

# Appendix

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# 1 Basic properties of $CP_2$

## 1.1 $CP_2$ as a manifold

$CP_2$ , the complex projective space of two complex dimensions, is obtained by identifying the points of complex 3-space  $C^3$  under the projective equivalence

$$(z^1, z^2, z^3) \equiv \lambda(z^1, z^2, z^3) . \quad (1)$$

Here  $\lambda$  is any nonzero complex number. Note that  $CP_2$  can also be regarded as the coset space  $SU(3)/U(2)$ . The pair  $z^i/z^j$  for fixed  $j$  and  $z^i \neq 0$  defines a complex coordinate chart for  $CP_2$ . As  $j$  runs from 1 to 3 one obtains an atlas of three charts covering  $CP_2$ , the charts being holomorphically related to each other (e.g.  $CP_2$  is a complex manifold). The points  $z^3 \neq 0$  form a subset of  $CP_2$  homeomorphic to  $R^4$  and the points with  $z^3 = 0$  a set homeomorphic to  $S^2$ . Therefore  $CP_2$  is obtained by "adding the 2-sphere at infinity to  $R^4$ ".

Besides the standard complex coordinates  $\xi^i = z^i/z^3$ ,  $i = 1, 2$  the coordinates of Eguchi and Freund [2] will be used and their relation to the complex coordinates is given by

$$\begin{aligned} \xi^1 &= z + it , \\ \xi^2 &= x + iy . \end{aligned} \quad (2)$$

These are related to the "spherical coordinates" via the equations

$$\begin{aligned} \xi^1 &= r \exp\left(i \frac{(\Psi + \Phi)}{2}\right) \cos\left(\frac{\Theta}{2}\right) , \\ \xi^2 &= r \exp\left(i \frac{(\Psi - \Phi)}{2}\right) \sin\left(\frac{\Theta}{2}\right) . \end{aligned} \quad (3)$$

The ranges of the variables  $r, \Theta, \Phi, \Psi$  are  $[0, \infty], [0, \pi], [0, 4\pi], [0, 2\pi]$  respectively.

Considered as a real four-manifold  $CP_2$  is compact and simply connected, with Euler number 3, Pontryagin number 3 and second Betti number  $b = 1$ .

## 1.2 Metric and Kähler structures of $CP_2$

In order to obtain a natural metric for  $CP_2$ , observe that  $CP_2$  can be thought of as a set of the orbits of the isometries  $z^i \rightarrow \exp(i\alpha)z^i$  on the sphere  $S^5$ :  $\sum z^i \bar{z}^i = R^2$ . The metric of  $CP_2$  is obtained by projecting the metric of  $S^5$  orthogonally to the orbits of the isometries. Therefore the distance between the points of  $CP_2$  is that between the representative orbits on  $S^5$ . The line element has the following form in the complex coordinates

$$ds^2 = g_{a\bar{b}} d\xi^a d\bar{\xi}^b , \quad (4)$$

where the Hermitian, in fact Kähler, metric  $g_{a\bar{b}}$  is defined by

$$g_{a\bar{b}} = R^2 \partial_a \partial_{\bar{b}} K , \quad (5)$$

where the function  $K$ , Kähler function, is defined as

$$\begin{aligned} K &= \ln F , \\ F &= 1 + r^2 . \end{aligned} \quad (6)$$

The representation of the metric is given by

$$\frac{ds^2}{R^2} = \frac{(dr^2 + r^2 \sigma_3^2)}{F^2} + \frac{r^2(\sigma_1^2 + \sigma_2^2)}{F} , \quad (7)$$

where the quantities  $\sigma_i$  are defined as

$$\begin{aligned} r^2 \sigma_1 &= \text{Im}(\xi^1 d\xi^2 - \xi^2 d\xi^1) , \\ r^2 \sigma_2 &= -\text{Re}(\xi^1 d\xi^2 - \xi^2 d\xi^1) , \\ r^2 \sigma_3 &= -\text{Im}(\xi^1 d\bar{\xi}^1 + \xi^2 d\bar{\xi}^2) . \end{aligned} \quad (8)$$

The vierbein forms, which satisfy the defining relation

$$s_{kl} = R^2 \sum_A e_k^A e_l^A , \quad (9)$$

are given by

$$\begin{aligned} e^0 &= \frac{dr}{F} , & e^1 &= \frac{r\sigma_1}{\sqrt{F}} , \\ e^2 &= \frac{r\sigma_2}{\sqrt{F}} , & e^3 &= \frac{r\sigma_3}{F} . \end{aligned} \quad (10)$$

The explicit representations of vierbein vectors are given by

$$\begin{aligned} e^0 &= \frac{dr}{F} , & e^1 &= \frac{r(\sin\Theta \cos\Psi d\Phi + \sin\Psi d\Theta)}{2\sqrt{F}} , \\ e^2 &= \frac{r(\sin\Theta \sin\Psi d\Phi - \cos\Psi d\Theta)}{2\sqrt{F}} , & e^3 &= \frac{r(d\Psi + \cos\Theta d\Phi)}{2F} . \end{aligned} \quad (11)$$

The explicit representation of the line element is given by the expression

$$ds^2/R^2 = dr^2/F^2 + (r^2/4F^2)(d\Psi + \cos\Theta d\Phi)^2 + (r^2/4F)(d\Theta^2 + \sin^2\Theta d\Phi^2) . \quad (12)$$

The vierbein connection satisfying the defining relation

$$de^A = -V_B^A \wedge e^B , \quad (13)$$

is given by

$$\begin{aligned} V_{01} &= -\frac{e^1}{r_2} , & V_{23} &= \frac{e^1}{r_2} , \\ V_{02} &= -\frac{e^2}{r} , & V_{31} &= \frac{e^2}{r} , \\ V_{03} &= (r - \frac{1}{r})e^3 , & V_{12} &= (2r + \frac{1}{r})e^3 . \end{aligned} \quad (14)$$

The representation of the covariantly constant curvature tensor is given by

$$\begin{aligned} R_{01} &= e^0 \wedge e^1 - e^2 \wedge e^3 , & R_{23} &= e^0 \wedge e^1 - e^2 \wedge e^3 , \\ R_{02} &= e^0 \wedge e^2 - e^3 \wedge e^1 , & R_{31} &= -e^0 \wedge e^2 + e^3 \wedge e^1 , \\ R_{03} &= 4e^0 \wedge e^3 + 2e^1 \wedge e^2 , & R_{12} &= 2e^0 \wedge e^3 + 4e^1 \wedge e^2 . \end{aligned} \quad (15)$$

Metric defines a real, covariantly constant, and therefore closed 2-form  $J$

$$J = -ig_{a\bar{b}} d\xi^a d\bar{\xi}^b , \quad (16)$$

the so called Kähler form. Kähler form  $J$  defines in  $CP_2$  a symplectic structure because it satisfies the condition

$$J^k{}_r J^{rl} = -s^{kl} . \quad (17)$$

The form  $J$  is integer valued and by its covariant constancy satisfies free Maxwell equations. Hence it can be regarded as a curvature form of a  $U(1)$  gauge potential  $B$  carrying a magnetic charge of unit  $1/2g$  ( $g$  denotes the gauge coupling). Locally one has therefore

$$J = dB , \quad (18)$$

where  $B$  is the so called Kähler potential, which is not defined globally since  $J$  describes magnetic monopole.

It should be noticed that the magnetic flux of  $J$  through a 2-surface in  $CP_2$  is proportional to its homology equivalence class, which is integer valued. The explicit representations of  $J$  and  $B$  are given by

$$\begin{aligned} B &= 2re^3 , \\ J &= 2(e^0 \wedge e^3 + e^1 \wedge e^2) = \frac{r^2}{F^2} dr \wedge (d\Psi + \cos\Theta d\Phi) + \frac{r^2}{2F} \sin\Theta d\Theta d\Phi . \end{aligned} \quad (19)$$

The vielbein curvature form and Kähler form are covariantly constant and have in the complex coordinates only components of type (1,1).

Useful coordinates for  $CP_2$  are the so called canonical coordinates in which Kähler potential and Kähler form have very simple expressions

$$\begin{aligned} B &= \sum_{k=1,2} P_k dQ_k , \\ J &= \sum_{k=1,2} dP_k \wedge dQ_k . \end{aligned} \quad (20)$$

The relationship of the canonical coordinates to the "spherical" coordinates is given by the equations

$$\begin{aligned} P_1 &= -\frac{1}{1+r^2} , \\ P_2 &= \frac{r^2 \cos \Theta}{2(1+r^2)} , \\ Q_1 &= \Psi , \\ Q_2 &= \Phi . \end{aligned} \quad (21)$$

### 1.3 Spinors in $CP_2$

$CP_2$  doesn't allow spinor structure in the conventional sense [5]. However, the coupling of the spinors to a half odd multiple of the Kähler potential leads to a respectable spinor structure. Because the delicacies associated with the spinor structure of  $CP_2$  play a fundamental role in TGD, the arguments of Hawking are repeated here.

To see how the space can fail to have an ordinary spinor structure consider the parallel transport of the vierbein in a simply connected space  $M$ . The parallel propagation around a closed curve with a base point  $x$  leads to a rotated vierbein at  $x$ :  $e^A = R_B^A e^B$  and one can associate to each closed path an element of  $SO(4)$ .

Consider now a one-parameter family of closed curves  $\gamma(v) : v \in (0,1)$  with the same base point  $x$  and  $\gamma(0)$  and  $\gamma(1)$  trivial paths. Clearly these paths define a sphere  $S^2$  in  $M$  and the element  $R_B^A(v)$  defines a closed path in  $SO(4)$ . When the sphere  $S^2$  is contractible to a point e.g., homologically trivial, the path in  $SO(4)$  is also contractible to a point and therefore represents a trivial element of the homotopy group  $\Pi_1(SO(4)) = Z_2$ .

For a homologically nontrivial 2-surface  $S^2$  the associated path in  $SO(4)$  can be homotopically nontrivial and therefore corresponds to a nonclosed path in the covering group  $Spin(4)$  (leading from the matrix 1 to -1 in the matrix representation). Assume this is the case.

Assume now that the space allows spinor structure. Then one can parallelly propagate also spinors and by the above construction associate a closed path

of  $\text{Spin}(4)$  to the surface  $S^2$ . Now, however this path corresponds to a lift of the corresponding  $\text{SO}(4)$  path and cannot be closed. Thus one ends up with a contradiction.

From the preceding argument it is clear that one could compensate the nonallowed  $-1$ -factor associated with the parallel transport of the spinor around the sphere  $S^2$  by coupling it to a gauge potential in such a way that in the parallel transport the gauge potential introduces a compensating  $-1$ -factor. For a  $U(1)$  gauge potential this factor is given by the exponential  $\exp(i2\Phi)$ , where  $\Phi$  is the magnetic flux through the surface. This factor has the value  $-1$  provided the  $U(1)$  potential carries half odd multiple of Dirac charge  $1/2g$ . In case of  $CP_2$  the required gauge potential is half odd multiple of the Kähler potential  $B$  defined previously. In the case of  $M^4 \times CP_2$  one can in addition couple the spinor components with different chiralities independently to an odd multiple of  $B/2$ .

#### 1.4 Geodesic submanifolds of $CP_2$

Geodesic submanifolds are defined as submanifolds having common geodesic lines with the imbedding space. As a consequence the second fundamental form of the geodesic manifold vanishes, which means that the tangent vectors  $h_\alpha^k$  (understood as vectors of  $H$ ) are covariantly constant quantities with respect to the covariant derivative taking into account that the tangent vectors are vectors both with respect to  $H$  and  $X^4$ .

In [3] a general characterization of the geodesic submanifolds for an arbitrary symmetric space  $G/H$  is given. Geodesic submanifolds are in 1-1-correspondence with the so called Lie triple systems of the Lie-algebra  $g$  of the group  $G$ . The Lie triple system  $t$  is defined as a subspace of  $g$  characterized by the closedness property with respect to double commutation

$$[X, [Y, Z]] \in t \text{ for } X, Y, Z \in t . \quad (22)$$

$SU(3)$  allows, besides geodesic lines, two nonequivalent (not isometry related) geodesic spheres. This is understood by observing that  $SU(3)$  allows two nonequivalent  $SU(2)$  algebras corresponding to subgroups  $SO(3)$  (orthogonal  $3 \times 3$  matrices) and the usual isospin group  $SU(2)$ . By taking any subset of two generators from these algebras, one obtains a Lie triple system and by exponentiating this system, one obtains a 2-dimensional geodesic submanifold of  $CP_2$ .

Standard representatives for the geodesic spheres of  $CP_2$  are given by the equations

$$S_I^2 : \xi^1 = \bar{\xi}^2 \text{ or equivalently } (\Theta = \pi/2, \Psi = 0) ,$$

$$S_{II}^2 : \xi^1 = \xi^2 \text{ or equivalently } (\Theta = \pi/2, \Phi = 0) .$$

The nonequivalence of these submanifolds is clear from the fact that isometries act as holomorphic transformations in  $CP_2$ . The vanishing of the second fundamental form is also easy to verify. The first geodesic manifold is homologically trivial: in fact, the induced Kähler form vanishes identically for  $S_I^2$ .  $S_{II}^2$  is homologically nontrivial and the flux of the Kähler form gives its homology equivalence class.

## 2 Identification of the electroweak couplings

The delicacies of the spinor structure of  $CP_2$  make it a unique candidate for space  $S$ . First, the coupling of the spinors to the  $U(1)$  gauge potential defined by the Kähler structure provides the missing  $U(1)$  factor in the gauge group. Secondly, it is possible to couple different  $H$ -chiralities independently to a half odd multiple of the Kähler potential. Thus the hopes of obtaining a correct spectrum for the electromagnetic charge are considerable. In the following it will be demonstrated that the couplings of the induced spinor connection are indeed those of the GWS model [4] and in particular that the right handed neutrinos decouple completely from the electroweak interactions.

To begin with, recall that the space  $H$  allows to define three different chiralities for spinors. Spinors with fixed  $H$ -chirality  $e = \pm 1$ ,  $CP_2$ -chirality  $l, r$  and  $M^4$ -chirality  $L, R$  are defined by the condition

$$\begin{aligned}\Gamma\Psi &= e\Psi, \\ e &= \pm 1,\end{aligned}\tag{23}$$

where  $\Gamma$  denotes the matrix  $\Gamma_9 = \gamma_5 \times \gamma_5$ ,  $1 \times \gamma_5$  and  $\gamma_5 \times 1$  respectively. Clearly, for a fixed  $H$ -chirality  $CP_2$ - and  $M^4$ -chiralities are correlated.

The spinors with  $H$ -chirality  $e = \pm 1$  can be identified as quark and lepton like spinors respectively. The separate conservation of baryon and lepton numbers can be understood as a consequence of generalized chiral invariance if this identification is accepted. For the spinors with a definite  $H$ -chirality one can identify the vielbein group of  $CP_2$  as the electroweak group:  $SO(4) = SU(2)_L \times SU(2)_R$ .

The covariant derivatives are defined by the spinorial connection

$$A = V + \frac{B}{2}(n_+1_+ + n_-1_-) .\tag{24}$$

Here  $V$  and  $B$  denote the projections of the vielbein and Kähler gauge potentials respectively and  $1_{+(-)}$  projects to the spinor  $H$ -chirality  $+(-)$ . The integers  $n_{\pm}$  are odd from the requirement of a respectable spinor structure.

The explicit representation of the vielbein connection  $V$  and of  $B$  are given by the equations

$$\begin{aligned}
V_{01} &= -\frac{e^1}{r} , & V_{23} &= \frac{e^1}{r^2} , \\
V_{02} &= -\frac{e^2}{r} , & V_{31} &= \frac{e^2}{r} , \\
V_{03} &= (r - \frac{1}{r})e^3 , & V_{12} &= (2r + \frac{1}{r})e^3 ,
\end{aligned} \tag{25}$$

and

$$B = 2re^3 , \tag{26}$$

respectively. The explicit representation of the vielbein is not needed here.

Let us first show that the charged part of the spinor connection couples purely left handedly. Identifying  $\Sigma_3^0$  and  $\Sigma_2^1$  as the diagonal (neutral) Lie-algebra generators of  $SO(4)$ , one finds that the charged part of the spinor connection is given by

$$A_{ch} = 2V_{23}I_L^1 + 2V_{13}I_L^2 , \tag{27}$$

where one have defined

$$\begin{aligned}
I_L^1 &= \frac{(\Sigma_{01} - \Sigma_{23})}{2} , \\
I_L^2 &= \frac{(\Sigma_{02} - \Sigma_{13})}{2} .
\end{aligned} \tag{28}$$

$A_{ch}$  is clearly left handed so that one can perform the identification

$$W^\pm = \frac{2(e^1 \pm ie^2)}{r} , \tag{29}$$

where  $W^\pm$  denotes the charged intermediate vector boson.

Consider next the identification of the neutral gauge bosons  $\gamma$  and  $Z^0$  as appropriate linear combinations of the two functionally independent quantities

$$\begin{aligned}
X &= re^3 , \\
Y &= \frac{e^3}{r} ,
\end{aligned} \tag{30}$$

appearing in the neutral part of the spinor connection. We show first that the mere requirement that photon couples vectorially implies the basic coupling structure of the GWS model leaving only the value of Weinberg angle undetermined.

To begin with let us define



$$\begin{aligned}\bar{\gamma} &= aX + bY , \\ \bar{Z}^0 &= cX + dY ,\end{aligned}\tag{31}$$

where the normalization condition

$$ad - bc = 1 ,$$

is satisfied. The physical fields  $\gamma$  and  $Z^0$  are related to  $\bar{\gamma}$  and  $\bar{Z}^0$  by simple normalization factors.

Expressing the neutral part of the spinor connection in term of these fields one obtains

$$\begin{aligned}A_{nc} &= [(c + d)2\Sigma_{03} + (2d - c)2\Sigma_{12} + d(n_+1_+ + n_-1_-)]\bar{\gamma} \\ &+ [(a - b)2\Sigma_{03} + (a - 2b)2\Sigma_{12} - b(n_+1_+ + n_-1_-)]\bar{Z}^0 .\end{aligned}\tag{32}$$

Identifying  $\Sigma_{12}$  and  $\Sigma_{03} = 1 \times \gamma_5 \Sigma_{12}$  as vectorial and axial Lie-algebra generators, respectively, the requirement that  $\gamma$  couples vectorially leads to the condition

$$c = -d .\tag{33}$$

Using this result plus previous equations, one obtains for the neutral part of the connection the expression

$$A_{nc} = \gamma Q_{em} + Z^0 (I_L^3 - \sin^2 \theta_W Q_{em}) .\tag{34}$$

Here the electromagnetic charge  $Q_{em}$  and the weak isospin are defined by

$$\begin{aligned}Q_{em} &= \Sigma^{12} + \frac{(n_+1_+ + n_-1_-)}{6} , \\ I_L^3 &= \frac{(\Sigma^{12} - \Sigma^{03})}{2} .\end{aligned}\tag{35}$$

The fields  $\gamma$  and  $Z^0$  are defined via the relations

$$\begin{aligned}\gamma &= 6d\bar{\gamma} = \frac{6}{(a + b)}(aX + bY) , \\ Z^0 &= 4(a + b)\bar{Z}^0 = 4(X - Y) .\end{aligned}\tag{36}$$

The value of the Weinberg angle is given by

$$\sin^2\theta_W = \frac{3b}{2(a+b)} , \quad (37)$$

and is not fixed completely. Observe that right handed neutrinos decouple completely from the electroweak interactions.

The determination of the value of Weinberg angle is a dynamical problem. The angle is completely fixed once the YM action is fixed by requiring that action contains no cross term of type  $\gamma Z^0$ . Pure symmetry non-broken electroweak YM action leads to a definite value for the Weinberg angle. One can however add a symmetry breaking term proportional to Kähler action and this changes the value of the Weinberg angle.

To evaluate the value of the Weinberg angle one can express the neutral part  $F_{nc}$  of the induced gauge field as

$$F_{nc} = 2R_{03}\Sigma^{03} + 2R_{12}\Sigma^{12} + J(n_{+1+} + n_{-1-}) , \quad (38)$$

where one has

$$\begin{aligned} R_{03} &= 2(2e^0 \wedge e^3 + e^1 \wedge e^2) , \\ R_{12} &= 2(e^0 \wedge e^3 + 2e^1 \wedge e^2) , \\ J &= 2(e^0 \wedge e^3 + e^1 \wedge e^2) , \end{aligned} \quad (39)$$

in terms of the fields  $\gamma$  and  $Z^0$  (photon and  $Z$ - boson)

$$F_{nc} = \gamma Q_{em} + Z^0(I_L^3 - \sin^2\theta_W Q_{em}) . \quad (40)$$

Evaluating the expressions above one obtains for  $\gamma$  and  $Z^0$  the expressions

$$\begin{aligned} \gamma &= 3J - \sin^2\theta_W R_{03} , \\ Z^0 &= 2R_{03} . \end{aligned} \quad (41)$$

For the Kähler field one obtains

$$J = \frac{1}{3}(\gamma + \sin^2\theta_W Z^0) . \quad (42)$$

Expressing the neutral part of the symmetry broken YM action

$$\begin{aligned} L_{ew} &= L_{sym} + f J^{\alpha\beta} J_{\alpha\beta} , \\ L_{sym} &= \frac{1}{4g^2} Tr(F^{\alpha\beta} F_{\alpha\beta}) , \end{aligned} \quad (43)$$

where the trace is taken in spinor representation, in terms of  $\gamma$  and  $Z^0$  one obtains for the coefficient  $X$  of the  $\gamma Z^0$  cross term (this coefficient must vanish) the expression

$$\begin{aligned} X &= -\frac{K}{2g^2} + \frac{fp}{18} , \\ K &= \text{Tr} [Q_{em}(I_L^3 - \sin^2\theta_W Q_{em})] , \end{aligned} \quad (44)$$

In the general case the value of the coefficient  $K$  is given by

$$K = \sum_i \left[ -\frac{(18 + 2n_i^2)\sin^2\theta_W}{9} \right] , \quad (45)$$

where the sum is over the spinor chiralities, which appear as elementary fermions and  $n_i$  is the integer describing the coupling of the spinor field to the the Kähler potential. The cross term vanishes provided the value of the Weinberg angle is given by

$$\sin^2\theta_W = \frac{9 \sum_i 1}{(fg^2 + 2 \sum_i (18 + n_i^2))} . \quad (46)$$

In the scenario where both leptons and quarks are elementary fermions the value of the Weinberg angle is given by

$$\sin^2\theta_W = \frac{9}{(\frac{fg^2}{2} + 28)} . \quad (47)$$

The bare value of the Weinberg angle is 9/28 in this scenario, which is quite close to the typical value 9/24 of GUTs [6].

### 3 Discrete symmetries

The treatment of discrete symmetries C, P, and T is based on the following requirements:

- a) Symmetries must be realized as purely geometric transformations.
- b) Transformation properties of the field variables should be essentially the same as in the conventional quantum field theories [1].

The action of the reflection  $P$  on spinors of is given by

$$\Psi \rightarrow P\Psi = \gamma^0 \times \gamma^0 \Psi . \quad (48)$$

in the representation of the gamma matrices for which  $\gamma^0$  is diagonal. It should be noticed that  $W$  and  $Z^0$  bosons break parity symmetry as they should since their charge matrices do not commute with the matrix of P.

The guess that a complex conjugation in  $CP_2$  is associated with T transformation of the physicist turns out to be correct. One can verify by a direct calculation that pure Dirac action is invariant under T realized according to

$$\begin{aligned} m^k &\rightarrow T(M^k) , \\ \xi^k &\rightarrow \bar{\xi}^k , \\ \Psi &\rightarrow \gamma^1 \gamma^3 \times 1 \times \Psi . \end{aligned} \tag{49}$$

The operation bearing closest resemblance to the ordinary charge conjugation corresponds geometrically to complex conjugation in  $CP_2$ :

$$\begin{aligned} \xi^k &\rightarrow \bar{\xi}^k , \\ \Psi &\rightarrow \gamma^2 \gamma^0 \times 1 \times \Psi^\dagger . \end{aligned} \tag{50}$$

As one might have expected symmetries CP and T are exact symmetries of the pure Dirac action.

## 4 Space-time surfaces with vanishing em, $Z^0$ , Kähler, or $W$ fields

In the sequel it is shown that space-times for which either em,  $Z^0$ , or Kähler field vanishes decompose into regions characterized by six vacuum parameters: two of these quantum numbers ( $\omega_1$  and  $\omega_2$ ) are frequency type parameters, two ( $k_1$  and  $k_2$ ) are wave vector like quantum numbers, two of the quantum numbers ( $n_1$  and  $n_2$ ) are integers. The parameters  $\omega_i$  and  $n_i$  will be referred as electric and magnetic quantum numbers. The existence of these quantum numbers is not a feature of these solutions alone but represents a much more general phenomenon differentiating in a clear cut manner between TGD and Maxwell's electrodynamics.

The simplest manner to avoid surface Kähler charges and discontinuities or infinities in the derivatives of  $CP_2$  coordinates on the common boundary of two neighboring regions with different vacuum quantum numbers is topological field quantization, 3-space decomposes into disjoint topological field quanta, 3-surfaces having outer boundaries with possibly macroscopic size.

If one requires that space-time surface is an extremal of Kähler action and has a 2-dimensional  $CP_2$  projection, only vacuum extremals and space-time surfaces for which  $CP_2$  projection is a geodesic sphere, are allowed. Homologically non-trivial geodesic sphere correspond to vanishing  $W$  fields and homologically non-trivial sphere to non-vanishing  $W$  fields but vanishing  $\gamma$  and  $Z^0$ . For vacuum extremals all electro-weak gauge fields are in general non-vanishing although the net gauge field has  $U(1)$  holonomy.

### 4.1 Em neutral space-times

Em and  $Z^0$  neutral spacetimes are especially interesting space-times as far as applications of TGD are considered. Consider first the electromagnetically neutral space-times. Using spherical coordinates  $(r, \Theta, \Psi, \Phi)$  for  $CP_2$ , the expression of Kähler form reads as

$$\begin{aligned} J &= \frac{r}{F^2} dr \wedge (d\Psi + \cos(\Theta)d\Phi) + \frac{r^2}{2F} \sin(\Theta) d\Theta \wedge d\Phi , \\ F &= 1 + r^2 . \end{aligned} \quad (51)$$

The general expression of electromagnetic field reads as

$$\begin{aligned} F_{em} &= (3 + 2p) \frac{r}{F^2} dr \wedge (d\Psi + \cos(\Theta)d\Phi) + (3 + p) \frac{r^2}{2F} \sin(\Theta) d\Theta \wedge d\Phi , \\ p &= \sin^2(\Theta_W) , \end{aligned} \quad (52)$$

where  $\Theta_W$  denotes Weinberg angle.

The vanishing of the electromagnetic fields is guaranteed, when the conditions

$$\Psi = k\Phi ,$$

$$(3 + 2p)\frac{1}{r^2 F}(d(r^2)/d\Theta)(k + \cos(\Theta)) + (3 + p)\sin(\Theta) = 0 , \quad (53)$$

hold true. The conditions imply that  $CP_2$  projection of the electromagnetically neutral space-time is 2-dimensional. Solving the differential equation one obtains

$$r = \sqrt{\frac{X}{1-X}} ,$$

$$X = D \left[ \frac{(k+u)}{C} \right]^\epsilon ,$$

$$u \equiv \cos(\Theta) , \quad C = k + \cos(\Theta_0) , \quad D = \frac{r_0^2}{1+r_0^2} , \quad \epsilon = \frac{3+p}{3+2p} , \quad (54)$$

where  $C$  and  $D$  are integration constants.  $0 \leq X \leq 1$  is required by the reality of  $r$ .  $r = 0$  would correspond to  $X = 0$  giving  $u = -k$  achieved only for  $|k| \leq 1$  and  $r = \infty$  to  $X = 1$  giving  $|u+k| = [(1+r_0^2)/r_0^2]^{(3+2p)/(3+p)}$  achieved only for

$$\text{sign}(u+k) \times \left[ \frac{1+r_0^2}{r_0^2} \right]^{\frac{3+2p}{3+p}} \leq k+1 ,$$

where  $\text{sign}(x)$  denotes the sign of  $x$ .

Under rather general conditions the coordinates  $\Psi$  and  $\Phi$  can be written in the form

$$\Psi = \omega_2 m^0 + k_2 m^3 + n_2 \phi + \text{Fourier expansion} ,$$

$$\Phi = \omega_1 m^0 + k_1 m^3 + n_1 \phi + \text{Fourier expansion} . \quad (55)$$

$m^0, m^3$  and  $\phi$  denote the coordinate variables of the cylindrical  $M^4$  coordinates) so that one has  $k = \omega_2/\omega_1 = n_2/n_1 = k_2/k_1$ . The regions of the space-time surface with given values of the vacuum parameters  $\omega_i, k_i$  and  $n_i$  and  $m$  and  $C$  are bounded by the surfaces at which the electromagnetically neutral imbeddings become ill-defined, say by  $r > 0$  or  $r < \infty$  surfaces.

The space-time surface decomposes into regions characterized by different values of the vacuum parameters  $r_0$  and  $\Theta_0$ . At  $r = \infty$  surfaces  $n_2, \omega_2$  and  $m$  can change since all values of  $\Psi$  correspond to the same point of  $CP_2$ : at  $r = 0$  surfaces also  $n_1$  and  $\omega_1$  can change since all values of  $\Phi$  correspond to same point of  $CP_2$ , too. If  $r = 0$  or  $r = \infty$  is not in the allowed range space-time surface develops a boundary.

This implies what might be called topological quantization since in general it is not possible to find a smooth global imbedding for, say a constant magnetic field. Although global imbedding exists it decomposes into regions with different

values of the vacuum parameters and the coordinate  $u$  in general possesses discontinuous derivative at  $r = 0$  and  $r = \infty$  surfaces. A possible manner to avoid edges of space-time is to allow field quantization so that 3-space (and field) decomposes into disjoint quanta, which can be regarded as structurally stable units a 3-space (and of the gauge field). This doesn't exclude partial join along boundaries for neighboring field quanta provided some additional conditions guaranteing the absence of edges are satisfied.

The vanishing of the electromagnetic fields implies that the condition

$$\Omega \equiv \frac{\omega_2}{n_2} - \frac{\omega_1}{n_1} = 0 \quad , \quad (56)$$

is satisfied. In particular, the ratio  $\omega_2/\omega_1$  is rational number for the electromagnetically neutral regions of space-time surface. The change of the parameter  $n_1$  and  $n_2$  ( $\omega_1$  and  $\omega_2$ ) in general generates magnetic field and therefore these integers will be referred to as magnetic (electric) quantum numbers.

The expression for the Kähler form and  $Z^0$  field of the electromagnetically neutral space-time surface will be needed in sequel and is given by

$$\begin{aligned} J &= -\frac{p}{3+2p} X du \wedge d\Phi \quad , \\ Z^0 &= -\frac{6}{p} J \quad . \end{aligned} \quad (57)$$

The components of the electromagnetic field generated by varying vacuum parameters are proportional to the components of the Kähler field: in particular, the magnetic field is parallel to the Kähler magnetic field. The generation of a long range  $Z^0$  vacuum field is a purely TGD based feature not encountered in the standard gauge theories.

The effective form of the  $CP_2$  metric is given by

$$\begin{aligned} ds_{eff}^2 &= (s_{rr}(\frac{dr}{d\Theta})^2 + s_{\Theta\Theta})d\Theta^2 + (s_{\Phi\Phi} + 2ks_{\Phi\Psi})d\Phi^2 = \frac{R^2}{4}[s_{\Theta\Theta}^{eff}d\Theta^2 + s_{\Phi\Phi}^{eff}d\Phi^2] \quad , \\ s_{\Theta\Theta}^{eff} &= X \times \left[ \frac{\epsilon^2(1-u^2)}{(k+u)^2} \times \frac{1}{1-X} + 1 - X \right] \quad , \\ s_{\Phi\Phi}^{eff} &= X \times [(1-X)(k+u)^2 + 1 - u^2] \quad , \end{aligned} \quad (58)$$

and is useful in the construction of electromagnetically neutral imbedding of, say Schwarzschild metric. Note however that in general these imbeddings are not extremals of Kähler action.

## 4.2 Space-times with vanishing $Z^0$ or Kähler fields

The results just derived generalize to the  $Z^0$  neutral case as such. The only modification is the replacement of the parameter  $\epsilon$  with  $\epsilon = 1/2$  as becomes

clear by considering the condition stating that  $Z^0$  field vanishes identically also the relationship  $F_{em} = 3J = -\frac{3}{4}\frac{r^2}{F} du \wedge d\Phi$  is useful.

Also the generalization to the case of vacuum extremals is straightforward and corresponds to  $\epsilon = 1, p = 0$  in the formula for em neutral space-times. In this case classical em and  $Z^0$  fields are proportional to each other:

$$\begin{aligned} Z^0 &= 4e^0 \wedge e^3 = 2\frac{r}{F^2}(k + \cos(\Theta))dr \wedge d\Phi \ , \\ \gamma &= -\frac{p}{2}Z^0 \ . \end{aligned} \tag{59}$$

For vanishing value of Weinberg angle ( $p = 0$ ) em field vanishes and only  $Z^0$  field remains as a long range gauge field. Vacuum extremals for which long range  $Z^0$  field vanishes but em field is non-vanishing are not possible.

### 4.3 Induced gauge fields for space-times for which $CP_2$ projection is a geodesic sphere

For space-time sheets for which  $CP_2$  projection is  $r = \infty$  homologically non-trivial geodesic sphere of  $CP_2$  one has

$$\gamma = \left(\frac{3}{4} - \frac{\sin^2(\theta_W)}{2}\right)Z^0 \simeq \frac{5Z^0}{8} \ .$$

The induced  $W$  fields vanish in this case and they vanish also for all geodesic sphere obtained by  $SU(3)$  rotation.

For homologically trivial geodesic sphere a standard representative is obtained by using for the phase angles of standard complex  $CP_2$  coordinates constant values. In this case induced em,  $Z^0$ , and Kähler fields vanish but induced  $W$  fields are non-vanishing. This holds also for surfaces obtained by color rotation. Hence one can say that for non-vacuum extremals with 2-D  $CP_2$  projection color rotations and weak symmetries commute.



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