roots of unity possible in four-dimensional case correspond to $k = 2, 4, 6, 8, 10, 12$. Corresponding cyclic groups are products of $Z_2^i$, $Z_3$ and $Z_5$. $Z_2$, $Z_2$ and $Z_3$ and act as symmetries of the root lattices of Cartan algebras.

$Z_3$ gives rise to the Cartan algebra of $SU(3)$ and an interesting question is whether color symmetry is generated dynamically or whether it can be regarded as a basic symmetry with the lattice of integer quaternions providing scaled-up version for the root lattice of color group. Note that in TGD quark color is not spin like quantum number but corresponds to $CP_2$ partial waves for quark like spinors.

### 7.5.2 Permutations of the real roots of the minimal polynomial of $\theta$

The replacements of the primitive element $\theta$ of $K(\theta)$ with a new one obtained by acting in it with the elements of Galois group of the minimal polynomial of $\theta$ generate different internal states of number theoretic fermions and bosons. The subgroup $G_1$ of Galois group permuting the real roots of the minimal polynomial with each other acts also as a symmetry. The number of equivalent primitive elements is $n_1 = n - 2r_1$, where $r_2$ is the number of complex root pairs. For instance, for 2-dimensional extensions these symmetries permute the real roots of a second order polynomial irreducible in the set of rationals. Since the entire polynomial has rational coefficients, kind of $G_1$-confinement is realized. One could say that kind of algebraically confined n-color is in question.

## 7.6 Quaternionic primes

Primeness makes sense for quaternions and octonions. The following considerations are however restricted to quaternionic primes but can be easily generalized to the octonionic case. Quaternionic primes have Euclidian norm squared equal to a rational prime. The number $N(p)$ of primes associated with a given rational $p$ depends on $p$ and each $p$ allows at least two primes. Quaternionic primes correspond to points of 3-sphere with prime-valued radius squared. Prime-valued radius squared is consistent with p-adic length scale hypothesis, and one can indeed reduce p-adic length scale hypothesis to the assumption that the Euclidian region associated with $CP_2$ type extremal has prime-valued radius squared.

It is interesting to count the number of quaternionic primes with same prime valued length squared.

1. In the case of algebraic extensions the first definition of quaternionic norm is by using number theoretic norm either for entire quaternion squared or for each component of quaternion separately. The construction of infinite primes suggests that the first definition is more appropriate. Both definitions of norm are natural for four-momentum squared since they give integer valued mass squared spectrum associated with super-conformally invariant systems. One could also decompose quaternion to two parts as $q = (q_0 + Iq_1) + J(q_2 + Iq_3)$ and define number theoretic norm with respect to the algebraic extension $Q(\theta, I)$.

2. Quaternionic primes with the same norm are related by $SO(4)$ rotation plus a change of sign of the real component of quaternion. The components of integer quaternion are analogous to components of four-momentum.

3. There are $2^4$ quaternionic $\pm E_i$ and multiplication by these units defines symmetries. Non-commutativity of the quaternionic multiplication makes the interpretation of units as parity like quantum numbers somewhat problematic since the net parity associated with a product of primes representing physical particles associated with the infinite primes depends on the

order of quaternionic primes. For real algebraic extensions $K = Q(\theta)$ there is also the units defining a 'momentum' lattice with dimension $n-1$, where $n$ is the degree of the minimal polynomial $P(\theta)$.

4. Quaternionic primes cannot be real so that a given quaternionic prime with $k \geq 2$ components has $2^k$ conjugates obtained by changing the signs of the components of quaternion. Basic conjugation changes the signs of imagy components of quaternion. This corresponds to group $Z_2^k \subset Z_2^4$, $2 \leq k \leq 4$.

5. The group $S_4$ of $4! = 24$ permutations of four objects preserves the norm of a prime quaternion: these permutations are representable as a multiplication with non-prime quaternion and thus identifiable as subgroup of $SO(4)$ and also as a subgroup of $SO(3)$ (invariance group of tedrahedron). In degenerate cases (say when some components of $q$ are identical), some subgroup of $S_4$ leaves quaternionic prime invariant and the rotational degeneracy reduces from $D = 24$ to some smaller number which is some factor of 24 and equals to $4, 6$ or 12 as is easy to see. There are 16 quaternionic conjugations corresponding to change of sign of any quaternion unit but all these conjugations are obtained from single quaternionic conjugation changing the sign of the imaginary part of quaternion by combining them with a multiplication with unit and its inverse. Thus the restricted group of symmetries is $S_4 \times Z_2$.

6. It is possible to find for every prime $p$ at least two quaternionic ( primes with norm squared equal to $p$. For a given prime $p$ there are in general several quaternionic primes not obtainable from each other by transformations of $S_4$. There must exist some discrete subgroup of $SO(4)$ relating these quaternionic primes to each other.

7. The maximal number of quaternionic primes generated by $S_4 \times Z_2$ is $24 \times 2$. In noncommutative situation it is not clear whether units can be regarded as parity type quantum numbers. In any case, one can divide the entire group with $Z_2^4$ to obtain $Z_3$. This group corresponds to cyclic permutations of imaginary quaternion units.

$D = 24$ is the number of physical dimensions in bosonic string model. In TGD framework a possible interpretation is based on the observation that infinite primes constructed from rational primes the product of all primes contains the first power of each prime having interpretation as a representation for a single filled state of the fermionic sea. In the case of quaternions the Fock vacuum defined as a product of all quaternionic primes gives rise to a vacuum state

$$X = \prod_p p^{N(p)/2} \ ,$$

since each prime and its quaternionic conjugate contribute one power of $p$.

## 7.7 Imbedding space metric and vielbein must involve only rational functions

Algebraization requires that imbedding space exists in the algebraic sense containing only points for which preferred coordinate variables have values in some algebraic extension of rationals. Imbedding space metric at the algebraic level can be defined as a quadratic form without any reference to metric concepts like line element or distance. The metric tensors of both $M_+^4$ and $CP_2$ are indeed represented by algebraic functions in the preferred coordinates dictated by the symmetries of these spaces.

One should also construct spinor structure and this requires the introduction of an algebraic extension containing square roots since vielbein vectors appearing in the definition of the gamma matrices involve square roots of the components of the metric. In $CP_2$ degrees of freedom this forces the introduction of square root function, and thus all square roots, unless one restricts the values of the radial $CP_2$ coordinate appearing in the vielbein in such a manner that rationals result. What is interesting is that all components of spinor curvature and Kähler form of $CP_2$ are quadratic with respect to vierbein and algebraic functions of $CP_2$ complex coordinates. Also the square root of the determinant of the induce metric appears only as a multiplicative factor in the Euler-Lagrange equations so that one can get rid of the square roots.

Induced spinor structure and Dirac equation relies on the notion of the induced gamma matrices and here the projections of the vierbein of $CP_2$ containing square roots are unavoidable. In complex coordinates the components of $CP_2$ vielbein in complex coordinates $\xi_1, \xi_2$, in which the action of $U(2)$ is linear holomorphic transformation, involve the square roots $r = \sqrt{|\xi|^2 + |\xi_2|^2}$ and $\sqrt{1+r^2}$ (for detailed formulas see Appendix at the end of the book). If one has $r = m/n$, the requirement that $\sqrt{1+r^2}$ is rational, implies $m^2 + n^2 = k^2$ so that $(m,n)$ defines Pythagorean square. Thus induced Dirac equation is rationalized if the allowed values of $r$ correspond to Pythagorean phases. The notion of the phase preserving canonical identification [E6], crucial for the earlier formulation of TGD, is consistent with this assumption. The metric of $S^2 = CP_1$ is a simplified example of what happens. One can write the metric as $g_{z\bar{z}=r^2} = \frac{1}{1+r^2}$ and vielbein component is proportional to $1/\sqrt{1+r^2}$, this exists for $r = m/n$ as rational number if one has $m^2 + n^2 = k^2$, which indeed defines Pythagorean triangle.

The restriction of the phases associated with the $CP_2$ coordinates to Pythagorean ones has deeper coordinate-invariant meaning. Rational $CP_2$ can be defined as a coset space $SU_Q(3)/U_Q(2)$ of rational groups $SU_Q(3)$ and $U_Q(2)$: rationality is required in the linear matrix representation of these groups.

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