Hyper-Finite Factors and Construction of S-Matrix

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Abstract

During years I have spent a lot of time and effort to attempts to imagine various options for the construction of S-matrix. Contrary to my original belief, the real problem has not been the lack of my analytic skills but the failure of ordinary QFT based thinking in TGD framework.

Super-conformal symmetries generalized from string model context to TGD framework are symmetries of S-matrix. This is very powerful constraint to S-matrix but useless unless one has precisely defined ontology translated to a rigorous mathematical framework. The zero energy ontology of TGD is now rather well understood but differs dramatically from that of standard quantum field theories. Second deep difference is that path integral formalism is given up and the goal is to construct S-matrix as a generalization of braiding S-matrices with reaction vertices replaced with the replication of number theoretic braids associated with partonic 2-surfaces taking the role of vertices. Also number theoretic universality requiring fusion of real physics and various p-adic physics to single coherent whole is a completely new element.

The most recent vision about S-matrix combines ideas scattered in various chapters of various books and often drowned with details. A very brief summary would be as follows.

a) In TGD framework functional integral formalism is given up. S-matrix should be constructible as a generalization of braiding S-matrix in such a manner that the number theoretic braids assignable to light-like partonic 3-surfaces glued along their ends at 2-dimensional partonic 2-surfaces representing reaction vertices replicate in the vertex. This means a replacement of the free dynamics of point particles of quantum field theories with braiding dynamics associated with partonic 2-surfaces carrying braids and the replacement of particle creation with the creation of partons and replication of braids.

b) The construction of braiding S-matrices assignable to the incoming and outgoing partonic 2-surfaces is not a problem. The problem is to express mathematically what happens in the vertex. Here the observation that the tensor product of hyper-finite factors (HFFs) of type II is HFF of type II is the key observation. Many-parton vertex can be identified as a unitary isomorphism between the tensor product of incoming *resp.* outgoing HFFs. A reduction to HFF of type II_1 occurs because only a finite-dimensional projection of S-matrix in bosonic degrees of freedom defines a normalizable state. Most importantly, unitarity and non-triviality of S-matrix follows trivially.

c) HFFs of type III could also appear at the level of field operators used to create states but that at the level of quantum states everything reduces to HFFs of type II_1 and their tensor products giving these factors back. If braiding automorphisms reduce to the purely intrinsic unitary automorphisms of HFFs of type III then for certain values of the time parameter of automorphism having interpretation as a scaling parameter these automorphisms are trivial. These time scales could correspond to p-adic time scales so that p-adic length scale hypothesis would emerge at the fundamental level. In this kind of situation the braiding S-matrices associated with the incoming and outgoing partons could be trivial so that everything would reduce to this unitary isomorphism: a counterpart for the elimination of external legs from Feynman diagram in QFT. p-Adic thermodynamics and particle massivation could be also obtained when the time parameter of the automorphism is allowed to be complex as a generalization of thermal QFT.

d) One might hope that all complications related to what happens for *space-like* 3-surfaces could be eliminated by quantum classical correspondence stating that space-time view about particle reaction is only a space-time correlate for what happens in quantum fluctuating degrees of freedom associated with partonic 2-surfaces. This turns out to be the case only in non-perturbative phase. The reason is that the arguments of n-point function appear as continuous moduli of Kähler function. In non-perturbative phases the dependence of the maximum of Kähler function on the arguments of n-point function cannot be regarded as negligible and Kähler function becomes the key to the understanding of these effects including formation of bound states and color confinement.

e) In this picture light-like 3-surface would take the dual role as a correlate for both state and time evolution of state and this dual role allows to understand why the restriction of time

like entanglement to that described by S-matrix must be made. For fixed values of moduli each reaction would correspond to a minimal braid diagram involving exchanges of partons being in one-one correspondence with a maximum of Kähler function. By quantum criticality and the requirement of ideal quantum-classical correspondence only one such diagram would contribute for given values of moduli. Coupling constant evolution would not be however lost: it would be realized as p-adic coupling constant at the level of free states via the $log(p)$ scaling of eigen modes of the modified Dirac operator.

f) A completely unexpected prediction deserving a special emphasis is that number theoretic braids replicate in vertices. This means classical replication of the number theoretic information carried by them. This allows to interpret one of the TGD inspired models of genetic code in terms of number theoretic braids representing at deeper level the information carried by DNA. This picture provides also further support for the proposal that DNA acts as topological quantum computer utilizing braids associated with partonic light-like 3-surfaces (which can have arbitrary size). In the reverse direction one must conclude that even elementary particles could be information processing and communicating entities in TGD Universe.

To sum up, my personal feeling is that the constraints identified hitherto might lead to a more or less unique final result and I can only hope that some young analytically blessed brain would bother to transform this picture to concrete calculational recipes.

1 Introduction

The purpose of this chapter is to provide a view about construction of S-matrix with a special emphasis on the role of hyper-finite factors of type II_1 (HFF for shortly). I do not pretend of having handle about the huge technical complexities and can only recommend the works of von Neumann [20, 21, 22, 23], Tomita [48, 49, 50, 51], the work of Powers and Araki and Woods which served as starting point for the work of Connes [46, 47], the work of Jones [25, 64], and other leading figures in the field. What is may main contribution is fresh physical interpretation of this mathematics which also helps to make mathematical conjectures. The book of Connes [47] available in web provides an excellent overall view about von Neumann algebras and non-commutative geometry.

1.1 About the general conceptual framework behind quantum TGD

Let us first list the basic conceptual framework in which I try to concretize the ideas about S-matrix.

1.1.1 $N = 4$ super-conformal invariance and light-like 3-surfaces as fundamental dynamical objects

Super-conformal symmetries generalized from string model context to TGD framework are symmetries of S-matrix. This is very powerful constraint to S-matrix but useless unless one has precisely defined ontology translated to a rigorous mathematical framework. The zero energy ontology of TGD is now rather well understood but differs dramatically from that of standard quantum field theories. Second deep difference is that path integral formalism is given up and the goal is to construct S-matrix as a generalization of braiding S-matrices with reaction vertices replaced with the replication of number theoretic braids associated with partonic 2-surfaces taking the role of vertices.

The path leading to the understanding of super-conformal invariance in TGD framework was long but the final outcome is briefly described. The are two kinds of super-conformal symmetries.

1. The first super-conformal invariance is associated with light-cone boundary and is due to its metric 2-dimensionality putting 4-D Minkowski space in a unique position. The canonical transformations of $\delta H_{\pm} = \delta M^4_{\pm} \times CP_2$ are identified as isometries of the configuration

space. The super-generators of super-canonical algebra correspond to the gamma matrices of configuration space.

2. Light-like partonic 3-surfaces X^3 are the basic dynamical objects and light-likeness is respected by the 3-D variant of Kac-Moody algebra of conformal transformations of imbedding space made local with respect to X^3. Ordinary 1-D Kac-Moody algebra with complex coordinate z replaced with a light-like radial coordinate r takes a special role and super Kac-Moody symmetry is associated with this. The conformal symmetries associated with X^2 are counterpart of stringy conformal symmetries but have a role analogous to the conformal symmetries of critical statistical systems.

The light-likeness property allows Chern-Simons action for the induced Kähler gauge potential as the only possible action principle. The resulting almost topological conformal field theory has maximal $N = 4$ super-conformal symmetry with the inherent gauge group $SU(2) \times U(2)$ identified in terms of rotations and electro-weak symmetries acting on imbedding space spinors.

The constraints coming from p-adic mass calculations lead to the following overall view about the relationship between the two algebras. Mass squared is p-adic thermal expectation value of conformal weight meaning that four-momentum does not appear in the super-conformal generators: this option is excluded also by purely geometric considerations. p-Adic thermodynamics is justified by the fact that physical states are not annihilated by SKMV. Super-canonical Virasoro algebra (SCV) creates tachyonic ground states with vanishing conformal weight as null states annihilated by L_n, $n < 0$, and SC and SKM generate massless states to which p-adic thermodynamics in SKMV degrees of freedom applies. The commutators of SKM and SC algebras and their Virasoro counterparts annihilate the physical states.

1.1.2 Dirac determinant and zeta function associated with modified Dirac operator as coders of TGD physics

For long time the zeros of Riemann Zeta remained excellent candidates for the conformal weights labelling the generators of super-canonical algebra [B2, B3, A6]. The basic motivation was that the radial conformal weights have very naturally real part which equals to -1/2 as does also the negative of the real part of complex zeros of Riemann Zeta. Also other conformal weights are possible but not so natural.

1. Why Riemann Zeta does not work

The following observations have however changed the situation.

1. The almost defining property of zeta functions is that their complex zeros reside at the critical line. There exists a lot of zeta functions [E3] so that the spectrum of super-canonical conformal weights allows to consider also other zetas.

2. The zeta functions analogous to the basic building blocks of Riemann Zeta labelled by prime p are especially natural from the point of view of p-adic length scale hypothesis and they have automatically the nice algebraic properties required by the number theoretic universality whereas in the case of Riemann Zeta they must be conjectured.

3. The generalized eigenvalues of the modified Dirac operator define in a very natural manner zeta functions coding geometric information about partonic 2-surfaces whereas Riemann Zeta has no obvious interpretation of this kind.

These findings do not of course exclude Riemann zeta or zetas analogous to it. For instance, one can assign Riemann Zeta to the purely bosonic infinite primes very naturally. The spectrum of the scaling generator L_0 consists of non-negative integers and the positive part of spectrum defines a zeta function of form $\sum_{n>0} g(n)n^{-s}$, which might be relevant for quantum TGD. I do not known about the zeros of this zeta function.

A further natural speculation was that the zeros of polyzetas $\zeta(z_1,...,z_K)$ label the super-canonical conformal weights of K-particle bound states. The vanishing of loop corrections could be understood as being due to the fact that they are proportional to polyzetas having super-canonical conformal weights as arguments. This speculation was inspired by the fact that polyzetas with integer arguments emerge in loop corrections of quantum field theories.

2. Zeta functions assignable to the modified Dirac operator

In the case of the modified Dirac operator and super-canonical conformal weights Riemann Zeta is naturally replaced by a zeta function determined by purely physical considerations (detailed argument can be found in [A6, C1]).

1. The determinant of the modified Dirac operator D gives rise to the vacuum functional of TGD and the conjecture is that it reduces to a product of exponents of Kähler function and Chern-Simons action. The construction assigns to a given 3-D light-like surface X^3_l a 4-D space-time sheet conjectured to be a preferred extremal of Kähler action [A6].

2. The generalized eigenvalue λ of D is actually a scalar field depending on the coordinates of partonic 2-surface X^2 (and light-like 3-surface X^3_l). λ codes purely geometric information about the light-like 3-surface, and Higgs vacuum expectation is naturally proportional to λ.

3. The minima of the modulus of the holomorphic function λ in X^2 give rise to what I call number theoretic braids. Dirac determinant is product of the eigenvalues at the minima of $|\lambda|$ interpreted as a function X^3_l.

4. One can assign to the values of λ at the points of the number theoretic braid also zeta function, call it ζ. ζ codes geometric information about 3-surface and super-canonical conformal weights correspond naturally to its zeros. ζ is sum over a finite number of terms only, and if it is rational function of a suitable coordinate, it has all the required number theoretic properties whereas in the case of Riemann Zeta these properties require strong number theoretic conjectures.

The notion of polyzeta might generalize in a natural manner to a dynamical polyzeta. Suppose that one has a collection X^2_i of partonic 2-surfaces assignable to a connected space-like 3-surface defined by the intersection $X^3 = X^4 \cap \delta M^4_+ \times CP_2$. In this kind of situation one might hope that the notion of polyzeta generalizes and can be defined in terms of the generalized eigenvalues of the modified Dirac operator assigned with various partonic 2-surfaces X^2_i. If X^3 is connected, the polyzeta cannot be a mere product of independent zetas associated with X^2_i obtained by assigning separate space-time sheets to the light-like orbits of X^2_i. Even if it reduces to a product, the eigenvalues assignable to X^2_i are correlated by the constraint that the minimization of λ_i is consistent with the condition $X^2_i \subset X^3$. This polyzeta would naturally characterize the bound state character of the resulting state.

1.1.3 S-matrix as a functor

Almost topological QFT property of quantum allows to identify S-matrix as a functor from the category of generalized Feynman cobordisms to the category of operators mapping the Hilbert space of positive energy states to that for negative energy states: these Hilbert spaces are assignable to partonic 2-surfaces. Feynman cobordism is the generalized Feynman diagram having light-like 3-surfaces as lines glued together along their ends defining vertices as 2-surfaces. This picture differs dramatically from that of string models. There is a functional integral over the small deformations of Feynman cobordisms corresponding to maxima of Kähler function. Functor property generalizes the unitary condition and allows also thermal S-matrices which seem to be unavoidable since imbedding space degrees of freedom give rise to a factor of type I with $Tr(id) = \infty$.

1.1.4 S-matrix in zero energy ontology

Zero energy ontology allows to construct unitary S-matrix in fermionic degrees of freedom as unitary entanglement coefficients between positive and negative energy parts of zero energy state. The basic properties of hyper-finite factor II_1 are absolutely crucial. The inclusion of bosonic degrees of freedom lead to a replacement of HFF of type II_1 with HFF of type $II_\infty = II_1 \otimes I_\infty$. However, normalizability of the states allows only a projection of S-matrix to a finite-dimensional subspace of incoming or outgoing states. Hence the S-matrix is effectively restricted to $II_1 \otimes I_n = II_1$ factor so that at the level of physical states HFF of type II_1 results. This is absolutely crucial for the unitary of the S-matrix since it makes possible to have $Tr(SS^\dagger) = Tr(Id) = 1$. If factor of type I is present as a tensor factor, thermal S-matrix is the only possibility and later arguments in favor of the idea that thermodynamics is unavoidable part of quantum theory in zero energy ontology will be developed.

One can worry whether unitarity condition is consistent with the idea that fermionic degrees of freedom should allow to represent Boolean functions in terms of time-like entanglement. That unitary time evolution is able to represent this kind of functions in the case of quantum computers suggests that unitarity is not too strong a restriction. The basic question is whether only a "cognitive" representation of physical S-matrix in terms of time like entanglement or a genuine physical S-matrix is in question. It seems that the latter option is the only possible one so that physical systems would represent the laws of physics.

1.1.5 U-matrix

Besides S-matrix there is also U-matrix defining the unitary process associated with the quantum jump. S- *resp.* U-matrix characterizes quantum state *resp.* quantum jump so that they cannot be one and same thing.

1. There are good arguments supporting the view that U-matrix is almost trivial, and the real importance of U-matrix seems to be related to the to the description of intentional action identified as a transition between p-adic and real zero energy states and to the possibility to perceive states rather than only changes as quantum jumps leaving the state almost unchanged.

2. State function reduction corresponds to a projection sub-factor in TGD inspired quantum measurement theory whereas U process in some sense corresponds its reversal. Therefore U matrix might correspond to unitary isomorphism mapping factor to a larger factor containing it.

3. State function reduction must be consistent with the unitarity of S-matrix defining time-like entanglement. Since state function reduction means essentially multiplication by a projector to a sub-space it seems that state function reduction for both incoming and outgoing states are possible and would naturally correspond to projections to sub-factors of corresponding HFFs of type II_1.

1.1.6 Unitarity of S-matrix is not necessary in zero energy ontology

U-matrix is necessarily unitary. There are good reasons to believe that this condition combined with Lorentz invariance makes it almost trivial. In the case of S-matrix unitarity is not absolutely necessary.

The restriction of the time-like entanglement coefficients to a unitary S-matrix would conform with the idea that light-like partonic 2-surfaces represent a dynamical evolution at quantum level so that zero energy states must be orthogonal both with respect to positive and negative energy parts of the states. On the other hand, the light-like 3-surface can be chosen arbitrarily and its choice indeed affects S-matrix. Hence the theory cannot fully reduce to a 2-dimensional theory. The interpretation is that light-like 3-surfaces are in 1-1 correspondence with the ground states of super-conformal representations identifiable as light particles.

There are several arguments supporting the view that S-matrix need not be unitary. The simplest observation is that imbedding space degrees of freedom naturally give rise to a factor of type I so that only thermal S-matrix defines a normalizable zero energy state. S-matrix as functor from the category of Feynman cobordisms to the category operators defining entanglement coefficients implies that S-matrix in fermionic degrees of freedom for a product of cobordisms is product of the S-matrices for cobordisms. This implies that in fermionic degrees of freedom S-matrix is thermal S-matrix with time parameter replaced with complex time parameter whose imaginary part corresponds to inverse temperature. Also an argument based on the existence of universal thermal S-matrix with a complex time parameter for hyper-finite factors of type III_1 supports the view that unitarity is not necessary. A further argument is based on the finding that in dimensions $D < 4$ unitary S-matrix exists only if cobordism is trivial so that topology change would not be possible. This raises the fascinating possibility that thermodynamics - in particular p-adic thermodynamics - is an unavoidable and inherent property of quantum TGD.

1.1.7 Does Connes tensor product fix the allowed M-matrices?

Hyperfinite factors of type II_1 and the inclusion $\mathcal{N} \subset \mathcal{M}$ inclusions have been proposed to define quantum measurement theory with a finite measurement resolution characterized by $\mathcal{N}$ and with complex rays of state space replaced with $\mathcal{N}$ rays. What this really means is far from clear.

1. Naively one expects that matrices whose elements are elements of $\mathcal{N}$ give a representation for M. Now however unit operator has unit trace and one cannot visualize the situation in terms of matrices in case of $\mathcal{M}$ and $\mathcal{N}$.

2. The state space with $\mathcal{N}$ resolution would be formally $\mathcal{M}/\mathcal{N}$ consisting of $\mathcal{N}$ rays. For $\mathcal{M}/\mathcal{N}$ one has finite-D matrices with non-commuting elements of $\mathcal{N}$. In this case quantum matrix elements should be multiplets of selected elements of $\mathcal{N}$, **not all** possible elements of $\mathcal{N}$. One cannot therefore think in terms of the tensor product of $\mathcal{N}$ with $\mathcal{M}/\mathcal{N}$ regarded as a finite-D matrix algebra.

3. What does this mean? Obviously one must pose a condition implying that $\mathcal{N}$ action commutes with matrix action just like C: this poses conditions on the matrices that one can allow. Connes tensor product [43] does just this. Note I have proposed already earlier the reduction of interactions to Connes tensor product (see the section *"Could Connes tensor product...."* later in this chapter) but without reference to zero energy ontology as a fundamental manner to define measurement resolution with respect time and assuming unitarity.

The starting point is the Jones inclusion sequence

$$\mathcal{N} \subset \mathcal{M} \subset \mathcal{M} \otimes_N \mathcal{M}...$$

Here $\mathcal{M} \otimes_N \mathcal{M}$ is Connes tensor product which can be seen as elements of the ordinary tensor product commuting with $\mathcal{N}$ action so that $\mathcal{N}$ indeed acts like complex numbers in $\mathcal{M}$. $\mathcal{M}/\mathcal{N}$ is in this picture represented with $\mathcal{M}$ in which operators defined by Connes tensor products of elements of $\mathcal{M}$. The replacement $\mathcal{M} \rightarrow \mathcal{M}/\mathcal{N}$ corresponds to the replacement of the tensor product of elements of $\mathcal{M}$ defining matrices with Connes tensor product.

One can try to generalize this picture to zero energy ontology.

1. $\mathcal{M} \otimes_{\mathcal{N}} \mathcal{M}$ would be generalized by $\mathcal{M}_+ \otimes_{\mathcal{N}} \mathcal{M}_-$. Here $\mathcal{M}_+$ would create positive energy states and $\mathcal{M}_-$ negative energy states and $\mathcal{N}$ would create zero energy states in some shorter time scale resolution: this would be the precise meaning of finite measurement resolution.

2. Connes entanglement with respect to $\mathcal{N}$ would define a non-trivial and unique recipe for constructing M-matrices as a generalization of S-matrices expressible as products of square root of density matrix and unitary S-matrix but it is not how clear how many M-matrices this allows. In any case M-matrices would depend on the triplet $(\mathcal{N}, \mathcal{M}_+, \mathcal{M}_-)$ and this would correspond to p-adic length scale evolution giving replacing coupling constant evolution in TGD framework. Thermodynamics would enter the fundamental quantum theory via the square root of density matrix.

3. Zero energy ontology is a key element of this picture and the most compelling argument for zero energy ontology is the possibility of describing coherent states of Cooper pairs without giving up fermion number, charge, etc. conservation and automatic emerges of length scale dependent notion of quantum numbers (quantum numbers identified as those associated with positive energy factor).

To sum up, interactions would be an outcome of a finite measurement resolution and at the never-achievable limit of infinite measurement resolution the theory would be free: this would be the counterpart of asymptotic freedom.

1.1.8 Quantum classical correspondence

Quantum classical correspondence states that there is a correspondence between quantum fluctuating degrees of freedom associated with partonic 2-surfaces and classical dynamics. The weakest form of this principle is that the ground states of partonic super-conformal representations (massless states which generate light masses observed in laboratory) correspond to the interior dynamics of space-time sheets containing the partonic 2-surfaces. At the space-time level there would be 1-1 correspondence with the maxima of Kähler function giving rise to the analog of spin glass energy landscape.

One could protest by saying that excited states of super-conformal representations have no space-time correlate in this picture. Quantum states are replaced with states in which the projection of S-matrix to a finite-dimensional space in bosonic degrees of freedom appears as time-like entanglement coefficients so that quantum classical correspondence is obtained in strict sense after all. These states states are formally analogous which raises the question whether an actual relationship exists. For HFFs of type III unitary time evolution and thermal equilibrium are indeed closely related aspects of states [47]. $I_\infty \rightarrow I_n$ cutoff in the bosonic degrees of freedom would naturally have the discretization represented by number theoretic braids as a space-time correlate.

The effective elimination of the degrees of freedom associated with the space-time interior implied by the 1-1 correlation would allow to forget 4-D space-time degrees of freedom more or less completely as far as calculation of S-matrix is considered and everything would reduce to Fock space level as it does in quantum field theories. The functional integral around the maximum of Kähler function would select a set of preferred light-like partonic 3-surfaces. Quantum criticality suggests that the functional integral can be carried out exactly.

1.1.9 How TGD differs from string models

An important detail which deserves to be mentioned separately is one crucial deviation from string model picture: the stringy decays of partonic 2-surfaces or 3-surfaces are space-time correlates for the propagation of particle via several different routes rather than genuine particle decay. Note that partonic 2-surfaces can have arbitrarily large size and the outer boundary of any physical system represents the basic example of this kind of surface. Particle reactions correspond to branchings of light-like partonic 2-surfaces so that incoming and outgoing partons are glued together along their ends. This picture makes sense because quantum TGD reduces to almost topological conformal QFT at parton level (only light-likeness brings in the notion of metric).

Quantum classical correspondence allows to interpret light-like partonic 3-surface either as a time evolution of a highly non-deterministic 2-D system or as a 3-D system. This state-dynamics duality was discovered already in [E9], where it was realized that topological quantum computation has interpretation either as a program (state) or running of program (dynamics). Complete reduction to 2-D dynamics is not possible since the light-like 3-surfaces associated with maxima of Kähler action define spin glass energy landscape such that each maximum corresponds to its own S-matrix.

In this picture particle reactions correspond classically to branchings of partonic 2-surfaces generalizing the branchings for lines in Feynman diagrams. The stringy vertices for decays of surfaces correspond in TGD framework to the classical space-time correlate for a particle travelling along different paths and the particle creation and annihilation is a generalization of what occurs in Feynman diagrams with vertices replaced with 2-dimensional partonic surfaces along which light-like partonic 3-surfaces meet.

1.1.10 Physics as a generalized number theory vision

TGD as a generalized number theory vision gives powerful constraints. New view about space-time involves p-adic space-time sheets as space-time correlates for cognitive representations in fermionic case and for intentions in the bosonic case. This leads to the notion of number theoretic braid belonging to the algebraic intersection of real and p-adic partonic surfaces obeying same algebraic equations.

The implication is that the data characterizing S-matrix elements should come from discrete algebraic points of number theoretic braids. The Galois groups for braids occupying regions of partonic 2-surface emerge as a new element and relate closely to the representations of braid groups in HFFs of type II_1. Number theoretic universality leads to the condition that S-matrix elements are algebraic numbers in the extension of rational defined by the extension of p-adic numbers involved.

1.1.11 The role of hyper-finite factors of type II_1

The Clifford algebra of configuration space ("world of classical worlds") spinors is very naturally a hyper-finite factor of type II_1. During the last few years I have gradually learned something about the magnificent mathematical beauty of these objects.

1. TGD inspired quantum measurement theory with measurement resolution characterized in terms of Jones inclusion and based on HFFs of type II_1 brings in non-commutative quantum physics and leads to powerful general predictions [C7, C2, H2]. The basic idea is that complex rays of the state space are replaced with $\mathcal{N}$ rays for Jones inclusion $\mathcal{N} \subset \mathcal{M}$. $\mathcal{N}$ defines the measurement resolution in the sense that the group G leaving elements of $\mathcal{N}$ invariant characterizes the measured quantum numbers.

2. Hyper-finite factors have the property that they are isomorphic with their tensor powers. This makes possible the construction of vertices as unitary isomorphisms between tensor products of HFFs of type II_1 associated with incoming and outgoing states. The core part of S-matrix boils down to a unitary isomorphism between tensor products of hyper-finite

factors of type II_1 associated with incoming *resp.* outgoing partonic 3-surfaces whose ends meet at the partonic 2-surface representing reaction vertex.

3. The study of Jones inclusions leads to the idea that Planck constant is dynamical and quantized. The predicted hierarchy of Planck constants involving a generalization of imbedding space concept and an explanation of dark matter as macroscopic quantum phases [A9]. Here the special mathematical role of Jones inclusions with index $r \leq 4$ is crucial.

4. The properties of HFFs inspire also the idea that TGD based physics should able to mimic any imaginable quantum physical system defined by gauge theory or conformal field theory involving Kac-Moody symmetry. Thus the ultimate physics would be kind of analog for Turing machine. The prediction inspired by TGD based explanation of McKay correspondence [27] is that TGD Universe is indeed able to simulate gauge and Kac-Moody dynamics of a very large subset of ADE type groups. In fact, also much more general prediction that simulation should be possible for any compact Lie group emerges.

5. HFFs of type II_1 lead also to deep connections with number theory [27] and number theoretical braids can be interpreted in terms of representations of Galois groups assignable with partonic 2-surfaces in terms of HFFs of type II_1. Particle decay represents a replication of number theoretical braids and this together with p-adic fractality and hierarchy of Planck constants suggests strongly direct connections with genetic code and DNA.

1.1.12 Could TGD emerge from a local version of infinite-dimensional Clifford algebra?

A crucial step in the progress was the realization that TGD emerges from the mere idea that a local version of hyper-finite factor of type II_1 represented as an infinite-dimensional Clifford algebra must exist (as analog of say local gauge groups). This implies a connection with the classical number fields. Quantum version of complexified octonions defining the coordinate with respect to which one localizes is unique by its non-associativity allowing to uniquely separate the powers of octonionic coordinate from the associative infinite-dimensional Clifford algebra elements appearing as Taylor coefficients in the expansion of Clifford algebra valued field.

Associativity condition implies the classical and quantum dynamics of TGD. Space-time surfaces are hyper-quaternionic of co-hyper-quatenionic sub-manifolds of hyper-octonionic imbedding space HO. Also the interpretation as a four-surface in $H = M^4 \times CP_2$ emerges and implies $HO - H$ duality. What is also nice that Minkowski spaces correspond to the spectra for the eigenvalues of maximal set of commuting quantum coordinates of suitably defined quantum spaces. Thus Minkowski signature has quantal explanation.

1.2 Summary about the construction of S-matrix

It is perhaps wise to summarize briefly the vision about S-matrix.

1. S-matrix defines entanglement between positive and negative energy parts of zero energy states. This kind of S-matrix need not be unitary unlike the U-matrix associated with unitary process forming part of quantum jump. There are several good arguments suggesting that that S-matrix cannot be unitary but can be regarded as thermal S-matrix so that thermodynamics would become an essential part of quantum theory. In TGD framework path integral formalism is given up although functional integral over the 3-surfaces is present.

2. Almost topological QFT property of quantum allows to identify S-matrix as a functor from the category of generalized Feynman cobordisms to the category of operators mapping the

Hilbert space of positive energy states to that for negative energy states: these Hilbert spaces are assignable to partonic 2-surfaces. It is difficult to overestimate the importance of this result bringing category theory absolutely essential part of quantum TGD. One can assign to S-matrix a complex parameter whose real part has interpretation as interaction time and imaginary part as the inverse temperature. S-matrices and thus also quantum states in zero energy ontology possess a semigroup like structure and in the product time and inverse temperature are additive. This suggests that the cosmological evolution of temperature as $T \propto 1/t$ could be understood at the level of fundamental quantum theory. The most general identification of the time like entanglement coefficients would be as a "square root" of density matrix thus satisfying the condition $\rho^+ = SS^\dagger$, $\rho^- = SS^\dagger$, $Tr(\rho^{pm}) = 1$. $\rho^\pm$ has interpretation as density matrix for positive/negative energy states. Physical intuition suggest that S can be written as a product of universal unitary matrix and square root of state dependent density matrix.

3. S-matrix should be constructible as a generalization of braiding S-matrix in such a manner that the number theoretic braids assignable to light-like partonic 3-surfaces glued along their ends at 2-dimensional partonic 2-surfaces representing reaction vertices replicate in the vertex [C6].

4. The construction of braiding S-matrices assignable to the incoming and outgoing partonic 2-surfaces is not a problem [C6]. The problem is to express mathematically what happens in the vertex. Here the observation that the tensor product of HFFs of type II is HFF of type II is the key observation. Many-parton vertex can be identified as a unitary isomorphism between the tensor product of incoming *resp.* outgoing HFFs. A reduction to HFF of type II_1 occurs because only a finite-dimensional projection of S-matrix in bosonic degrees of freedom defines a normalizable state. In the case of factor of type II_∞ only thermal S-matrix is possible without finite-dimensional projection and thermodynamics would thus emerge as an essential part of quantum theory.

5. HFFs of type III could also appear at the level of field operators used to create states but at the level of quantum states everything reduces to HFFs of type II_1 and their tensor products giving these factors back. If braiding automorphisms reduce to the purely intrinsic unitary automorphisms of HFFs of type III then for certain values of the time parameter of automorphism having interpretation as a scaling parameter these automorphisms are trivial. These time scales could correspond to p-adic time scales so that p-adic length scale hypothesis would emerge at the fundamental level. In this kind of situation the braiding S-matrices associated with the incoming and outgoing partons could be trivial so that everything would reduce to this unitary isomorphism: a counterpart for the elimination of external legs from Feynman diagram in QFT.

6. One might hope that all complications related to what happens for *space-like* 3-surfaces could be eliminated by quantum classical correspondence stating that space-time view about particle reaction is only a space-time correlate for what happens in quantum fluctuating degrees of freedom associated with partonic 2-surfaces. This turns out to be the case only in non-perturbative phase. The reason is that the arguments of n-point function appear as continuous moduli of Kähler function. In non-perturbative phases the dependence of the maximum of Kähler function on the arguments of n-point function cannot be regarded as negligible and Kähler function becomes the key to the understanding of these effects including formation of bound states and color confinement.

7. In this picture light-like 3-surface would take the dual role as a correlate for both state and time evolution of state and this dual role allows to understand why the restriction of time

like entanglement to that described by S-matrix must be made. For fixed values of moduli each reaction would correspond to a minimal braid diagram involving exchanges of partons being in one-one correspondence with a maximum of Kähler function. By quantum criticality and the requirement of ideal quantum-classical correspondence only one such diagram would contribute for given values of moduli.

i) A completely unexpected prediction deserving a special emphasis is that number theoretic braids replicate in vertices. This is of course the braid counterpart for the introduction of annihilation and creation of particles in the transition from free QFT to an interacting one. This means classical replication of the number theoretic information carried by them. This allows to interpret one of the TGD inspired models of genetic code [L4] in terms of number theoretic braids representing at deeper level the information carried by DNA. This picture provides also further support for the proposal that DNA acts as topological quantum computer utilizing braids associated with partonic light-like 3-surfaces (which can have arbitrary size) [E9]. In the reverse direction one must conclude that even elementary particles could be information processing and communicating entities in TGD Universe.

1.3 Topics of the chapter

In this chapter the role of HFFs of type II_1 and possibly also that of type III is discussed. What makes the latter factors attractive is that they possess a unique one parameter group of outer automorphisms defining a natural candidate for a unitary S-matrix if inner automorphisms act as gauge transformations. Only thermal S-matrix defines a normalizable state in zero energy ontology with complex value of time parameter giving rise to thermalization.

Also number theoretical ideas are considered. The notion of number theoretic braid is central and the vision that quantum physics in TGD Universe provides physical representations of Galois groups for the algebraic extensions of rationals is discussed. The reader wishing for a brief summary of TGD might find the e three articles about TGD, TGD inspired theory of consciousness, and TGD based view about quantum biology helpful [16, 17, 18].

2 Basic facts about hyper-finite factors

2.1 Von Neumann algebras

In this section basic facts about von Neumann algebras are summarized using as a background material the concise summary given in the lecture notes of Longo [24].

2.1.1 Basic definitions

A formal definition of von Neumann algebra [21, 22, 23] is as a *-subalgebra of the set of bounded operators $\mathcal{B}(\mathcal{H})$ on a Hilbert space $\mathcal{H}$ closed under weak operator topology, stable under the conjugation $J = ^*$: $x \rightarrow x^*$, and containing identity operator Id. This definition allows also von Neumann algebras for which the trace of the unit operator is not finite.

Identity operator is the only operator commuting with a simple von Neumann algebra. A general von Neumann algebra allows a decomposition as a direct integral of simple algebras, which von Neumann called factors. Classification of von Neumann algebras reduces to that for factors.

$\mathcal{B}(\mathcal{H})$ has involution * and is thus a *-algebra. $\mathcal{B}(\mathcal{H})$ has order order structure $A \geq 0 : (Ax, x) \geq 0$. This is equivalent to $A = BB^*$ so that order structure is determined by algebraic structure. $\mathcal{B}(\mathcal{H})$ has metric structure in the sense that norm defined as supremum of $||Ax||$, $||x|| \leq 1$ defines the notion of continuity. $||A||^2 = inf\{\lambda > 0 : AA^* \leq \lambda I\}$ so that algebraic structure determines metric structure.

There are also other topologies for $\mathcal{B}(\mathcal{H})$ besides norm topology.

1. $A_i \rightarrow A$ strongly if $||Ax - A_ix|| \rightarrow 0$ for all x. This topology defines the topology of C^* algebra. $\mathcal{B}(\mathcal{H})$ is a Banach algebra that is $||AB|| \leq ||A|| \times ||B||$ (inner product is not necessary) and also C^* algebra that is $||AA^*|| = ||A||^2$.

2. $A_i \rightarrow A$ weakly if $(A_ix, y) \rightarrow (Ax, y)$ for all pairs (x, y) (inner product is necessary). This topology defines the topology of von Neumann algebra as a sub-algebra of $\mathcal{B}(\mathcal{H})$.

Denote by M' the commutant of $\mathcal{M}$ which is also algebra. Von Neumann's bicommutant theorem says that $\mathcal{M}$ equals to its own bi-commutant. Depending on whether the identity operator has a finite trace or not, one distinguishes between algebras of type II_1 and type II_∞. II_1 factor allow trace with properties $tr(Id) = 1$, $tr(xy) = tr(yx)$, and $tr(x^*x) > 0$, for all $x \neq 0$. Let $L^2(\mathcal{M})$ be the Hilbert space obtained by completing $\mathcal{M}$ respect to the inner product defined $\langle x|y\rangle = tr(x^*y)$ defines inner product in $\mathcal{M}$ interpreted as Hilbert space. The normalized trace induces a trace in M', natural trace $Tr_{M'}$, which is however not necessarily normalized. JxJ defines an element of M': if $\mathcal{H} = L^2(\mathcal{M})$, the natural trace is given by $Tr_{M'}(JxJ) = tr_M(x)$ for all $x \in M$ and bounded.

2.1.2 Basic classification of von Neumann algebras

Consider first some definitions. First of all, Hermitian operators with positive trace expressible as products xx^* are of special interest since their sums with positive coefficients are also positive.

In quantum mechanics Hermitian operators can be expressed in terms of projectors to the eigen states. There is a natural partial order in the set of isomorphism classes of projectors by inclusion: $E < F$ if the image of $\mathcal{H}$ by E is contained to the image of $\mathcal{H}$ by a suitable isomorph of F. Projectors are said to be metrically equivalent if there exist a partial isometry which maps the images $\mathcal{H}$ by them to each other. In the finite-dimensional case metric equivalence means that isomorphism classes are identical $E = F$.

The algebras possessing a minimal projection E_0 satisfying $E_0 \leq F$ for any F are called type I algebras. Bounded operators of n-dimensional Hilbert space define algebras I_n whereas the bounded operators of infinite-dimensional separable Hilbert space define the algebra I_∞. I_n and I_∞ correspond to the operator algebras of quantum mechanics. The states of harmonic oscillator correspond to a factor of type I.

The projection F is said to be finite if $F < E$ and $F \equiv E$ implies $F = E$. Hence metric equivalence means identity. Simple von Neumann algebras possessing finite projections but no minimal projections so that any projection E can be further decomposed as $E = F + G$, are called factors of type II.

Hyper-finiteness means that any finite set of elements can be approximated arbitrary well with the elements of a finite-dimensional sub-algebra. The hyper-finite II_∞ algebra can be regarded as a tensor product of hyper-finite II_1 and I_∞ algebras. Hyper-finite II_1 algebra can be regarded as a Clifford algebra of an infinite-dimensional separable Hilbert space sub-algebra of I_∞.

Hyper-finite II_1 algebra can be constructed using Clifford algebras $C(2n)$ of $2n$-dimensional spaces and identifying the element x of $2^n \times 2^n$ dimensional $C(n)$ as the element $diag(x, x)/2$ of $2^{n+1} \times 2^{n+1}$-dimensional $C(n+1)$. The union of algebras $C(n)$ is formed and completed in the weak operator topology to give a hyper-finite II_1 factor. This algebra defines the Clifford algebra of infinite-dimensional separable Hilbert space and is thus a sub-algebra of I_∞ so that hyper-finite II_1 algebra is more regular than I_∞.

von Neumann algebras possessing no finite projections (all traces are infinite or zero) are called algebras of type III. It was later shown by Connes [46] that these algebras are labelled by a parameter varying in the range $[0, 1]$, and referred to as algebras of type III_x. III_1 category contains a unique hyper-finite algebra. It has been found that the algebras of observables associated with bounded regions of 4-dimensional Minkowski space in quantum field theories correspond to hyper-finite factors of type III_1 [24]. Also statistical systems at finite temperature correspond to factors of type III and temperature parameterizes one-parameter set of automorphisms of this algebra [50]. Zero temperature limit correspond to I_∞ factor and infinite temperature limit to II_1 factor.

2.1.3 Non-commutative measure theory and non-commutative topologies and geometries

von Neumann algebras and C^* algebras give rise to non-commutative generalizations of ordinary measure theory (integration), topology, and geometry. It must be emphasized that these structures are completely natural aspects of quantum theory. In particular, for the hyper-finite type II_1 factors quantum groups and Kac Moody algebras [70] emerge quite naturally without any need for ad hoc modifications such as making space-time coordinates non-commutative. The effective 2-dimensionality of quantum TGD (partonic or stringy 2-surfaces code for states) means that these structures appear completely naturally in TGD framework.

2.1.4 Non-commutative measure theory

Von Neumann algebras define what might be a non-commutative generalization of measure theory and probability theory [24].

1. Consider first the commutative case. Measure theory is something more general than topology since the existence of measure (integral) does not necessitate topology. Any measurable function f in the space $L^\infty(X,\mu)$ in measure space (X,μ) defines a bounded operator M_f in the space $\mathcal{B}(L^2(X,\mu))$ of bounded operators in the space $L^2(X,\mu)$ of square integrable functions with action of M_f defined as $M_f g = fg$.

2. Integral over $\mathcal{M}$ is very much like trace of an operator $f_{x,y} = f(x)\delta(x,y)$. Thus trace is a natural non-commutative generalization of integral (measure) to the non-commutative case and defined for von Neumann algebras. In particular, generalization of probability measure results if the case $tr(Id) = 1$ and algebras of type I_n and II_1 are thus very natural from the point of view of non-commutative probability theory.

The trace can be expressed in terms of a cyclic vector Ω or vacuum/ground state in physicist's terminology. Ω is said to be cyclic if the completion $\overline{M\Omega} = H$ and separating if $x\Omega$ vanishes only for $x = 0$. Ω is cyclic for $\mathcal{M}$ if and only if it is separating for M'. The expression for the trace given by

$$Tr(ab) = \left(\frac{(ab+ba)}{2},\Omega\right) \tag{1}$$

is symmetric and allows to defined also inner product as $(a,b) = Tr(a^*b)$ in $\mathcal{M}$. If Ω has unit norm $(\Omega,\Omega) = 1$, unit operator has unit norm and the algebra is of type II_1. Fermionic oscillator operator algebra with discrete index labelling the oscillators defines II_1 factor. Group algebra is second example of II_1 factor.

The notion of probability measure can be abstracted using the notion of state. State ω on a C^* algebra with unit is a positive linear functional on $\mathcal{U}$, $\omega(1) = 1$. By so called KMS construction [24] any state ω in C^* algebra $\mathcal{U}$ can be expressed as $\omega(x) = (\pi(x)\Omega,\Omega)$ for some cyclic vector Ω and π is a homomorphism $\mathcal{U} \to \mathcal{B}(\mathcal{H})$.

2.1.5 Non-commutative topology and geometry

C^* algebras generalize in a well-defined sense ordinary topology to non-commutative topology.

1. In the Abelian case Gelfand Naimark theorem [24] states that there exists a contravariant functor F from the category of unital abelian C^* algebras and category of compact topological spaces. The inverse of this functor assigns to space X the continuous functions f on X with norm defined by the maximum of f. The functor assigns to these functions having interpretation as eigen states of mutually commuting observables defined by the function algebra. These eigen states are delta functions localized at single point of X. The points of X label the eigenfunctions and thus define the spectrum and obviously span X. The connection with topology comes from the fact that continuous map $Y \rightarrow X$ corresponds to homomorphism $C(X) \rightarrow C(Y)$.

2. In non-commutative topology the function algebra $C(X)$ is replaced with a general C^* algebra. Spectrum is identified as labels of simultaneous eigen states of the Cartan algebra of C^* and defines what can be observed about non-commutative space X.

3. Non-commutative geometry can be very roughly said to correspond to *-subalgebras of C^* algebras plus additional structure such as symmetries. The non-commutative geometry of Connes [47] is a basic example here.

2.2 Basic facts about hyper-finite factors of type III

Hyper-finite factors of type II_1, II_∞ and III_1, III_0, III_λ, $\lambda \in (0,1)$, allow by definition hierarchy of finite approximations and are unique as von Neumann algebras. Also hyper-finite factors of type II_∞ and type III could be relevant for the formulation of TGD. HFFs of type II_∞ and III could appear at the level operator algebra but that at the level of quantum states one would obtain HFFs of type II_1. These extended factors inspire highly non-trivial conjectures about quantum TGD. The book of Connes [47] provides a detailed view about von Neumann algebras in general.

2.2.1 Basic definitions and facts

A highly non-trivial result is that HFFs of type II_∞ are expressible as tensor products $II_\infty = II_1 \otimes I_\infty$, where II_1 is hyper-finite [47].

1. *The existence of one-parameter family of outer automorphisms*

The unique feature of factors of type III is the existence of one-parameter unitary group of outer automorphisms. The automorphism group originates in the following manner.

1. Introduce the notion of linear functional in the algebra as a map $\omega : \mathcal{M} \rightarrow C$. ω is said to be hermitian it respects conjugation in $\mathcal{M}$; positive if it is consistent with the notion of positivity for elements of $\mathcal{M}$ in which case it is called weight; state if it is positive and normalized meaning that $\omega(1) = 1$, faithful if $\omega(A) > 0$ for all positive A; a trace if $\omega(AB) = \omega(BA)$, a vector state if $\omega(A)$ is "vacuum expectation" $\omega_\Omega(A) = (\Omega, \omega(A)\Omega)$ for a non-degenerate representation $(\mathcal{H}, \pi)$ of $\mathcal{M}$ and some vector $\Omega \in \mathcal{H}$ with $||\Omega|| = 1$.

2. The existence of trace is essential for hyper-finite factors of type II_1. Trace does not exist for factors of type III and is replaced with the weaker notion of state. State defines inner product via the formula $(x, y) = \phi(y^* x)$ and $*$ is isometry of the inner product. $*$-operator has property known as pre-closedness implying polar decomposition $S = J\Delta^{1/2}$ of its closure. Δ is positive definite unbounded operator and J is isometry which restores the symmetry between $\mathcal{M}$ and its commutant $\mathcal{M}'$ in the Hilbert space $\mathcal{H}_\phi$, where $\mathcal{M}$ acts via left multiplication: $\mathcal{M}' = J\mathcal{M}J$.

3. The basic result of Tomita-Takesaki theory is that Δ defines a one-parameter group $\sigma_\phi^t(x) = \Delta^{it}x\Delta^{-it}$ of automorphisms of $\mathcal{M}$ since one has $\Delta^{it}\mathcal{M}\Delta^{-it} = \mathcal{M}$. This unitary evolution is an automorphism fixed apart from unitary automorphism $A \rightarrow UAU^*$ related with the choice of ϕ. For factors of type I and II this automorphism reduces to inner automorphism so that the group of outer automorphisms is trivial as is also the outer automorphism associated with ω. For factors of type III the group of these automorphisms divided by inner automorphisms gives a one-parameter group of $Out(\mathcal{M})$ of outer automorphisms, which does not depend at all on the choice of the state ϕ.

More precisely, let ω be a normal faithful state: $\omega(x^*x) > 0$ for any x. One can map $\mathcal{M}$ to $L^2(\mathcal{M})$ defined as a completion of $\mathcal{M}$ by $x \rightarrow x\Omega$. The conjugation * in $\mathcal{M}$ has image at Hilbert space level as a map $S_0 : x\Omega \rightarrow x^*\Omega$. The closure of S_0 is an anti-linear operator and has polar decomposition $S = J\Delta^{1/2}$, $\Delta = SS^*$. Δ is positive self-adjoint operator and J anti-unitary involution. The following conditions are satisfied

$$\begin{aligned} \Delta^{it}\mathcal{M}\Delta^{-it} &= \mathcal{M} \ , \\ J\mathcal{M}J &= \mathcal{M}' \ . \end{aligned} \tag{2}$$

Δ^{it} is obviously analogous to the time evolution induced by positive definite Hamiltonian and induces also the evolution of the expectation ω as $\pi \rightarrow \Delta^{it}\pi\Delta^{-it}$. What makes this result thought provoking is that it might mean a universal quantum dynamics apart from inner automorphisms and thus a realization of general coordinate invariance and gauge invariance at the level of Hilbert space.

2. Classification of HFFs of type III

Connes achieved an almost complete classification of hyper-finite factors of type III completed later by others. He demonstrated that they are labelled by single parameter $0 \leq \lambda \leq 1]$ and that factors of type III_λ, $0 \leq \lambda < 1$ are unique. Haagerup showed the uniqueness for $\lambda = 1$. The idea was that the the group has an invariant, the kernel $T(M)$ of the map from time like R to $Out(M)$, consisting of those values of the parameter t for which σ_ϕ^t reduces to an inner automorphism and to unity as outer automorphism. Connes also discovered also an invariant, which he called spectrum $S(\mathcal{M})$ of $\mathcal{M}$ identified as the intersection of spectra of $\Delta_\phi\backslash\{0\}$, which is closed multiplicative subgroup of R^+.

Connes showed that there are three cases according to whether $S(\mathcal{M})$ is

1. R^+, type III_1
2. $\{\lambda^n, n \in Z\}$, type III_λ.
3. $\{1\}$, type III_0.

   The value range of λ is this by convention. For the reversal of the automorphism it would be that associated with $1/\lambda$.

Connes constructed also an explicit representation of the factors $0 < \lambda < 1$ as crossed product II_∞ factor $\mathcal{N}$ and group Z represented as powers of automorphism of II_∞ factor inducing the scaling of trace by λ. The classification of HFFs of type III reduced thus to the classification of automorphisms of $\mathcal{N} \otimes \mathcal{B}(\mathcal{H}$. In this sense the theory of HFFs of type III was reduced to that for HFFs of type II_∞ or even II_1. The representation of Connes might be also physically interesting.

2.2.2 Probabilistic view about factors of type III

Second very concise representation of HFFs relies on thermodynamical thinking and realizes factors as infinite tensor product of finite-dimensional matrix algebras acting on state spaces of finite state systems with a varying and finite dimension n such that one assigns to each factor a density matrix characterized by its eigen values. Intuitively one can think the finite matrix factors as associated with n-state system characterized by its energies with density matrix ρ defining a thermodynamics. The logarithm of the ρ defines the single particle quantum Hamiltonian as $H = log(\rho)$ and $\Delta = \rho = exp(H)$ defines the automorphism σ_ϕ for each finite tensor factor as $exp(iHt)$. Obviously free field representation is in question.

Depending on the asymptotic behavior of the eigenvalue spectrum one obtains different factors [47].

1. Factor of type I corresponds to ordinary thermodynamics for which the density matrix as a function of matrix factor approaches sufficiently fast that for a system for which only ground state has non-vanishing Boltzmann weight.

2. Factor of type II_1 results if the density matrix approaches to identity matrix sufficiently fast. This means that the states are completely degenerate which for ordinary thermodynamics results only at the limit of infinite temperature. Spin glass could be a counterpart for this kind of situation.

3. Factor of type III results if one of the eigenvalues is above some lower bound for all tensor factors in such a manner that neither factor of type I or II_1 results but thermodynamics for systems having infinite number of degrees of freedom could yield this kind of situation.

This construction demonstrates how varied representations factors can have, a fact which might look frustrating for a novice in the field. In particular, the infinite tensor power of $M(2,C)$ with state defined as an infinite tensor power of $M(2,C)$ state assigning to the matrix A the complex number $(\lambda^{1/2}A_{11} + \lambda^{-1/2}\ \phi(A) = A_{22})/(\lambda^{1/2} + \lambda^{-1/2})$ defines HFF III_λ [66, 47]. Formally the same algebra which for $\lambda = 1$ gives ordinary trace and HFF of type II_1, gives III factor only by replacing trace with state. This simple model was discovered by Powers in 1967.

It is indeed the notion of state or thermodynamics is what distinguishes between factors. This looks somewhat weird unless one realizes that the Hilbert space inner product is defined by the "thermodynamical" state ϕ and thus probability distribution for operators and for their thermal expectation values. Inner product in turn defines the notion of norm and thus of continuity and it is this notion which differs dramatically for $\lambda = 1$ and $\lambda < 1$ so that the completions of the algebra differ dramatically.

In particular, there is no sign about I_∞ tensor factor or crossed product with Z represented as automorphisms inducing the scaling of trace by λ. By taking tensor product of I_∞ factor represented as tensor power with induces running from $-\infty$ to 0 and II_1 HFF with indices running from 1 to ∞ one can make explicit the representation of the automorphism of II_∞ factor inducing scaling of trace by λ and transforming matrix factors possessing trace given by square root of index $\mathcal{M} : \mathcal{N}$ to those with trace 2.

2.3 Joint modular structure and sectors

Let $\mathcal{N} \subset \mathcal{M}$ be an inclusion. The unitary operator $\gamma = J_N J_M$ defines a canonical endomorphisms $M \rightarrow N$ in the sense that it depends only up to inner automorphism on $\mathcal{N}$, γ defines a sector of $\mathcal{M}$. The sectors of $\mathcal{M}$ are defined as $Sect(\mathcal{M}) = End(\mathcal{M})/Inn(\mathcal{M})$ and form a semi-ring with respected to direct sum and composition by the usual operator product. It allows also conjugation.

$L^2(\mathcal{M})$ is a normal bi-module in the sense that it allows commuting left and right multiplications. For $a, b \in M$ and $x \in L^2(\mathcal{M})$ these multiplications are defined as $axb = aJb^*Jx$ and it is easy to verify the commutativity using the factor $Jy^*J \in \mathcal{M}'$. Connes [47] has shown that all normal bi-modules arise in this way up to unitary equivalence so that representation concepts make sense. It is possible to assign to any endomorphism ρ index $Ind(\rho) \equiv M : \rho(\mathcal{M})$. This means that the sectors are in 1-1 correspondence with inclusions. For instance, in the case of hyper-finite II_1 they are labelled by Jones index. Furthermore, the objects with non-integral dimension $\sqrt{[\mathcal{M} : \rho(\mathcal{M})]}$ can be identified as quantum groups, loop groups, infinite-dimensional Lie algebras, etc...

2.4 About inclusions of hyper-finite factors of type II_1

Many names have been assigned to inclusions: Jones, Wenzl, Ocneacnu, Pimsner-Popa, Wasserman [69]. It would seem to me that the notion Jones inclusion includes them all so that various names would correspond to different concrete realizations of the inclusions conjugate under outer automorphisms.

1. According to [69] for inclusions with $\mathcal{M} : \mathcal{N} \leq 4$ (with $A_1^{(1)}$ excluded) there exists a countable infinity of sub-factors with are pairwise non inner conjugate but conjugate to $\mathcal{N}$.
2. Also for any finite group G and its outer action there exists uncountably many sub-factors which are pairwise non inner conjugate but conjugate to the fixed point algebra of G [69]. For any amenable group G the the inclusion is also unique apart from outer automorphism [60].

Thus it seems that not only Jones inclusions but also more general inclusions are unique apart from outer automorphism.

Any *-endomorphism σ, which is unit preserving, faithful, and weakly continuous, defines a sub-factor of type II_1 factor [69]. The construction of Jones leads to a atandard inclusion sequence $\mathcal{N} \subset \mathcal{M} \subset \mathcal{M}^1 \subset ....$ This sequence means addition of projectors e_i, $i < 0$, having visualization as an addition of braid strand in braid picture. This hierarchy exists for all factors of type II. At the limit $\mathcal{M}^\infty = \cup_i \mathcal{M}^i$ the braid sequence extends from $-\infty$ to ∞. Inclusion hierarchy can be understood as a hierarchy of Connes tensor powers $\mathcal{M} \otimes_\mathcal{N} \mathcal{M} .... \otimes_\mathcal{N} \mathcal{M}$. Also the ordinary tensor powers of hyper-finite factors of type II_1 (HFF) as well as their tensor products with finite-dimensional matrix algebras are isomorphic to the original HFF so that these objects share the magic of fractals.

Under certain assumptions the hierarchy can be continued also in opposite direction. For a finite index an infinite inclusion hierarchy of factors results with the same value of index. σ is said to be basic if it can be extended to *-endomorphisms from $\mathcal{M}^1$ to $\mathcal{M}$. This means that the hierarchy of inclusions can be continued in the opposite direction: this means elimination of strands in the braid picture. For finite factors (as opposed to hyper-finite ones) there are no basic *-endomorphisms of $\mathcal{M}$ having fixed point algebra of non-abelian G as a sub-factor [69].

1. Jones inclusions

For hyper-finite factors of type II_1 Jones inclusions allow basic *-endomorphism. They exist for all values of $\mathcal{M} : \mathcal{N} = r$ with $r \in \{4cos^2(\pi/n)|n \geq 3\} \cap [4, \infty)$ [69]. They are defined for an algebra defined by projectors e_i, $i \geq 1$. All but nearest neighbor projectors commute. $\lambda = 1/r$ appears in the relations for the generators of the algebra given by $e_i e_j e_i = \lambda e_i$, $|i-j| = 1$. $\mathcal{N} \subset \mathcal{M}$ is identified as the double commutator of algebra generated by e_i, $i \geq 2$.

This means that principal graph and its dual are equivalent and the braid defined by projectors can be continued not only to $-\infty$ but that also the dropping of arbitrary number of strands is possible [69]. It would seem that ADE property of the principal graph meaning single root length codes for the duality in the case of $r \leq 4$ inclusions.

Irreducibility holds true for $r < 4$ in the sense that the intersection of $Q' \cap P = P' \cap P = C$. For $r \geq 4$ one has $dim(Q' \cap P) = 2$. The operators commuting with Q contain besides identify operator of Q also the identify operator of P. Q would contain a single finite-dimensional matrix factor less than P in this case. Basic *-endomorphisms with $\sigma(P) = Q$ is $\sigma(e_i) = e_{i+1}$. The difference between genuine symmetries of quantum TGD and symmetries which can be mimicked by TGD could relate to the irreducibility for $r < 4$ and raise these inclusions in a unique position. This difference could partially justify the hypothesis [A9] that only the groups $G_a \times G_b \subset SU(2) \times SU(2) \subset SL(2,C) \times SU(3)$ define orbifold coverings of $H_\pm = M^4_\pm \times CP_2 \rightarrow H_\pm/G_a \times G_b$.

2. Wasserman's inclusion

Wasserman's construction of $r = 4$ factors clarifies the role of the subgroup of $G \subset SU(2)$ for these inclusions. Also now $r = 4$ inclusion is characterized by a discrete subgroup $G \subset SU(2)$ and is given by $(1 \otimes \mathcal{M})^G \subset (M_2(C) \times \mathcal{M})^G$. According to [69] Jones inclusions are irreducible also for $r = 4$. The definition of Wasserman inclusion for $r = 4$ seems however to imply that the identity matrices of both $\mathcal{M}^G$ and $(M(2,C) \otimes \mathcal{M})^G$ commute with $\mathcal{M}^G$ so that the inclusion should be reducible for $r = 4$.

Note that G leaves both the elements of $\mathcal{N}$ and $\mathcal{M}$ invariant whereas $SU(2)$ leaves the elements of $\mathcal{N}$ invariant. $M(2,C)$ is effectively replaced with the orbifold $M(2,C)/G$, with G acting as automoprhisms. The space of these orbits has complex dimension $d = 4$ for finite G.

For $r < 4$ inclusion is defined as $M^G \subset M$. The representation of G as outer automorphism must change step by step in the inclusion sequence ... $\subset \mathcal{N} \subset \mathcal{M} \subset$ ... since otherwise G would act trivially as one proceeds in the inclusion sequence. This is true since each step brings in additional finite-dimensional tensor factor in which G acts as automorphisms so that although $\mathcal{M}$ can be invariant under $G_{\mathcal{M}}$ it is not invariant under $G_{\mathcal{N}}$.

These two inclusions might accompany each other in TGD based physics. One could consider $r < 4$ inclusion $\mathcal{N} = \mathcal{M}^G \subset \mathcal{M}$ with G acting non-trivially in $\mathcal{M}/\mathcal{N}$ quantum Clifford algebra. $\mathcal{N}$ would decompose by $r = 4$ inclusion to $\mathcal{N}_1 \subset \mathcal{N}$ with $SU(2)$ taking the role of G. $\mathcal{N}/\mathcal{N}_1$ quantum Clifford algebra would transform non-trivially under $SU(2)$ but would be G singlet.

In TGD framework the G-invariance for $SU(2)$ representations means a reduction of S^2 to the orbifold S^2/G. The coverings $H_\pm \rightarrow H_\pm/G_a \times G_b$ should relate to these double inclusions and $SU(2)$ inclusion could mean Kac-Moody type gauge symmetry for $\mathcal{N}$. Note that the presence of the factor containing only unit matrix should relate directly to the generator d in the generator set of affine algebra in the McKay construction [27]. The physical interpretation of the fact that almost all ADE type extended diagrams ($D^{(1)}_n$ must have $n \geq 4$) are allowed for $r = 4$ inclusions whereas D_{2n+1} and E_6 are not allowed for $r < 4$, remains open.

3 Hyper-finite factors and TGD

The basic question is whether only hyper-finite factors of type II_1 appear in TGD framework. Affirmative answer would allow to interpret physical S-matrix as time like entanglement coefficients rather than only a cognitive representation of S-matrix in fermionic degrees of freedom analogous to representations of Boolean functions.

3.1 Generalization of the notion of imbedding space?

The original idea was that the proposed modification of the imbedding space could explain naturally phenomena like quantum Hall effect involving fractionization of quantum numbers like spin and charge. This does not however seem to be the case. $G_a \times G_b$ implies just the opposite if these quantum numbers are assigned with the symmetries of the imbedding space. For instance, quantization unit for orbital angular momentum becomes n_a where Z_{n_a} is the maximal cyclic subgroup of G_a.

One can however imagine of obtaining fractionization at the level of imbedding space for space-time sheets, which are analogous to multi-sheeted Riemann surfaces (say Riemann surfaces associated with $z^{1/n}$ since the rotation by 2π understood as a homotopy of M^4 lifted to the space-time sheet is a non-closed curve. Continuity requirement indeed allows fractionization of the orbital quantum numbers and color in this kind of situation.

3.1.1 Both covering spaces and factor spaces are possible

The observation above stimulates the question whether it might be possible in some sense to replace H or its factors by their multiple coverings.

1. This is certainly not possible for M^4, CP_2, or H since their fundamental groups are trivial. On the other hand, the fixing of quantization axes implies a selection of the sub-space $H_4 = M^2 \times S^2 \subset M^4 \times CP_2$, where S^2 is a geodesic sphere of CP_2. $\hat{M}^4 = M^4 \backslash M^2$ and $\hat{CP}_2 = CP_2 \backslash S^2$ have fundamental group Z since the codimension of the excluded sub-manifold is equal to two and homotopically the situation is like that for a punctured plane. The exclusion of these sub-manifolds defined by the choice of quantization axes could naturally give rise to the desired situation.

2. H_4 represents a straight cosmic string. Quantum field theory phase corresponds to Jones inclusions with Jones index $\mathcal{M} : \mathcal{N} < 4$. Stringy phase would by previous arguments correspond to $\mathcal{M} : \mathcal{N} = 4$. Also these Jones inclusions are labelled by finite subgroups of $SO(3)$ and thus by Z_n identified as a maximal Abelian subgroup.

   One can argue that cosmic strings are not allowed in QFT phase. This would encourage the replacement $\hat{M}^4 \times \hat{CP}_2$ implying that surfaces in $M^4 \times S^2$ and $M^2 \times CP_2$ are not allowed. In particular, cosmic strings and CP_2 type extremals with M^4 projection in M^2 and thus light-like geodesic without zitterwebegung essential for massivation are forbidden. This brings in mind instability of Higgs=0 phase.

3. The covering spaces in question would correspond to the Cartesian products $\hat{M}^4{}_{n_a} \times \hat{CP}_{2 n_b}$ of the covering spaces of $\hat{M}^4$ and $\hat{CP}_2$ by Z_{n_a} and Z_{n_b} with fundamental group is $Z_{n_a} \times Z_{n_b}$. One can also consider extension by replacing M^2 and S^2 with its orbit under G_a (say tedrahedral, octahedral, or icosahedral group). The resulting space will be denoted by $\hat{M}^4 \hat{\times} G_a$ *resp.* $\hat{CP}_2 \hat{\times} G_b$.

4. One expects the discrete subgroups of $SU(2)$ emerge naturally in this framework if one allows the action of these groups on the singular sub-manifolds M^2 or S^2. This would replace the singular manifold with a set of its rotated copies in the case that the subgroups have genuinely 3-dimensional action (the subgroups which corresponds to exceptional groups in the ADE correspondence). For instance, in the case of M^2 the quantization axes for angular momentum would be replaced by the set of quantization axes going through the vertices of tedrahedron, octahedron, or icosahedron. This would bring non-commutative homotopy groups into the picture in a natural manner.

5. Also the orbifolds $\hat{M}^4/G_a \times \hat{CP}_2/G_b$ can be allowed as also the spaces $\hat{M}^4/G_a \times (\hat{CP}_2 \hat{\times} G_b)$ and $(\hat{M}^4 \hat{\times} G_a) \times \hat{CP}_2/G_b$. Hence the previous framework would generalize considerably by the allowance of both coset spaces and covering spaces.

There are several non-trivial questions related to the details of the gluing procedure and phase transition as motion of partonic 2-surface from one sector of the imbedding space to another one.

1. How the gluing of copies of imbedding space at $M^2 \times CP_2$ takes place? It would seem that the covariant metric of M^4 factor proportional to $\hbar^2$ must be discontinuous at the singular manifold since only in this manner the idea about different scaling factor of M^4 metric can make sense. This is consistent with the identical vanishing of Chern-Simons action in $M^2 \times S^2$.

2. One might worry whether the phase transition changing Planck constant means an instantaneous change of the size of partonic 2-surface in M^4 degrees of freedom. This is not the case. Light-likeness in $M^2 \times S^2$ makes sense only for surfaces $X^1 \times D^2 \subset M^2 \times S^2$, where X^1 is light-like geodesic. The requirement that the partonic 2-surface X^2 moving from one sector of H to another one is light-like at $M^2 \times S^2$ irrespective of the value of Planck constant requires that X^2 has single point of M^2 as M^2 projection. Hence no sudden change of the size X^2 occurs.

3. A natural question is whether the phase transition changing the value of Planck constant can occur purely classically or whether it is analogous to quantum tunnelling. Classical non-vacuum extremals of Chern-Simons action have two-dimensional CP_2 projection to homologically non-trivial geodesic sphere S^2_I. The deformation of the entire S^2_I to homologically trivial geodesic sphere S^2_{II} is not possible so that only combinations of partonic 2-surfaces with vanishing total homology charge (Kähler magnetic charge) can in principle move from sector to another one, and this process involves fusion of these 2-surfaces such that CP_2 projection becomes single homologically trivial 2-surface. A piece of a non-trivial geodesic sphere S^2_I of CP_2 can be deformed to that of S^2_{II} using 2-dimensional homotopy flattening the piece of S^2 to curve. If this homotopy cannot be chosen to be light-like, the phase transitions changing Planck constant take place only via quantum tunnelling. Obviously the notions of light-like homotopies (cobordisms) and classical light-like homotopies (cobordisms) are very relevant for the understanding of phase transitions changing Planck constant.

3.1.2 Do factor spaces and coverings correspond to the two kinds of Jones inclusions?

What could be the interpretation of these two kinds of spaces?

1. Jones inclusions appear in two varieties corresponding to $\mathcal{M} : \mathcal{N} < 4$ and $\mathcal{M} : \mathcal{N} = 4$ and one can assign a hierarchy of subgroups of $SU(2)$ with both of them. In particular, their maximal Abelian subgroups Z_n label these inclusions. The interpretation of Z_n as invariance group is natural for $\mathcal{M} : \mathcal{N} < 4$ and it naturally corresponds to the coset spaces. For $\mathcal{M} : \mathcal{N} = 4$ the interpretation of Z_n has remained open. Obviously the interpretation of Z_n as the homology group defining covering would be natural.

2. $\mathcal{M} : \mathcal{N} = 4$ should correspond to the allowance of cosmic strings and other analogous objects. Does the introduction of the covering spaces bring in cosmic strings in some controlled manner? Formally the subgroup of $SU(2)$ defining the inclusion is $SU(2)$ would mean that states are $SU(2)$ singlets which is something non-physical. For covering spaces one would however obtain the degrees of freedom associated with the discrete fiber and the degrees of freedom in question would not disappear completely and would be characterized by the discrete subgroup of $SU(2)$.

   For anyons the non-trivial homotopy of plane brings in non-trivial connection with a flat curvature and the non-trivial dynamics of topological QFTs. Also now one might expect similar non-trivial contribution to appear in the spinor connection of $\hat{M}^2 \hat{\times} G_a$ and $\hat{CP}_2 \hat{\times} G_b$. In conformal field theory models non-trivial monodromy would correspond to the presence of punctures in plane.

3. For factor spaces the unit for quantum numbers like orbital angular momentum is multiplied by n_a *resp.* n_b and for coverings it is divided by this number. These two kind of spaces are in a well defined sense obtained by multiplying and dividing the factors of $\hat{H}$ by G_a *resp.* G_b and multiplication and division are expected to relate to Jones inclusions with $\mathcal{M}:\mathcal{N}<4$ and $\mathcal{M}:\mathcal{N}=4$, which both are labelled by a subset of discrete subgroups of SU(2).

4. The discrete subgroups of $SU(2)$ with fixed quantization axes possess a well defined multiplication with product defined as the group generated by forming all possible products of group elements as elements of $SU(2)$. This product is commutative and all elements are idempotent and thus analogous to projectors. Trivial group G_1, two-element group G_2 consisting of reflection and identity, the cyclic groups Z_p, p prime, and tedrahedral, octahedral, and icosahedral groups are the generators of this algebra.

   By commutativity one can regard this algebra as an 11-dimensional module having natural numbers as coefficients ("rig"). The trivial group G_1, two-element group G_2¡ generated by reflection, and tedrahedral, octahedral, and icosahedral groups define 5 generating elements for this algebra. The products of groups other than trivial group define 10 units for this algebra so that there are 11 units altogether. The groups Z_p generate a structure analogous to natural numbers acting as analog of coefficients of this structure. Clearly, one has effectively 11-dimensional commutative algebra in 1-1 correspondence with the 11-dimensional "half-lattice" N^{11} (N denotes natural numbers). Leaving away reflections, one obtains N^7. The projector representation suggests a connection with Jones inclusions. An interesting question concerns the possible Jones inclusions assignable to the subgroups containing infinitely manner elements. Reader has of course already asked whether dimensions 11, 7 and their difference 4 might relate somehow to the mathematical structures of M-theory with 7 compactified dimensions. One could introduce generalized configuration space spinor fields in the configuration space labelled by sectors of H with given quantization axes. By introducing Fourier transform in N^{11} one would formally obtain an infinite-component field in 11-D space.

5. How do the Planck constants associated with factors and coverings relate? One might argue that Planck constant defines a homomorphism respecting the multiplication and division (when possible) by G_i. If so, then Planck constant in units of $\hbar_0$ would be equal to n_a/n_b for $\hat{H}/G_a\times G_b$ option and n_b/n_a for $\hat{H}\hat{\times}(G_a\times G_b)$ with obvious formulas for hybrid cases. This option would put M^4 and CP_2 in a very symmetric role and allow much more flexibility in the identification of symmetries associated with large Planck constant phases.

3.1.3 Fractional Quantum Hall effect

The generalization of the imbedding space allows to understand fractional quantum Hall effect [80]. The formula for the quantized Hall conductance is given by

$$\begin{aligned} \sigma &= \nu\times\frac{e^2}{h} \ , \\ \nu &= \frac{n}{m} \ . \end{aligned} \tag{3}$$

Series of fractions in $\nu = 1/3, 2/5, 3/7, 4/9, 5/11, 6/13, 7/15..., 2/3, 3/5, 4/7, 5/9, 6/11, 7/13..., 5/3, 8/5, 11/7, 14/9...4/$ $1/5, 2/9, 3/13..., 2/7, 3/11..., 1/7....$ with odd denominator have been observed as are also $\nu = 1/2$ and $\nu = 5/2$ states with even denominator [80].

The model of Laughlin [78, 79] cannot explain all aspects of FQHE. The best existing model proposed originally by Jain is based on composite fermions resulting as bound states of electron and even number of magnetic flux quanta [81]. Electrons remain integer charged but due to the effective magnetic field electrons appear to have fractional charges. Composite fermion picture predicts all the observed fractions and also their relative intensities and the order in which they appear as the quality of sample improves.

In [E9] I have proposed a possible TGD based model of FQHE not involving hierarchy of Planck constants. The generalization of the notion of imbedding space suggests also the possibility to interpret these states in terms of fractionized charge and electron number.

1. The easiest manner to understand the observed fractions is by assuming that both M^4 and CP_2 correspond to covering spaces so that both spin and electric charge and fermion number are quantized. With this assumption the expression for the Planck constant becomes $\hbar/\hbar_0 = n_b/n_a$ and charge and spin units are equal to $1/n_b$ and $1/n_a$ respectively. This gives $\nu = nn_a/n_b^2$. The values $m = 2, 3, 5, 7, ..$ are observed. Planck constant can have arbitrarily large values. There are general arguments stating that also spin is fractionized in FQHE and for $n_a = kn_b$ required by the observed values of ν charge fractionization occurs in units of k/n_b and forces also spin fractionization. For factor space option in M^4 degrees of freedom one would have $\nu = n/n_a n_b^2$.

2. The appearance of $n_b = 2$ would suggest that also Z_2 appears as the homotopy group of the covering space: filling fraction 1/2 corresponds in the composite fermion model and also experimentally to the limit of zero magnetic field [81]. Also $\nu = 5/2$ has been observed [82].

3. A possible problematic aspect of the TGD based model is the experimental absence of even values of n_b except $n_b = 2$. A possible explanation is that by some symmetry condition possibly related to fermionic statistics kn/n_b must reduce to a rational with an odd denominator for $n_b > 2$. In other words, one has $k \propto 2^r$, where 2^r the largest power of 2 divisor of n_b smaller than n_b.

4. Large values of n_b emerge as B increases. This can be understood from flux quantization. One has $eBS = n\hbar = n(n_b/n_a)\hbar_0$. The interpretation is that each of the n_b sheets contributes n/n_a units to the flux. As B increases also the flux increases for a fixed value of n_a and area S. Note that the value of magnetic field in given sheet is not affected so that the build-up of multiple covering seems to keep magnetic field strength below critical value. For $n_a = kn_b$ one obtains $eBS/\hbar_0 = n/k$ so that a fractionization of magnetic flux results and each sheet contributes $1/kn_b$ units to the flux. $\nu = 1/2$ corresponds to $k = 1, n_b = 2$ and to non-vanishing magnetic flux unlike in the case of composite fermion model.

5. The understanding of the thermal stability is not trivial. The original FQHE was observed in 80 mK temperature corresponding roughly to a thermal energy of $T \sim 10^{-5}$ eV. For graphene the effect is observed at room temperature. Cyclotron energy for electron is (from $f_e = 6 \times 10^5$ Hz at $B = .2$ Gauss) of order thermal energy at room temperature in a magnetic field varying in the range 1-10 Tesla. This raises the question why the original FQHE requires so low temperature. The magnetic energy of a flux tube of length L is by flux quantization roughly $e^2B^2S \sim E_c(e)m_eL$ ($\hbar_0 = c = 1$) and exceeds cyclotron roughly by a factor L/L_e, L_e electron Compton length so that thermal stability of magnetic flux quanta is not the explanation. A possible explanation is that since FQHE involves several values of Planck constant, it is quantum critical phenomenon and is characterized by a critical temperature. The differences of the energies associated with the phase with ordinary Planck constant and phases with different Planck constant would characterize the transition temperature.

3.1.4 What is the role of dimensions?

Could the dimensions of M^4 and CP_2 and the dimensions of spaces defined by the choice of the quantization axes play a fundamental role in the construction from the constraint that the fundamental group is non-trivial?

1. Suppose that the sub-manifold in question is geodesic sub-manifold containing the orbits of its points under Cartan subgroup defining quantization axes. A stronger assumption would be that the orbit of maximal compact subgroup is in question.

2. For M^{2n} Cartan group contains translations in time direction with orbit M^1 and Cartan subgroup of $SO(2n-1)$ and would be M^n so that $\hat{M^{2n}}$ would have a trivial fundamental group for $n > 2$. Same result applies in massless case for which one has $SO(1,1) \times SO(2n-2)$ acts as Cartan subgroup. The orbit under maximal compact subgroup would not be in question.

3. For CP_2 homologically non-trivial geodesic sphere CP_1 contains orbits of the Cartan subgroup. For $CP_n = SU(n+1)/SU(n) \times U(1)$ having real dimension $2n$ the sub-manifold CP_{n-1} contains orbits of the Cartan subgroup and defines a sub-manifold with codimension 2 so that the dimensional restriction does not appear.

4. For spheres $S^{n-1} = SO(n)/SO(n-1)$ the dimension is $n-1$ and orbit of $SO(n-1)$ of point left fixed by Cartan subgroup $SO(2) \times ..$ would for $n = 2$ consist of two points and S_{n-2} in more general case. Again co-dimension 2 condition would be satisfied.

3.1.5 What about holes of the configuration space?

One can raise analogous questions at the level of configuration space geometry. Vacuum extremals correspond to Lagrangian sub-manifolds $Y^2 \subset CP_2$ with vanishing induced Kähler form. They correspond to singularities of the configuration space (”world of classical worlds”) and configuration space spinor fields should vanish for the vacuum extremals. Effectively this would mean a hole in configuration space, and the question is whether this hole could also naturally lead to the introduction of covering spaces and factor spaces of the configuration spaces. How much information about the general structure of the theory just this kind of decomposition might allow to deduce? This kind of singularities are infinite-dimensional variants of those discussed in catastrophe theory and this suggests that their understanding might be crucial.

3.2 What kind of hyper-finite factors one can imagine in TGD?

The working hypothesis has been that only hyper-finite factors of type II_1 appear in TGD. The basic motivation has been that they allow a new view about S-matrix as an operator representable as time-like entanglement coefficients of zero energy states so that physical states would represent laws of physics in their structure. They allow also the introduction of the notion of measurement resolution directly to the definition of reaction probabilities by using Jones inclusion and the replacement of state space with a finite-dimensional state space defined by quantum spinors. This hypothesis is of course just an attractive working hypothesis and deserves to be challenged.

3.2.1 Configuration space spinors

For configuration space spinors the HFF II_1 property is very natural because of the properties of infinite-dimensional Clifford algebra and the inner product defined by the configuration space geometry does not allow other factors than this. A good guess is that the values of conformal weights label the factors appearing in the tensor power defining configuration space spinors. Because of the non-degeneracy and super-canonical symmetries the density matrix representing metric must be essentially unit matrix for each conformal weight which would be the defining characteristic of hyper-finite factor of type II_1.

3.2.2 Bosonic degrees of freedom

The bosonic part of the super-canonical algebra consists of Hamiltonians of CH in one-one correspondence with those of $\delta M^4_\pm \times CP_2$. Also the Kac-Moody algebra acting leaving the light-likeness of the partonic 3-surfaces intact contributes to the bosonic degrees of freedom. The commutator of these algebras annihilates physical states and there are also Virasoro conditions associated with ordinary conformal symmetries of partonic 2-surface [C1]. The labels of Hamiltonians of configuration space and spin indices contribute to bosonic degrees of freedom.

Hyper-finite factors of type II_1 result naturally if the system is an infinite tensor product finite-dimensional matrix algebra associated with finite dimensional systems [47]. Unfortunately, neither Virasoro, canonical nor Kac-Moody algebras do have decomposition into this kind of infinite tensor product. If bosonic degrees for super-canonical and super-Kac Moody algebra indeed give I_∞ factor one has HFF if type II_∞. This looks the most natural option but threatens to spoil the beautiful idea about S-matrix as time-like entanglement coefficients between positive and negative energy parts of zero energy state.

The resolution of the problem is surprisingly simple and trivial after one has discovered it. The requirement that state is normalizable forces to project S-matrix to a finite-dimensional sub-space in bosonic degrees of freedom so that the reduction $I_\infty \rightarrow I_n$ occurs and one has the reduction $II_\infty \rightarrow II_1 \times I_n = II_1$ to the desired HFF.

One can consider also the possibility of taking the limit $n \rightarrow \infty$. One could indeed say that since I_∞ factor can be mapped to an infinite tensor power of $M(2, C)$ characterized by a state which is not trace, it is possible to map this representation to HFF by replacing state with trace [47]. The question is whether the forcing the bosonic foot to fermionic shoe is physically natural. One could also regard the II_1 type notion of probability as fundamental and also argue that it is required by full super-symmetry realized also at the level of many-particle states rather than mere single particle states.

3.2.3 How the bosonic cutoff is realized?

Normalizability of state requires that projection to a finite-dimensional bosonic sub-space is carried out for the bosonic part of the S-matrix. This requires a cutoff in quantum numbers of super-conformal algebras. The cutoff for the values of conformal weight could be formulated by replacing integers with Z_n or with some finite field $G(p, 1)$. The cutoff for the labels associated with Hamiltonians defined as an upper bound for the dimension of the representation looks also natural.

Number theoretical braids which are discrete and finite structures would define space-time correlate for this cutoff. p-Adic length scale $p \simeq 2^k$ hypothesis could be interpreted as stating the fact that only powers of p up to p^k are significant in p-adic thermodynamics which would correspond to finite field $G(k, 1)$ if k is prime. This has no consequences for p-adic mass calculations since already the first two terms give practically exact results for the large primes associated with elementary particles [6].

Finite number of strands for the theoretical braids would serve as a correlate for the reduction of the representation of Galois group S_∞ of rationals to an infinite produce of diagonal copies of finite-dimensional Galois group so that same braid would repeat itself like a unit cell of lattice i condensed matter [27].

3.2.4 HFF of type III for field operators and HFF of type II_1 for states?

One could also argue that the Hamiltonians with fixed conformal weight are included in fermionic II_1 factor and bosonic factor I_∞ factor, and that the inclusion of conformal weights leads to a factor of type III. Conformal weight could relate to the integer appearing in the crossed product representation $III = Z \times_{cr} II_\infty$ of HFF of type III [47].

The value of conformal weight is non-negative for physical states which suggests that Z reduces to semigroup N so that a factor of type III would reduce to a factor of type II_∞ since trace would become finite. If unitary process corresponds to an automorphism for II_∞ factor, the action of automorphisms affecting scaling must be uni-directional. Also thermodynamical irreversibility suggests the same. The assumption that state function reduction for positive energy part of state implies unitary process for negative energy state and vice versa would only mean that the shifts for positive and negative energy parts of state are opposite so that $Z \rightarrow N$ reduction would still hold true.

3.2.5 HFF of type II_1 for the maxima of Kähler function?

Probabilistic interpretation allows to gain heuristic insights about whether and how hyper-finite factors of type type II_1 might be associated with configuration space degrees of freedom. They can appear both in quantum fluctuating degrees of freedom associated with a given maximum of Kähler function and in the discrete space of maxima of Kähler function.

Spin glass degeneracy is the basic prediction of classical TGD and means that instead of a single maximum of Kähler function analogous to single free energy minimum of a thermodynamical system there is a fractal spin glass energy landscape with valleys inside valleys. The discretization of the configuration space in terms of the maxima of Kähler function crucial for the p-adicization problem, leads to the analog of spin glass energy landscape and hyper-finite factor of type II_1 might be the appropriate description of the situation.

The presence of the tensor product structure is a powerful additional constraint and something analogous to this should emerge in configuration space degrees of freedom. Fractality of the many-sheeted space-time is a natural candidate here since the decomposition of the original geometric structure to parts and replacing them with the scaled down variant of original structure is the geometric analog of forming a tensor power of the original structure.

3.3 Direct sum of HFFs of type II_1 as a minimal option

HFF II_1 property for the Clifford algebra of the configuration space means a definite distinction from the ordinary Clifford algebra defined by the fermionic oscillator operators since the trace of the unit matrix of the Clifford algebra is normalized to one. This does not affect the anti-commutation relations at the basic level and delta functions can appear in them at space-time level. At the level of momentum space I_∞ property requires discrete basis and anti-commutators involve only Kronecker deltas. This conforms with the fact that HFF of type II_1 can be identified as the Clifford algebra associated with a separable Hilbert space.

3.3.1 II_∞ factor or direct sum of HFFs of type II_1?

The expectation is that super-canonical algebra is a direct sum over HFFs of type II_1 labelled by the radial conformal weight. In the same manner the algebra defined by fermionic anti-commutation relations at partonic 2-surface would decompose to a direct sum of algebras labelled by the conformal weight associated with the light-like coordinate of X_l^3. Super-conformal symmetry suggests that also the configuration space degrees of freedom correspond to a direct sum of HFFs of type II_1.

One can of course ask why not $II_\infty = I_\infty \times II_1$ structures so that one would have single factor rather than a direct sum of factors.

1. The physical motivation is that the direct sum property allow to decompose M-matrix to direct summands associated with various sectors with weights whose moduli squared have an interpretation in terms of the density matrix. This is also consistent with p-adic thermodynamics where conformal weights take the place of energy eigen values.

2. II_∞ property would predict automorphisms scaling the trace by an arbitrary positive real number $\lambda \in R_+$. These automorphisms would require the scaling of the trace of the projectors of Clifford algebra having values in the range $[0,1]$ and it is difficult to imagine how these automorphisms could be realized geometrically.

3.3.2 How HFF property reflects itself in the construction of geometry of WCW?

The interesting question is what HFF property and finite measurement resolution realizing itself as the use of projection operators means concretely at the level of the configuration space geometry.

Super-Hamiltonians define the Clifford algebra of the configuration space. Super-conformal symmetry suggests that the unavoidable restriction to projection operators instead of complex rays is realized also configuration space degrees of freedom. Of course, infinite precision in the determination of the shape of 3-surface would be physically a completely unrealistic idea.

In the fermionic situation the anti-commutators for the gamma matrices associated with configuration space individual Hamiltonians in 3-D sense are replaced with anti-commutators where Hamiltonians are replaced with projectors to subspaces of the space spanned by Hamiltonians. This projection is realized by restricting the anti-commutator to partonic 2-surfaces so that the anti-commutator depends only the restriction of the Hamiltonian to those surfaces.

What is interesting that the measurement resolution has a concrete particle physical meaning since the parton content of the system characterizes the projection. The larger the number of partons, the better the resolution about configuration space degrees of freedom is. The degeneracy of configuration space metric would be interpreted in terms of finite measurement resolution inherent to HFFs of type II_1, which is not due to Jones inclusions but due to the fact that one can project only to infinite-D subspaces rather than complex rays.

Effective 2-dimensionality in the sense that configuration space Hamiltonians reduce to functionals of the partonic 2-surfaces of X^3_l rather than functionals of X^3_l could be interpreted in this manner. For a wide class of Hamiltonians actually effective 1-dimensionality holds true in accordance with conformal invariance.

The generalization of configuration space Hamiltonians and super-Hamiltonians by allowing integrals over the 2-D boundaries of the patches of X^3_l would be natural and is suggested by the requirement of discretized 3-dimensionality at the level of configuration space.

By quantum classical correspondence the inclusions of HFFs related to the measurement resolution should also have a geometric description. Measurement resolution corresponds to braids in given time scale and as already explained there is a hierarchy of braids in time scales coming as negative powers of two corresponding to the addition of zero energy components to positive/negative energy state. Note however that particle reactions understood as decays and fusions of braid strands could also lead to a notion of measurement resolution.

3.4 Could HFFs of type III have application in TGD framework?

One can imagine several manners for how HFFs of type III could emerge in TGD although the proposed view about S-matrix in zero energy ontology suggests that HFFs of type III_1 should be only an auxiliary tool at best. Both TGD inspired quantum measurement theory, the idea about a variant of HFF of type II_1 analogous to a local gauge algebra, and some other arguments, suggest that HFFs of type III could be seen as a useful idealization allowing to make non-trivial conjectures both about quantum TGD and about HFFs of type III. Quantum fields would correspond to HFFs of type III and II_∞ whereas physical states (S-matrix) would correspond to HFF of type II_1.

3.4.1 Quantum measurement theory and HFFs of type III

The attempt to interpret the HFFs of type III in terms of quantum measurement theory based on Jones inclusions leads to highly non-trivial conjectures about these factors.

1. Could the scalings of trace relate to quantum measurements?

What should be understood is the physical meaning of the automorphism inducing the scaling of trace. In the representation based of factors based on infinite tensor powers the action of g should transform single $n \times n$ matrix factor with density matrix Id/n to a density matrix e_{11} of a pure state.

Obviously the number of degrees of freedom is affected and this can be interpreted in terms of appearance or disappearance of correlations. Quantization and emergence of non-commutativity indeed implies the emergence of correlations and effective reduction of degrees of freedom. In particular, the fundamental quantum Clifford algebra has reduced dimension $\mathcal{M} : \mathcal{N} = r \leq 4$ instead of $r = 4$ since the replacement of complex valued matrix elements with $\mathcal{N}$ valued ones implies non-commutativity and correlations.

The transformation would be induced by the shift of finite-dimensional state to right or left so that the number of matrix factors overlapping with I_∞ part increases or is reduced. Could it have interpretation in terms of quantum measurement for a quantum Clifford factor? Could quantum measurement for $\mathcal{M}/\mathcal{N}$ degrees of freedom reducing the state in these degrees of freedom to a pure state be interpreted as a transformation of single finite-dimensional matrix factor to a type I factor inducing the scaling of the trace and could the scalings associated with automorphisms of HFFs of type III also be interpreted in terms of quantum measurement?

This interpretation does not as such say anything about HFF factors of type III since only a decomposition of II_1 factor to I_2^k factor and II_1 factor with a reduced trace of projector to the latter. However, one can ask whether the scaling of trace for HFFs of type III could correspond to a situation in which infinite number of finite-dimensional factors have been quantum measured. This would correspond to the inclusion $\mathcal{N} \subset \mathcal{M}_\infty = \cup_n \mathcal{M}_n$ where $\mathcal{N} \subset \mathcal{M} \subset ...\mathcal{M}_n...$ defines the canonical inclusion sequence. Physicist can of course ask whether the presence of infinite number of I_2-, or more generally, I_n-factors is at all relevant to quantum measurement and it has already become clear that situation at the level of S-matrix reduces to I_n.

2. Could the theory of HHFs of type III relate to the theory of Jones inclusions?

The idea about a connection of HFFs of type III and quantum measurement theory seems to be consistent with the basic facts about inclusions and HFFs of type III_1.

1. Quantum measurement would scale the trace by a factor $2^k/\sqrt{\mathcal{M} : \mathcal{N}}$ since the trace would become a product for the trace of the projector to the newly born $M(2,C)^{\otimes k}$ factor and the trace for the projection to $\mathcal{N}$ given by $1/\sqrt{\mathcal{M} : \mathcal{N}}$. The continuous range of values $\mathcal{M} : \mathcal{N} \geq 4$ gives good hopes that all values of λ are realized. The prediction would be that $2^k\sqrt{\mathcal{M} : \mathcal{N}} \geq 1$ holds always true.

2. The values $\mathcal{M} : \mathcal{N} \in \{r_n = 4cos^2(\pi/n)\}$ for which the single $M(2,C)$ factor emerges in state function reduction would define preferred values of the inverse of $\lambda = \sqrt{\mathcal{M} : \mathcal{N}/4}$ parameterizing factors III_λ. These preferred values vary in the range $[1/2, 1]$.

3. $\lambda = 1$ at the end of continuum would correspond to HFF III_1 and to Jones inclusions defined by infinite cyclic subgroups dense in $U(1) \subset SU(2)$ and this group combined with reflection. These groups correspond to the Dynkin diagrams A_∞ and D_∞. Also the classical values of $\mathcal{M} : \mathcal{N} = n^2$ characterizing the dimension of the quantum Clifford $\mathcal{M} : \mathcal{N}$ are possible. In this case the scaling of trace would be trivial since the factor n to the trace would be compensated by the factor $1/n$ due to the disappearance of $\mathcal{M}/\mathcal{N}$ factor III_1 factor.

4. Inclusions with $\mathcal{M} : \mathcal{N} = \infty$ are also possible and they would correspond to $\lambda = 0$ so that also III_0 factor would also have a natural identification in this framework. These factors correspond to ergodic systems and one might perhaps argue that quantum measurement in this case would give infinite amount of information.

5. This picture makes sense also physically. p-Adic thermodynamics for the representations of super-conformal algebra could be formulated in terms of factors of type I_∞ and in excellent approximation using factors I_n. The generation of arbitrary number of type II_1 factors in quantum measurement allow this possibility.

3. The end points of spectrum of preferred values of λ are physically special

The fact that the end points of the spectrum of preferred values of λ are physically special, supports the hopes that this picture might have something to do with reality.

1. The Jones inclusion with $q = exp(i\pi/n)$, $n = 3$ (with principal diagram reducing to a Dynkin diagram of group SU(3)) corresponds to $\lambda = 1/2$, which corresponds to HFF III_1 differing in essential manner from factors III_λ, $\lambda < 1$. On the other hand, $SU(3)$ corresponds to color group which appears as an isometry group and important subgroup of automorphisms of octonions thus differs physically from the ADE gauge groups predicted to be realized dynamically by the TGD based view about McKay correspondence [27].

2. For $r = 4$ SU(2) inclusion parameterized by extended ADE diagrams $M(2, C)^{\otimes 2}$ would be created in the state function reduction and also this would give $\lambda = 1/2$ and scaling by a factor of 2. Hence the end points of the range of discrete spectrum would correspond to the same scaling factor and same HFF of type III. SU(2) could be interpreted either as electro-weak gauge group, group of rotations of th geodesic sphere of $\delta M^4_\pm$, or a subgroup of SU(3). In TGD interpretation for McKay correspondence a phase transition replacing gauge symmetry with Kac-Moody symmetry.

3. The scalings of trace by factor 2 seem to be preferred physically which should be contrasted with the fact that primes near prime powers of 2 and with the fact that quantum phases $q = exp(i\pi/n)$ with n equal to Fermat integer proportional to power of 2 and product of the Fermat primes (the known ones are 5, 17, 257, and $2^{16} + 1$) are in a special role in TGD Universe.

4. What could one say about II_1 automorphism associated with the II_∞ automorphism defining factor of type III?

An interesting question relates to the interpretation of the automorphisms of II_∞ factor inducing the scaling of trace.

1. If the automorphism for Jones inclusion involves the generator of cyclic automorphism subgroup Z_n of II_1 factor then it would seem that for other values of λ this group cannot be cyclic. SU(2) has discrete subgroups generated by arbitrary phase q and these are dense in $U(1) \subset SU(2)$ sub-group. If the interpretation in terms of Jones inclusion makes sense then the identification $\lambda = \sqrt{\mathcal{M} : \mathcal{N}}/2^k$ makes sense.

2. If HFF of type II_1 is realized as group algebra of infinite symmetric group [27], the outer automorphism induced by the diagonally imbedded finite Galois groups can induce only integer values of n and Z_n would correspond to cyclic subgroups. This interpretation conforms with the fact that the automorphisms in the completion of inner automorphisms of HFF of type II_1 induce trivial scalings. Therefore only automorphisms which do not belong to this completion can define HFFs of type III.

3.4.2 What could be the physical interpretation of two kinds of invariants associated with HFFs type III?

TGD predicts two kinds of S-matrices: S-matrix and U-matrix. Both are expected to be more or less universal. There are also *two* kinds of invariants and automorphisms associated with HFFs of type III.

1. The first invariant corresponds to the scaling $\lambda \in]0,1[$ of the trace associated with the automorphism of factor of II_∞. Also the end points of the interval make sense. The inverse of this scaling accompanies the inverse of this automorphism.

2. Second invariant corresponds to the time scales $t = T_0$ for which the outer automorphism σ_t reduces to inner automorphism. It turns out that T_0 and λ are related by the formula $\lambda^{iT_0} = 1$, which gives the allowed values of T_0 as $T_0 = n2\pi/log(\lambda)$ [47]. This formula can be understood intuitively by realizing that λ corresponds to the eigenvalue of the density matrix $\Delta = e^H$ in the simplest possible realization of the state ϕ.

The presence of two automorphisms and invariants brings in mind U matrix characterizing the unitary process occurring in quantum jump and S-matrix characterizing time like entanglement.

1. If one accepts the vision based on quantum measurement theory then λ corresponds to the scaling of the trace resulting when quantum Clifford algebra $\mathcal{M}/\mathcal{N}$ reduces to a tensor power of $M(2,C)$ factor in the state function reduction. The proposed interpretation for U process would be as the inverse of state function reduction transforming this factor back to $\mathcal{M}/\mathcal{N}$. Thus U process and state function reduction would correspond naturally to the scaling and its inverse. This picture might apply not only in single particle case but also for zero energy states which can be seen as states associated the a tensor power of HFFs of type II_1 associated with partons.

2. The implication is that U process can occur only in the direction in which trace is reduced. This would suggest that the full III_1 factor is not a physical notion and that one must restrict the group Z in the crossed product $Z \times_{cr} II_\infty$ to the group N of non-negative integers. In this kind of situation the trace is well defined since the traces for the terms in the crossed product comes as powers λ^{-n} so that the net result is finite. This would mean a reduction to II_∞ factor.

3. Since time t is a natural parameter in elementary particle physics experiment, one could argue that σ_t could define naturally S-matrix. Time parameter would most naturally correspond to a parameter of scaling affecting all $M^4_\pm$ coordinates rather than linear time. This conforms also with the fundamental role of conformal transformations and scalings in TGD framework.

The identification of the full S-matrix in terms of σ does not seem to make sense generally. It would however make sense for incoming and outgoing number theoretic braids so that σ could define universal braiding S-matrices. Inner automorphisms would bring in the dependence on experimental situation. The reduction of the braiding matrix to an inner automorphism for critical values of t which could be interpreted in terms of scaling by power of p. This trivialization would be a counterpart for the elimination of propagator legs from S-matrix element. Vertex itself could be interpreted as unitary isomorphism between tensor product of incoming and outgoing HFFs of type II_1 would code all what is relevant about the particle reaction.

3.4.3 Does the time parameter t represent time translation or scaling?

The connection $T_n = n2\pi/log(\lambda)$ would give a relationship between the scaling of trace and value of time parameter for which the outer automorphism represented by σ reduces to inner automorphism. It must be emphasized that the time parameter t appearing in σ need not have anything to do with time translation. The alternative interpretation is in terms of $M^4_\pm$ scaling (implying also time scaling) but one cannot exclude even preferred Lorentz boosts in the direction of quantization axis of angular momentum.

1. Could the time parameter correspond to scaling?

The central role of conformal invariance in quantum TGD suggests that t parameterizes scaling rather than translation. In this case scalings would correspond to powers of $(K\lambda)^n$. The numerical factor K which cannot be excluded a priori, seems to reduce to $K = 1$.

1. The scalings by powers of p have a simple realization in terms of the representation of HFF of type II_∞ as infinite tensor power of $M(p, C)$ with suitably chosen densities matrices in factors to get product of I_∞ and II_1 factor. These matrix algebras have the remarkable property of defining prime tensor power factors of finite matrix algebras. Thus p-adic fractality would reflect directly basic properties of matrix algebras as suggested already earlier. That scalings by powers of p would correspond to automorphism reducing to inner automorphisms would conform with p-adic fractality.
2. Also scalings by powers $[\sqrt{\mathcal{M} : \mathcal{N}}/2^k]^n$ would be physically preferred if one takes previous arguments about Jones inclusions seriously and if also in this case scalings are involved. For $q = exp(i\pi/n)$, $n = 5$ the minimal value of n allowing universal topological quantum computation would correspond to a scaling by Golden Mean and these fractal scalings indeed play a key role in living matter. In particular, Golden Mean makes it visible in the geometry of DNA.

2. Could the time parameter correspond to time translation?

One can consider also the interpretation of σ_t as time translation. TGD predicts a hierarchy of Planck constants parameterized by rational numbers such that integer multiples are favored. In particular, integers defining ruler and compass polygons are predicted to be in a very special role physically. Since the geometric time span associated with zero energy state should scale as Planck constant one expects that preferred values of time t associated with σ are quantized as rational multiples of some fundamental time scales, say the basic time scale defined by CP_2 length or p-adic time scales.

1. For $\lambda = 1/p$, p prime, the time scale would be $T_n = nT_1$, $T_1 = T_0 = 2\pi/log(p)$ which is not what p-adic length scale hypothesis would suggest.
2. For Jones inclusions one would have $T_n/T_0 = n2\pi/log(2^{2k}/\mathcal{M} : \mathcal{N})$. In the limit when λ becomes very small (the number k of reduced $M(2, C)$ factors is large one obtains $T_n = (n/k)t_1$, $T_1 = T_0\pi/log(2)$. Approximate rational multiples of the basic length scale would be obtained as also predicted by the general quantization of Planck constant.

3.4.4 p-Adic thermodynamics from first principles

Quantum field theory at non-zero temperature can be formulated in the functional integral formalism by replacing the time parameter associated with the unitary time evolution operator $U(t)$ with a complexified time containing as imaginary part the inverse of the temperature: $t \rightarrow t + i\hbar/T$. In the framework of standard quantum field theory this is a mere computational trick but the time parameter associated with the automorphisms σ_t of HFF of type III is a temperature like parameter from the beginning, and its complexification would naturally lead to the analog of thermal QFT.

Thus thermal equilibrium state would be a genuine quantum state rather than fictive but useful auxiliary notion. Thermal equilibrium is defined separately for each incoming parton braid and perhaps even braid (partons can have arbitrarily large size). At elementary particle level p-adic thermodynamics could be in question so that particle massivation would have first principle description. p-Adic thermodynamics is under relatively mild conditions equivalent with its real counterpart obtained by the replacement of p^{L_0} interpreted as a p-adic number with p^{-L_0} interpreted as a real number.

3.4.5 Could HFFs of type III be associated with the dynamics in $M^4_\pm$ degrees of freedom?

HFFs of type III could be also assigned with the poorly understood dynamics in $M^4_\pm$ degrees of freedom which should have a lot of to do with four-dimensional quantum field theory. Hyper-finite factors of type III_1 might emerge when one extends II_1 to a local algebra by multiplying it with hyper-octonions replaced as analog of matrix factor and considers hyper-quaternionic subalgebra. The resulting algebra would be the analog of local gauge algebra and the elements of algebra would be analogous to conformal fields with complex argument replaced with hyper-octonionic, -quaternionic, or -complex one. Since quantum field theory in M^4 gives rise to hyper-finite III_1 factors one might guess that the hyper-quaternionic restriction indeed gives these factors.

The expansion of the local HFF II_∞ element as $O(m) = \sum_n m^n O_n$, where M^4 coordinate m is interpreted as hyper-quaternion, could have interpretation as expansion in which O_n belongs to $\mathcal{N}g^n$ in the crossed product $\mathcal{N} \times_{cr} \{g^n, n \in Z\}$. The analogy with conformal fields suggests that the power g^n inducing λ^n fold scaling of trace increases the conformal weight by n.

One can ask whether the scaling of trace by powers of λ defines an inclusion hierarchy of sub-algebras of conformal sub-algebras as suggested by previous arguments. One such hierarchy would be the hierarchy of sub-algebras containing only the generators O_m with conformal weight $m \geq n$, $n \in Z$.

It has been suggested that the automorphism Δ could correspond to scaling inside light-cone. This interpretation would fit nicely with Lorentz invariance and TGD in general. The factors III_λ with λ generating semi-subgroups of integers (in particular powers of primes) could be of special physical importance in TGD framework. The values of t for which automorphism reduces to inner automorphism should be of special physical importance in TGD framework. These automorphisms correspond to scalings identifiable in terms of powers of p-adic prime p so that p-adic fractality would find an explanation at the fundamental level.

If the above mentioned expansion in powers of m^n of $M^4_\pm$ coordinate makes sense then the action of σ^t representing a scaling by p^n would leave the elements O invariant or induce a mere inner automorphism. Conformal weight n corresponds naturally to n-ary p-adic length scale by uncertainty principle in p-adic mass calculations.

The basic question is the physical interpretation of the automorphism inducing the scaling of trace by λ and its detailed action in HFF. This scaling could relate to a scaling in M^4 and to the appearance in the trace of an integral over M^4 or subspace of it defining the trace. Fractal structures suggests itself strongly here. At the level of construction of physical states one always selects some minimum non-positive conformal weight defining the tachyonic ground state and physical states have non-negative conformal weights. The interpretation would be as a reduction to HHF of type II_∞ or even II_1.

3.4.6 Could the continuation of braidings to homotopies involve Δ^{it} automorphisms

The representation of braidings as special case of homotopies might lead from discrete automorphisms for HFFs type II_1 to continuous outer automorphisms for HFFs of type III_1. The question is whether the periodic automorphism of II_1 represented as a discrete sub-group of $U(1)$ would be continued to $U(1)$ in the transition.

The automorphism of II_∞ HFF associated with a given value of the scaling factor λ is unique. If Jones inclusions defined by the preferred values of λ as $\lambda = \sqrt{\mathcal{M}:\mathcal{N}}/2^k$ (see the previous considerations), then this automorphism could involve a periodic automorphism of II_1 factor defined by the generator of cyclic subgroup Z_n for $\mathcal{M}:\mathcal{N} < 4$ besides additional shift transforming II_1 factor to I_∞ factor and inducing the scaling.

4 The construction of S-matrix and hyper-finite factors

I am more or less accepted as a fact that I will never be able to write explicit formulas for the S-matrix of TGD. I however feel that this is not solely due to my considerable personal limitations but also because a radically new conceptualization must be developed before one can start to think about calculating something. The reason is that the notion of functional integral must be given up in TGD framework and replaced with an approach inspired by the construction of braiding S-matrices which should be generalized to allow also reactions in which braids are created and disappear.

4.1 Jones inclusions in relation to S-matrix and U matrix

Zero energy ontology which reduces to the positive energy ontology of the standard model only as a limiting case [C2]. What I have called U-matrix characterizes the unitary process associated with the quantum jump (and followed by state function reduction and state preparation).

In the simplest scenario S-matrix would define time-like entanglement between positive and negative energy parts of the zero energy state and code the rates for particle reactions which in TGD framework correspond to quantum measurements reducing time-like entanglement. S-matrix is obviously not identifiable as U-matrix, which for real-real quantum transitions might be almost trivial by arguments of [C2] and interesting only for p-adic-to-real transitions. U process for positive energy states would induce state function reduction for negative energy states and vice versa and U process would correspond to extension of factor and thus define Jones inclusion.

4.1.1 S-matrix

In the following both the critics of earlier picture about S-matrix is discussed in more detail than in introduction.

1. Criticism of the original picture

For HFFs of type II_1 $Tr(SS^\dagger) = Tr(Id) = 1$ holds true. Hence in zero energy ontology and for HFFs of type II_1 time like entanglement coefficients between positive and negative energy part of the state could define a unitary S-matrix. For this interpretation S-matrix would code for the transition rates measured in particle physics experiments with particle reactions interpreted as quantum measurements reducing time like entanglement.

It is not difficult to criticize this picture.

1. Why time like entanglement should be always characterized by a unitary S-matrix? Why not some more general matrix? If one allows more general time like entanglement, the description of particle reaction rates in terms of a unitary S-matrix must be replaced with something more general and would require a profound revision of the vision about the relationship between experiment and theory. Also the consistency of the zero energy ontology with positive energy ontology in time scales shorter than the time scale determined by the geometric time interval between positive and negative energy parts of the zero energy state would be lost. Hence the easy way to proceed is to postulate that the universe is self-referential in the sense that quantum states represent the laws of physics by coding S-matrix as entanglement coefficients.

2. One might argue that the requirement of HFF of type II_1 is too strong one in bosonic degrees of freedom. The normalizability of the states in bosonic degrees of freedom however leads to the reduction $II_\infty \rightarrow II_1 \otimes I_n = II_1$ to a HFF of type II_1. Also the condition that Boolean functions are representable in terms of time like entanglement is consistent with the unitarity.

3. A further objection is that there might a huge number of unitary S-matrices so that it would not be possible to speak about quantum laws of physics anymore. This need not be the case since super-conformal symmetries and number theoretic universality pose extremely powerful constraints on S-matrix. Super-conformal symmetry in light-like radial degree of freedom for light-like partonic 3-surfaces is consistent with the identification as partonic 3-surface as space-time correlate for time evolution in turn forcing entanglement coefficients to define a unitary S-matrix. That only single zero energy state would be assignable to a partonic 3-surface would imply that quantum states are in 1-1 correspondence with space-time sheets required by a strict quantum classical correspondence.

Since the maxima of Kähler function correspond to different S-matrices, the dynamics does not reduce to genuinely 2-dimensional string model dynamics. Spin glass degeneracy indeed allows huge variety of dynamics: recall that the model of spin glass indeed involves an ensemble of Hamiltonians. Light-like non-determinism of partonic 3-surfaces would also explain the undeniable engineering aspect of the Universe by predicting its presence at the level of fundamental quantum dynamics.

Topological quantum computation represents certainly the highest level of engineering. The replication of number theoretic braids in partonic vertices at which the ends of light-like partonic 3-surfaces meet makes possible to understand not only topological quantum computation but also copying of information and its communication as parton exchange as fundamental processes.

2. Is S-matrix invariant under inclusions?

A highly attractive additional assumption is that S-matrix is universal in the sense that it is invariant under the inclusion sequences defined by Galois groups G associated with partonic 2-surfaces. Various constraints on S-matrix might actually imply the inclusion invariance. One might argue that zero energy states for which time-like entanglement is characterized by S-matrix invariant in the inclusion correspond to asymptotic self-organization patterns for which U-process and state function reduction do not affect the S-matrix in the relabelled basis. The analogy with a fractal asymptotic self-organization pattern is obvious.

3. Jones inclusions and physical states as representations of Galois groups

In TGD inspired quantum measurement theory measurement resolution is characterized by Jones inclusion (the group G defines the measured quantum numbers), $\mathcal{N} \subset \mathcal{M}$ takes the role of complex numbers, and state function reduction in fermionic degrees of freedom leads to $\mathcal{N}$ ray in the space $\mathcal{M}/\mathcal{N}$ regarded as $\mathcal{N}$ module and thus from a factor to a sub-factor [C2].

The finite number theoretic braid having Galois group G as its symmetries is the space-time correlate for both the finite measurement resolution and the effective reduction of HFF to that associated with a finite-dimensional quantum Clifford algebra $\mathcal{M}/\mathcal{N}$. $SU(2)$ inclusions would allow angular momentum and color quantum numbers in bosonic degrees of freedom and spin and electro-weak quantum numbers in spinorial degrees of freedom. McKay correspondence would allow to assign to G also compact ADE type Lie group so that also Lie group type quantum numbers could be included in the repertoire [27].

Galois group G would characterize sub-spaces of the configuration space ("world of classical worlds") number theoretically in a manner analogous to the rough characterization of physical states by using topological quantum numbers. Each braid associated with a given partonic 2-surface would correspond to a particular G that the state would be characterized by a collection of groups G. G would act as symmetries of zero energy states and thus of S-matrix. S-matrix would reduce to a direct integral of S-matrices associated with various collections of Galois groups characterizing the number theoretical properties of partonic 2-surfaces.

4.1.2 U-matrix

U-matrix is global notion as opposed to S-matrix and it is not clear how it relates to S-matrix, which reduces to a local notion in TGD framework. In a well-defined sense U process seems to be the reversal of state function reduction. Hence the natural guess is that U-matrix means a quantum transition in which a factor becomes a sub-factor whereas state function reduction would lead from a factor to a sub-factor. As found, the unitarity of S-matrix suggests that U process for positive energy part of state could induces state function reduction for negative energy part and vice versa.

The arguments of [C2] suggest that U matrix could be almost trivial and has as a basic building block the so called factorizing S-matrices for integrable quantum field theories in 2-dimensional Minkowski space. For these S-matrices particle scattering would mean only a permutation of momenta in momentum space. If the S-matrix is invariant under inclusion then U matrix should be in a well-defined sense almost trivial apart from a dispersion in zero modes leading to a superpositions of states characterized by different collections of Galois groups. One must be however very cautious since the idea about extension of factor might change the situation.

4.1.3 Relation to TGD inspired theory of consciousness

U-matrix could be almost trivial with respect to the transitions which are diagonal with respect to the number field. What would however make U highly interesting is that it would predict the rates for the transitions representing a transformation of intention to action identified as a p-adic-to-real transition. In this context almost triviality would translate to a precise correlation between intention and action.

The general vision about the dynamics of quantum jumps suggests that the extension of a sub-factor to a factor is followed by a reduction to a sub-factor which is not necessarily the same. Breathing would be an excellent metaphor for the process. Breathing is also a metaphor for consciousness and life. Perhaps the essence of living systems distinguishing them from sub-systems with a fixed state space could be cyclic breathing like process $\mathcal{N} \rightarrow \mathcal{M} \supset \mathcal{N} \rightarrow \mathcal{N}_1 \subset \mathcal{M} \rightarrow$ .. extending and reducing the state space of the sub-system by entanglement followed by de-entanglement.

One could even ask whether the unique role of breathing exercise in meditation practices relates directly to this basic dynamics of living systems and whether the effect of these practices is to increase the value of $\mathcal{M} : \mathcal{N}$ and thus the order of Galois group G describing the algebraic complexity of "partonic" 2-surfaces involved (they can have arbitrarily large sizes). The basic hypothesis of TGD inspired theory of cognition indeed is that cognitive evolution corresponds to the growth of the dimension of the algebraic extension of p-adic numbers involved.

If one is willing to consider generalizations of the existing picture about quantum jump, one can imagine that unitary process can occur arbitrary number of times before it is followed by state function reduction. Unitary process and state function reduction could compete in this kind of situation.

4.1.4 Fractality of S-matrix and translational invariance in the lattice defined by sub-factors

Fractality realized as the invariance of the S-matrix in Jones inclusion means that the S-matrices of $\mathcal{N}$ and $\mathcal{M}$ relate by the projection $P : \mathcal{M} \rightarrow \mathcal{N}$ as $S_{\mathcal{N}} = PS_{\mathcal{M}}P$. $S_{\mathcal{N}}$ should be equivalent with $S_{\mathcal{M}}$ with a trivial re-labelling of strands of infinite braid.

Inclusion invariance would mean translational invariance of the S-matrix with respect to the index n labelling strands of braid defined by the projectors e_i. Translations would act only as a semigroup and S-matrix elements would depend on the difference $m - n$ only. Transitions can occur only for $m-n \geq 0$, that is to the direction of increasing label of strand. The group G leaving $\mathcal{N}$ element-wise invariant would define the analog of a unit cell in lattice like condensed matter systems so that translational invariance would be obtained only for translations $m \rightarrow m + nk$, where one has $n \geq 0$ and k is the number of $M(2, C)$ factors defining the unit cell. As a matter fact, this picture might apply also to ordinary condensed matter systems.

4.2 S-matrix as a generalization of braiding S-matrix?

Consider now whether and how this framework could help to understand the construction of S-matrix when one adds ideas inspired by HFFs to the earlier ideas.

4.2.1 From path integral to a generalization of braiding S-matrix

The basic difference as compared to standard QFT framework is that there is no path integral and therefore there are excellent hopes that there are no infinities. Hence one can worry about whether coupling constant evolution is possible at all. The construction of M-matrix with finite measurement resolution realized in terms of inclusions of von Neumann algebras allows to understand coupling constant evolution in terms of analogs of radiative corrections resulting as the scale of time resolution is increased. Also p-adic length scale hypothesis follows as a prediction [C2].

There is of course functional integral (not path integral) over the small deformations of partonic 3-surface corresponding to a functional integral around the maximum of Kähler function. This functional integral is however free of standard infinities since Kähler function is a non-local functional of light-like 3-surface. Quantum criticality strongly suggests and p-adicization requires that it can be carried out explicitly (meaning that TGD is integrable quantum theory). If so, then the presence of these degrees of freedom boils down to the bosonic parts of Super Kac-Moody algebra and super-canonical algebra and selection of light-like partonic 3-surfaces corresponding to maxima of Kähler function as preferred ones.

In accordance with the proposal developed in detail [C6] S-matrix is a generalization of braiding S-matrix allowing also a fusion and replication of braids. This replication is actually very much like that of DNA and the replication of classical information associated with number theoretic braids might be what occurs at the deeper level in DNA replication.

This implies that S-matrix separates into a tensor product of braiding S-matrices associated with incoming and outgoing legs and to a unitary isomorphism between the tensor product of HFFs of type II_∞ associated with the incoming and with outgoing partons. Thus the result would be automatically unitary. The result is same for both or S-matrix in ordinary sense and in the sense of zero energy ontology.

II_∞ automorphisms can be interpreted as shifts along the lattice of tensor factors assignable to the factor. This encourages the interpretation as a reversal of state function reduction and counterpart of U-process. U-process would be realized at local level in this manner.

4.2.2 HFFs of type III as super-structures providing additional uniqueness?

If the braiding S-matrices are as such highly unique. One could however consider the possibility that they are induced from the automorphisms σ_t for the HFFs of type III restricted to HFFs of type II_∞. If a reduction to inner automorphism in HFF of type III implies same with respect to HFF of type II_∞ and even II_1, they could be trivial for special values of time scaling t assignable to the partons and identifiable as a power of prime p characterizing the parton. This would allow to eliminate incoming and outgoing legs. This elimination would be the counterpart of the division of propagator legs in quantum field theories. Particle masses would however play no role in this process now although the power of padic prime would fix the mass scale of the particle.

4.2.3 Inner automorphisms as universal gauge symmetries?

The continuous outer automorphisms Δ^{it} of HFFs of type III are not completely unique and one can worry about the interpretation of the inner automorphisms. A possible resolution of the worries is that inner automorphisms act as universal gauge symmetries containing various super-conformal symmetries as a special case. For hyper-finite factors of type II_1 in the representation as an infinite tensor power of $M_2(C)$ this would mean that the transformations non-trivial in a finite number of tensor factors only act as analogs of local gauge symmetries. In the representation as a group algebra of S_∞ all unitary transformations acting on a finite number of braid strands act as gauge transformations whereas the infinite powers $P \times P \times ...$, $P \in S_n$, would act as counterparts of global gauge transformations. In particular, the Galois group of the closure of rationals would act as local gauge transformations but diagonally represented finite Galois groups would act like global gauge transformations and periodicity would make possible to have finite braids as space-time correlates without a loss of information.

4.2.4 Unitary isomorphisms between tensor powers of II_1 define vertices

What would be left would be the construction of unitary isomorphisms between the tensor products of the HFFs of type $II_1 \otimes I_n = II_1$ at the partonic 2-surfaces defining the vertices. This would be the only new element added to the construction of braiding S-matrices.

As a matter fact, this element is actually not completely new since it generalizes the fusion rules of conformal field theories, about which standard example is the fusion rule $\phi_i = c_i{}^{jk}\phi_j\phi_k$ for primary fields. These fusion rules would tell how a state of incoming HFF decomposes to the states of tensor product of two outgoing HFFs.

These rules indeed have interpretation in terms of Connes tensor products $\mathcal{M} \otimes_\mathcal{N} ... \otimes_\mathcal{N} \mathcal{M}$ for which the sub-factor $\mathcal{N}$ takes the role of complex numbers [60] so that one has $\mathcal{M}$ becomes $\mathcal{N}$ bimodule and "quantum quantum states" have $\mathcal{N}$ as coefficients instead of complex numbers. In TGD framework this has interpretation as quantum measurement resolution characterized by $\mathcal{N}$ (the group G characterizing leaving the elements of $\mathcal{N}$ invariant defines the measured quantum numbers).

4.2.5 The relationship of TGD to the perturbative QFT

There is infinity of generalized braid diagrams involving exchanges of partons containing braids. The difference with respect to the ordinary Feynman diagrams is that the generalized braid diagrams correspond to the maxima of Kähler function labelled by discrete labels rather than continuously varying positions of interaction vertices. This is true for fixed moduli defined as continuous parameters labelling sectors of the configuration space ("world of classical worlds") and thus Kähler functions.

The reduction to a maximum of Kähler function for a given value of moduli excludes radiative corrections and leaves only the diagrams with the simplest topology allowing the reaction to proceed. For ordinary Feynman diagrams this restriction would mean taking just the minimal Feynman diagram and forgetting all more complex diagrams. This makes sense only if all radiative corrections to the minimal diagram (, which can contain a loop as in photon-photon scattering) vanish. Quantum criticality of TGD Universe would suggest that only the minimal braid diagram is realized as a maximum of Kähler function. The dependence of the scattering amplitudes on p-adic primes labelling particles would give rise to the coupling constant evolution.

1. Several moduli are present

There are several kinds of moduli spaces involved. The maximum of Kähler function depends on positions of tips of light-cones defining the arguments of n-point function since partonic 2-surfaces end up on the boundaries of these light-cones. This means that one cannot avoid a superposition over generalized braid diagrams. Also the presence of conformal moduli for partonic 2-surfaces implies this kind of integral (possibly reducing to a discrete sum in p-adic variant of moduli space).

The generalization of imbedding space inspired by the quantization of Planck constant in terms of Jones inclusions predicts that a preferred point of CP_2 defining a selection of $U(2)$ subgroup of $SU(3)$ defines a moduli space CP_2 analogous to the tip of the light-cone. The closely related moduli fixing quantization axes of Poincare and color quantum numbers should be also present meaning direction of quantization axis of angular momentum and time axis fixing rest system as well as preferred directions isospin and color hyper charge. The choice of tip of the light-cone fixes $SO(3,1)$ as a particular subgroup of Poincare group. Besides this $SO(3)$ subgroup (a particular decomposition $\delta M^4_+ = S^2 \times R_+$ defining rest system) and $SO(2) \subset SO(3)$ subgroup (a light like ray in δM^4_+ defining the direction of angular momentum quantization axis) must be fixed. The choice of $U(2)$ fixes color hypercharge and color isopin is fixed by the choice $U(1) \subset SU(2) \subset SU(3)$. These choices characterize sectors of the configuration space.

The presence of continuous moduli implies that single generalized braid diagram is not enough to describe particle reaction. What is fortunate, is that the unitarity of the fundamental S-matrices assignable to the maxima of Kähler function does not exclude unitary for the direct integral of S-matrices associated with a union of maxima provided that one convolutes the S-matrices with an orthonormal set of wave functions.

2. Relationship to ordinary Feynman diagrams

The following argument suggests that this general picture allows to understand heuristically how braid picture relates to QFT without being equivalent with the standard Feynman diagram type description of scattering.

1. The theory should should produce the counterpart for S-matrix based on multiple Fourier transform of n-point function in M^4. In TGD framework positive/negative energy states correspond to the tips of light-cones $M^4_\pm$ and Feynman diagram correspond classically a connected 4-surface having partons as incoming and outgoing legs. Zero energy states have wave functions with respect to these points and thus wave function in $M = (M^4)^{n_+} \times (M^4)^{n_-}$ with factors corresponding to positive and negative energy particles. One can also consider a discretization of M^4 factors to lattice like structure and p-adicization suggests that this kind of discretization must be allowed. It allows simpler conceptualization but is not essential for the argument to be represented.

2. Assume that the generalized braid diagrams associated with various points of M correspond to braid diagrams assignable to the maxima of Kähler function. These diagrams have the property that their lines start from/end up to the boundary of $\delta M^4_\pm$ associated with a particular point of n-point function. The condition that the partons end to the light-like boundaries of $M^4_\pm$ forces the interpretation of M as a continuous moduli space for the maxima of Kähler function.

3. The conclusion is that S-matrix can be regarded as a direct integral of unitary S-matrices having point of M as parameter. This allows to convolute the generalized braiding S-matrix $S(m)$ with a product of wave functions $f_p(m)$ corresponding to plane waves assignable to particles. Even more, one can allow also other orthonormal wave functions with respect to other, possibly continuous parameters.

This picture should reproduce something resembling Feynman diagrams of QFT approach also quantitatively.

1. Vertices should reduce to unitary isomorphisms of powers of HFFs of type II_1 associated with vertices. Also propagators should emerge from this picture from the unitary braiding matrices associated with the internal lines. Assume that the unitary braiding matrices identifiable as Δ^{it}, where $t \geq 0$ is the maximum value of the time parameter $t_{max} = t(|m_{12}|)$ for the internal line with M^4 length $|m_{12}|$, and Δ is determined by the inherent automorphism characterizing HFF of type III. A weaker condition is that the braiding S-matrix is expressible as exponent of more general Hamiltonian H. Of course, the mere assumption that braiding S-matrix is in question poses strong conditions on H.

2. The naive expectation is that the integral over the values of t forced by the integral over points of M implies effectively integral over $t = f(|m|)$ and gives propagator essentially as $1/(\Delta + i\epsilon)$. Mass squared would correspond to Δ. This would be consistent with the identification of Δ as a scaling of trace in II_∞ factor by a power p^n and with p-adic mass scale hypothesis.

3. This picture assumes that besides the arguments of n-point function also the M^4 distance associated with internal lines varies continuously. Assume that there is just one maximum of Kähler function for a given given point of M. The positions of vertices cannot vary completely freely as they do in quantum field theory. This is crucial for the absence of divergences but means that the idea about propagators as Fourier transforms of Δ^{it} with $t = t(|m_{12}|)$ might fail.

   It however seems safe to assume that for a given value of m and for each internal line of the generalized braid diagram having vertices at $m_1(m)$ and $m_2(m)$ in M^4 the value of t is in the first approximation a linear function of M^4 distance $|m_{12}|$. If this distance various continuously as function of m then the integral over M can give rise to propagators. For 2-particle scattering this is easy to believe. For $n_1 \rightarrow n_2$ scattering the values of $t_i(m_{12})$ are correlated unless the M^4 distance between the scattering events is large enough. This condition means that the field theory description fails for too short distances which is very reasonable.

4. Also massless particles are characterized by a p-adic prime p. It seems that they must correspond to $\Delta = p^{-n}$, $n \rightarrow \infty$, and thus to the limit when the quantum measurement becomes ideal and corresponds to a Jones inclusion with a reduction of trace by an infinite power of p. Thus the quantum measurement resolution could be seen as a characterizer of the particle mass scale. This should also relate to p-adic thermodynamics with temperature $T = 1/n$ implying that real thermal mass squared is of order p^{-n}. Exactly massless particles would emerge only at the limit vanishing p-adic temperature but in the physical situation the natural infrared cutoff due to the dynamics would imply finite mass even in case of photons.

3. What perturbative QFT limit means in TGD framework?

The allowance of continuous families of maxima of Kähler function implies that in S-matrix elements the value of Kähler function at maximum is not eliminated completely anymore. The maximum of K has a maximum for some value m_0 of m. One can expand K in power series around m_0 and a Gaussian integral results in the lowest order approximation and gives as an outcome a factor which is Gaussian of form $exp(K_{ij}p^{i)} \cdot p^{j)})$ in the four-momenta $p^{i)}$ of the particles. This factor has obviously nothing to do with propagator factors. The dependence is qualitatively similar to the dependence of stringy scattering amplitudes on momenta explaining the strong transversal cutoff in hadronic reactions having interpretation in terms of color confinement. K_{ij} should be small for perturbative phase so that Kähler function $K(m)$ must be effectively constant when perturbative QFT type description applies.

4. Quantum classical correspondence at the level of perturbative S-matrix

A longstanding question has been whether quantum-classical correspondence is ideal in these sense that 4-D classical dynamics would be a mere passive correlate of the parton dynamics dictated by light-like randomness or whether it might have some active role. It would seem that in perturbative phase quantum classical correspondence holds in strong sense but not when non-perturbative effects are important. If quantum classical correspondence is ideal, the quantum dynamics of partonic 3-surface should be all that is needed to understand scattering matrix. This option is very elegant computationally.

1. There would be no need to solve complex classical field equations since the only condition on dynamics would be light-likeness. The light-like 3-surfaces corresponding to the maxima of Kähler function could correspond to (possibly preferred) extrema of Chern-Simons action having at most 2-dimensional CP_2 projection.

2. As already found, one cannot restrict the consideration to a single maximum of Kähler function. Kähler function however effectively disappears in the case of perturbative QFT defined as the limit when Kähler function does not depend on positions of the tips of the light-cones associated with n-point function. For instance, a braid diagram involving an exchange of a parton could describe the scattering by the exchange of massless gauge bosons without any information about space-time surface itself since the knowledge about the distance between scattering partons appearing in Coulomb potential would be coded by the presence of the exchanged parton.

5. Quantum classical correspondence at the level of bound states

The methods of perturbative QFT do not provide a satisfactory description for the formation of bound states. Even the understanding of hydrogen atom in QED framework is far from satisfactory, and the failure of perturbative QED can be guessed from the $1/\hbar^2$ type dependence of bound state energy scale on Planck constant. The TGD inspired interpretation has been that bound states cannot be described satisfactorily in terms of boson exchanges since the very process means that two space-like 3-surfaces fuse to form a single space-like 3-surface so that degrees of freedom disappear and this loss of degrees of freedom is not taken into account in perturbative QFT description based on relativistic propagators.

This picture about bound states conforms with the above formulated picture in which non-perturbative effects emerge, when Kähler function cannot be regarded as a constant as a function of positions of interacting particles so that 4-D space-time sheet does not reduce to a mere passive correlate for the purely partonic dynamics. More explicitly, Kähler function as a minimum of Kähler action contains terms identifiable as interaction energies between partons and if this dependence is strong enough, perturbative approximation fails and bound states result. Besides hydrogen atom also hadron collisions seen in sufficiently long time and length scales provide an example about this situation.

The conclusion is that for bound states the quantum classical correspondence cannot make sense in the strongest sense of the word. This is expected to hold true even for gravitation and TGD indeed makes the quite dramatic prediction that dark matter correspond to phases macroscopic in astrophysical length scales and that the presence of dark matter can induce Bohr orbit type quantization of planetary orbits [C7, D6].

4.2.6 Planar algebras and generalized Feynman diagrams

Planar algebras [71] are a very general notion due to Vaughan Jones and a special class of them is known to characterize inclusion sequences of hyper-finite factors of type II_1 [72]. In the following an argument is developed that planar algebras might have interpretation in terms of planar projections of generalized Feynman diagrams (these structures are metrically 2-D by presence of one light-like direction so that 2-D representation is especially natural).

1. Planar algebra very briefly

First a brief definition of planar algebra.

1. One starts from planar k-tangles obtained by putting disks inside a big disk. Inner disks are empty. Big disk contains $2k$ braid strands starting from its boundary and returning back or ending to the boundaries of small empty disks in the interior containing also even number of incoming lines. It is possible to have also loops. Disk boundaries and braid strands connecting them are different objects. A black-white coloring of the disjoint regions of k-tangle is assumed and there are two possible options (photo and its negative). Equivalence of planar tangles under diffeomorphisms is assumed.

2. One can define a product of k-tangles by identifying k-tangle along its outer boundary with some inner disk of another k-tangle. Obviously the product is not unique when the number of inner disks is larger than one. In the product one deletes the inner disk boundary but if one interprets this disk as a vertex-parton, it would be better to keep the boundary.

3. One assigns to the planar k-tangle a vector space V_k and a linear map from the tensor product of spaces V_{k_i} associated with the inner disks such that this map is consistent with the decomposition k-tangles. Under certain additional conditions the resulting algebra gives rise to an algebra characterizing multi-step inclusion of HFFs of type II_1.

4. It is possible to bring in additional structure and in TGD framework it seems necessary to assign to each line of tangle an arrow telling whether it corresponds to a strand of a braid associated with positive or negative energy parton. One can also wonder whether disks could be replaced with closed 2-D surfaces characterized by genus if braids are defined on partonic surfaces of genus g. In this case there is no topological distinction between big disk and small disks. One can also ask why not allow the strands to get linked (as suggested by the interpretation as planar projections of generalized Feynman diagrams) in which case one would not have a planar tangle anymore.

2. General arguments favoring the assignment of a planar algebra to a generalized Feynman diagram

There are some general arguments in favor of the assignment of planar algebra to generalized Feynman diagrams.

1. Planar diagrams describe sequences of inclusions of HFF:s and assign to them a multi-parameter algebra corresponding indices of inclusions. They describe also Connes tensor powers in the simplest situation corresponding to Jones inclusion sequence. Suppose that also general Connes tensor product has a description in terms of planar diagrams. This might be trivial.

2. Generalized vertices identified geometrically as partonic 2-surfaces indeed contain Connes tensor products. The smallest sub-factor N would play the role of complex numbers meaning that due to a finite measurement resolution one can speak only about N-rays of state space and the situation becomes effectively finite-dimensional but non-commutative.

3. The product of planar diagrams could be seen as a projection of 3-D Feynman diagram to plane or to one of the partonic vertices. It would contain a set of 2-D partonic 2-surfaces. Some of them would correspond vertices and the rest to partonic 2-surfaces at future and past directed light-cones corresponding to the incoming and outgoing particles.

4. The question is how to distinguish between vertex-partons and incoming and outgoing partons. If one does not delete the disk boundary of inner disk in the product, the fact that lines arrive at it from both sides could distinguish it as a vertex-parton whereas outgoing partons would correspond to empty disks. The direction of the arrows associated with the lines of planar diagram would allow to distinguish between positive and negative energy partons (note however line returning back).

5. One could worry about preferred role of the big disk identifiable as incoming or outgoing parton but this role is only apparent since by compactifying to say S^2 the big disk exterior becomes an interior of a small disk.

3. A more detailed view

The basic fact about planar algebras is that in the product of planar diagrams one glues two disks with identical boundary data together. One should understand the counterpart of this in more detail.

1. The boundaries of disks would correspond to 1-D closed space-like stringy curves at partonic 2-surfaces along which fermionic anti-commutators vanish.

2. The lines connecting the boundaries of disks to each other would correspond to the strands of number theoretic braids and thus to braidy time evolutions. The intersection points of lines with disk boundaries would correspond to the intersection points of strands of number theoretic braids meeting at the generalized vertex.

   [Number theoretic braid belongs to an algebraic intersection of a real parton 3-surface and its p-adic counterpart obeying same algebraic equations: of course, in time direction algebraicity allows only a sequence of snapshots about braid evolution].

3. Planar diagrams contain lines, which begin and return to the same disk boundary. Also "vacuum bubbles" are possible. Braid strands would disappear or appear in pairwise manner since they correspond to zeros of a polynomial and can transform from complex to real and vice versa under rather stringent algebraic conditions.

4. Planar diagrams contain also lines connecting any pair of disk boundaries. Stringy decay of partonic 2-surfaces with some strands of braid taken by the first and some strands by the second parton might bring in the lines connecting boundaries of any given pair of disks (if really possible!).

5. There is also something to worry about. The number of lines associated with disks is even in the case of k-tangles. In TGD framework incoming and outgoing tangles could have odd number of strands whereas partonic vertices would contain even number of k-tangles from fermion number conservation. One can wonder whether the replacement of boson lines with fermion lines could imply naturally the notion of half-k-tangle or whether one could assign half-k-tangles to the spinors of the configuration space ("world of classical worlds") whereas corresponding Clifford algebra defining HFF of type II_1 would correspond to k-tangles.

4.2.7 More concrete picture about vertices

The understanding about the construction of S-matrix has increased considerably and it is now possible to attack seriously the challenge of writing down the generalized Feynman rules.

1. Generalized Feynman diagrams

Let us first summarize the general picture.

1. Feynman diagrams are replaced with their higher dimensional variants with lines replaced with light-like 3-surfaces identifiable as partonic orbits and with vertices replaced with partonic 2-surfaces along which lines meet. Light-like 3-surfaces corresponding of maxima of Kähler function define generalized Feynman diagrams. There is no summation over the diagrams and each reaction corresponds to single minimal diagram. Quantum dynamics is 2-dimensional in the sense that vertices are defined by partonic 2-surfaces and 3-dimensional in the sense that different maxima of Kähler function defining points of spin glass energy landscape give rise to additional degeneracy essentially due to the presence of light-like direction.

2. In zero energy ontology M-matrix decomposes to a product of square root of density matrix and unitary unitary S-matrix depending parametrically on points of M^4 defining arguments of N-point function in QFT approach. The momentum representation of S-matrix is obtained by taking a Fourier transform of this and is also unitary.

3. S-matrix is a generalization of braiding S-matrix in the sense that one assigns to the incoming/outgoing and internal lines a unitary braiding matrix. To the vertices, where braids replicate, one assigns a unitary isomorphism between tensor product of hyper-finite II_1 factors (HFFs) associated with incoming *resp.* outgoing lines. A crucial element in the construction is that these tensor products are themselves HFFs of type II_1.

4. Since also bosons are fermion-antifermion states located at partonic 2-surfaces, the construction of vertices reduces basically to that in the fermionic Fock space associated with the vertex and the space of small deformations of the generalized Feynman diagram around the maximum of Kähler function. The discrete set of points defining number theoretic strand define the basic unitary S-matrix and these points carry various quantum numbers. The natural assumption is that one can use at the vertex same fermionic basis for all incoming and outgoing lines and that unitary braiding S-matrix associated with lines induces a unitary transformation of basis. Its presence in internal lines gives rise to propagators as one integrates over the positions for tips of future and past light-cones containing at their light-like boundaries incoming and outgoing partons.

One can proceed by making simple guesses about the unitary isomorphism associated with the vertex.

1. The simplest guess would be that vertices involve only simple Fock space inner product. This would be like old fashioned quark model in which the quarks of incoming hadrons are re-arranged to from outgoing hadrons without pair creation or gluon emission. This trial does not work since it would not allow bosons which can be regarded as fermion-antifermion pair with either of them having non-physical helicity. This observation however serves as a valuable guideline.

2. An alternative guess is based on the observation that partonic 2-surface with punctures defined by number theoretical braids is analogous to closed bosonic string emitting particles. This would suggest that unitary S-matrix could be assigned with some conformal field theory

or possibly string model. At least for non-specialist in conformal field theories this approach looks too abstract.

2. Vertices from free field theory defined by the modified Dirac operator

Something more concrete is required and to proceed one can try to apply the mathematical constraints from the basic definition of TGD.

1. The vertices should come out naturally from the modified Dirac action which contains the classical coupling of the gauge potentials (induced spinor connection) to fermions. Hence the modified Dirac action defining the analog of free field theory should appear as a basic building block in the definition of the inner product. Perturbation theory with respect to the induced gauge potential would conform with standard QFT but does not make sense. There is simply no decomposition of the modified Dirac operator D to "free" part and interaction term and the notions of on mass shell and off mass shell state must be reconsidered accordingly.

2. The vacuum expectation for the exponent of the modified Dirac action gives vacuum functional identified as the exponent of Kähler function. When one sandwiches the exponent of Dirac action between many-fermion states, one obtains an inner product analogous to that in free field theory Feynman rules. How however the states are not annihilated by D but are its generalized eigenstates with eigenvalues λ depending on p-adic prime by an overall scaling factor $log(p)$ responsible for the coupling constant evolution. The generalized eigenvalue equation reads as $D\Psi = \lambda t^k \Gamma_k \Psi$, where t^k is the light-like vector defining the tangent vector of partonic 3-surface or its M^4 dual fixed once rest system and quantization axis of angular momentum has been fixed (it is not yet quite clear which option is correct). The notion of generalized eigenmode allows also to define Dirac determinant without giving up the separate conservation of H-chiralities (B and L). The generalized eigenstates are analogs of solutions of massless wave equation in the sense that the square of D annihilates them. Between states created by a monomial of fermionic oscillator operators the inner product reduces to a product of propagators.

3. A strict correspondence with free field theory would require that the incoming and outgoing states correspond to zero modes with $\lambda = 0$ whereas internal lines as off mass shell states would correspond to non-vanishing eigenvalues λ. This assumption is however un-necessary since the four-momentum dependence comes only through the Fourier transform and one can regard all generalized eigenmodes as counterparts of massless modes. The restriction might be also inconsistent with unitarity.

4. For generalized eigenstates of D the modified Dirac propagator $1/D$ reduces to $o^k\Gamma_k/\lambda$. o^k is the light-like M^4 dual of the light-like vector t^k associated with the generalized eigenvalue equation. λ is the generalized eigenvalue of D proportional to $log(p)$. The propagator can be non-vanishing between vacuum and a boson consisting of fermion with physical helicity and anti-fermion with non-physical helicity so that non-trivial boson emission vertices are possible. At first it would seem that the inverse of the generalized eigenvalue λ contributes to the p-adic coupling constant evolution an overall $1/log(p)$ proportionality factor. However, since the inner product of un-normalized "bare" boson states (just fermion pair) is proportional to $1/log(p)$, the normalization of bosonic states cancels this factor so that algebraic number results. Thus fermionic contributions to the vertices are extremely simple since only the matrix $o^k\Gamma_k$ remains. The conclusion made already earlier is that the p-adic coupling constant evolution must be due to the time evolution along parton lines dictated by the modified Dirac operator.

5. The fermionic contribution to the vertex says nothings about gauge couplings. All gauge coupling strengths must be proportional to the RG invariant Kähler coupling strength α_K, which can emerge only from the functional integral over small fluctuations around maximum of Kähler function K when the operator inverse of the covariant configuration space Kähler metric defining propagator is contracted between bosonic vector fields generating Kac-Moody and super-canonical symmetries in terms of which the deformation of the partonic 3-surface can be expressed. Obviously the configuration space spinor fields representing bosonic states must vanish at the maximum of K: otherwise coupling strength is of order unity. Geometrically this means that the maxima of K correspond to fixed points of these isometries.

3. Number theoretical constraints

The condition that S-matrix elements are algebraic numbers is an additional powerful guideline.

1. The most straightforward manner to guarantee that S-matrix elements are algebraic numbers is that vertex factors and propagators are separately algebraic numbers. $log(p)$ -factors are obviously problematic number theoretically but normalization of the Fock space inner products cancels these factors. Thus coupling constant evolution can come only from the unitary time evolution with respect to the light-like coordinate of propagator lines dictated by the modified Dirac operator. Fermionic oscillator operators suffer a non-trivial unitary transformation depending on the p-adic prime p since (expressing it schematically) e^{iHt} is replaced by p^{iHt}.

2. The fundamental number theoretic conjecture is that the numbers p^{s_n}, where $s_n = 1/2 + iy_n$ correspond to non-trivial zeros of Riemann zeta (or of more general zetas possibly involved: as a matter fact zetas can assigned with the values of the generalized eigenvalues of the modified Dirac operator at points of defining number theoretic braid [C1]), are algebraic numbers. If this is the case, then also the products and sums involving finite number of nontrivial zeros of zeta are algebraic numbers and define a commutative algebra. The effect of the unitary time evolution operator should be expressible as an element of this algebra. Also larger algebraic extensions can be considered.

3. A simplified picture is provided by the dynamics of free number theoretic Hamiltonian for which eigenstates are labelled by primes and energy eigenvalues are given by $E_p = log(p)$. Time evolution gives rise to phase factors $exp(iE_pt) = p^{it}$ which are algebraic numbers in given extension of rationals for some quantized values of light-like coordinate t. If the conjectures about zeros of zeta hold true this is achieved if t is a linear combination of imaginary parts of zeros of zeta with integer coefficients: $t = \sum_n k(n)y_n$.

4.2.8 Comparison with the earlier views about S-matrix

It must be made clear that the proposed view about S-matrix is not completely identical with the earlier proposals for how to construct S-matrix.

1. In the recent proposal one starts from a minimal generalized braid diagram (not stringy diagram!) formed from partonic 2-surfaces containing exchanges and annihilations. The minimal diagrams correspond to maxima of Kähler function in spin energy landscape and are labelled by discrete labels and continous moduli such as arguments of n-point function and conformal moduli of partonic 2-surfaces. Quantum classical correspondence in strongest possible sense requires that one can assume a complete localization around single maximum and this is not possible in moduli space which implies non-perturbative effects such as formation of bound states.

2. In [C6] the idea that there is an infinite number of generalized braid diagrams which are equivalent, was developed. In particular, generalized braid diagrams with loops are equivalent to diagrams without loops. This view does not conform with spin energy landscape picture as such but is replaced with an equivalent picture in which Kähler function allows only the maximum corresponding to the minimal diagram making possible the reaction. What is nice in the new view is that it has concrete interpretation in terms of classical dynamics.

3. The zero energy ontology based view [C2] is that that S-matrix can be constructed by introducing single central 3-surface containing partons at which incoming and outgoing partons meet but do so only along their partonic 2-surfaces. The reason was that the assignment of single space-time surface to the entire transition would not conform with the classical conservation laws. One can criticize this view for not being completely consistent with quantum-classical correspondence. One can also ask why not allow arbitrary number of central 3-surfaces.

4.3 Finite measurement resolution: from S-matrix to M-matrix

The finite resolution of quantum measurement leads in TGD framework naturally to the notion of quantum S-matrix for which elements have values in sub-factor of HFF rather than being complex numbers. It is still possible to satisfy generalized unitarity condition but one can also consider the possibility that only probabilities are conserved.

4.3.1 Jones inclusion as characterizer of finite measurement resolution at the level of S-matrix

Jones inclusion $\mathcal{N} \subset \mathcal{M}$ characterizes naturally finite measurement resolution. This means following things.

1. Complex rays of state space resulting usually in an ideal state function reduction are replaced by $\mathcal{N}$-rays since $\mathcal{N}$ defines the measurement resolution and takes the role of complex numbers in ordinary quantum theory so that non-commutative quantum theory results. Non-commutativity corresponds to a finite measurement resolution rather than something exotic occurring in Planck length scales. The quantum Clifford algebra $\mathcal{M}/\mathcal{N}$ creates physical states modulo resolution. The fact that $\mathcal{N}$ takes the role of gauge algebra suggests that it might be necessary to fix a gauge by assigning to each element of $\mathcal{M}/\mathcal{N}$ a unique element of $\mathcal{M}$. Quantum Clifford algebra with fractal dimension $\beta = \mathcal{M} : \mathcal{N}$ creates physical states having interpretation as quantum spinors of fractal dimension $d = \sqrt{\beta}$. Hence direct connection with quantum groups emerges.

2. The notions of unitarity, hermiticity, and eigenvalue generalized. The elements of unitary and hermitian matrices and $\mathcal{N}$-valued. Eigenvalues are Hermitian elements of $\mathcal{N}$ and thus correspond entire spectra of Hermitian operators. The mutual non-commutativity of eigenvalues guarantees that it is possible to speak about state function reduction for quantum spinors. In the simplest case of a 2-component quantum spinor this means that second component of quantum spinor vanishes in the sense that second component of spinor annihilates physical state and second acts as element of $\mathcal{N}$ on it. The non-commutativity of spinor components implies correlations between then and thus fractal dimension is smaller than 2.

3. The intuition about ordinary tensor products suggests that one can decompose Tr in $\mathcal{M}$ as

$$Tr_{\mathcal{M}}(X) = Tr_{\mathcal{M}/\mathcal{N}}(Tr_{\mathcal{N}}(X)) \ . \quad (4)$$

Suppose one has fixed gauge by selecting basis $|r_k\rangle$ for $\mathcal{M}/\mathcal{N}$. In this case one expects that operator in $\mathcal{M}$ defines an operator in $\mathcal{M}/\mathcal{N}$ by a projection to the preferred elements of $\mathcal{M}$.

$$\langle r_1|X|r_2\rangle = \langle r_1|Tr_{\mathcal{N}}(X)|r_2\rangle \ . \tag{5}$$

4. Scattering probabilities in the resolution defined by $\mathcal{N}$ are obtained in the following manner. The scattering probability between states $|r_1\rangle$ and $|r_2\rangle$ is obtained by summing over the final states obtained by the action of $\mathcal{N}$ from $|r_2\rangle$ and taking the analog of spin average over the states created in the similar from $|r_1\rangle$. $\mathcal{N}$ average requires a division by $Tr(P_{\mathcal{N}}) = 1/\mathcal{M} : \mathcal{N}$ defining fractal dimension of $\mathcal{N}$. This gives

$$p(r_1 \rightarrow r_2) = \mathcal{M} : \mathcal{N} \times \langle r_1|Tr_{\mathcal{N}}(SP_{\mathcal{N}}S^\dagger)|r_2\rangle \ . \tag{6}$$

This formula is consistent with probability conservation since one has

$$\sum_{r_2} p(r_1 \rightarrow r_2) = \mathcal{M} : \mathcal{N} \times Tr_N(SS^\dagger) = \mathcal{M} : \mathcal{N} \times Tr(P_N) = 1 \ . \tag{7}$$

5. Unitary at the level of $\mathcal{M}/\mathcal{N}$ is obtained if the unit operator Id for $\mathcal{M}$ can be decomposed into an analog of tensor product for the unit operators of $\mathcal{M}/\mathcal{N}$ and $\mathcal{N}$.

4.3.2 Quantum M-matrix

The description of finite measurement resolution in terms of Jones inclusion $\mathcal{N} \subset \mathcal{M}$ seems to boil down to a simple rule. Replace ordinary quantum mechanics in complex number field C with that in $\mathcal{N}$. This means that the notions of unitarity, hermiticity, Hilbert space ray, etc.. are replaced with their $\mathcal{N}$ counterparts.

The full M-matrix in $\mathcal{M}$ should be reducible to a finite-dimensional quantum S-matrix in the state space generated by quantum Clifford algebra $\mathcal{M}/\mathcal{N}$ which can be regarded as a finite-dimensional matrix algebra with non-commuting $\mathcal{N}$-valued matrix elements. This suggests that full M-matrix can be expressed as M-matrix. In the case of quantum S-matrix tge $\mathcal{N}$-valued elements would satisfy $\mathcal{N}$-unitarity conditions.

Physical intuition also suggests that the transition probabilities defined by quantum S-matrix must be commuting hermitian $\mathcal{N}$-valued operators inside every row and column. The traces of these operators give $\mathcal{N}$-averaged transition probabilities. The eigenvalue spectrum of these Hermitian gives more detailed information about details below experimental resolution. $\mathcal{N}$-hermicity and commutativity pose powerful additional restrictions on the S-matrix.

Quantum M-matrix defines $\mathcal{N}$-valued entanglement coefficients between quantum states with $\mathcal{N}$-valued coefficients. How this affects the situation? The non-commutativity of quantum spinors has a natural interpretation in terms of fuzzy state function reduction meaning that quantum spinor corresponds effectively to a statistical ensemble which cannot correspond to pure state. Does this mean that predictions for transition probabilities must be averaged over the ensemble defined by "quantum quantum states"?

4.3.3 Does Connes tensor product fix the allowed M-matrices?

Hyperfinite factors of type II_1 and the inclusion $\mathcal{N} \subset \mathcal{M}$ inclusions have been proposed to define quantum measurement theory with a finite measurement resolution characterized by $\mathcal{N}$ and with complex rays of state space replaced with $\mathcal{N}$ rays. What this really means is far from clear.

1. Naively one expects that matrices whose elements are elements of $\mathcal{N}$ give a representation for M. Now however unit operator has unit trace and one cannot visualize the situation in terms of matrices in case of $\mathcal{M}$ and $\mathcal{N}$.

2. The state space with $\mathcal{N}$ resolution would be formally $\mathcal{M}/\mathcal{N}$ consisting of $\mathcal{N}$ rays. For $\mathcal{M}/\mathcal{N}$ one has finite-D matrices with non-commuting elements of $\mathcal{N}$. In this case quantum matrix elements should be multiplets of selected elements of $\mathcal{N}$, **not all** possible elements of $\mathcal{N}$. One cannot therefore think in terms of the tensor product of $\mathcal{N}$ with $\mathcal{M}/\mathcal{N}$ regarded as a finite-D matrix algebra.

3. What does this mean? Obviously one must pose a condition implying that $\mathcal{N}$ action commutes with matrix action just like C: this poses conditions on the matrices that one can allow. Connes tensor product [43] does just this.

The starting point is the Jones inclusion sequence

$$\mathcal{N} \subset \mathcal{M} \subset \mathcal{M} \otimes_N \mathcal{M} ...$$

Here $\mathcal{M} \otimes_N \mathcal{M}$ is Connes tensor product which can be seen as elements of the ordinary tensor product commuting with $\mathcal{N}$ action so that $\mathcal{N}$ indeed acts like complex numbers in $\mathcal{M}$. $\mathcal{M}/\mathcal{N}$ is in this picture represented with $\mathcal{M}$ in which operators defined by Connes tensor products of elements of $\mathcal{M}$. The replacement $\mathcal{M} \rightarrow \mathcal{M}/\mathcal{N}$ corresponds to the replacement of the tensor product of elements of $\mathcal{M}$ defining matrices with Connes tensor product.

One can try to generalize this picture to zero energy ontology.

1. $\mathcal{M} \otimes_\mathcal{N} \mathcal{M}$ would be generalized by $\mathcal{M}_+ \otimes_\mathcal{N} \mathcal{M}_-$. Here $\mathcal{M}_+$ would create positive energy states and $\mathcal{M}_-$ negative energy states and $\mathcal{N}$ would create zero energy states in some shorter time scale resolution: this would be the precise meaning of finite measurement resolution.

2. Connes entanglement with respect to $\mathcal{N}$ would define a non-trivial and unique recipe for constructing M-matrices as a generalization of S-matrices expressible as products of square root of density matrix and unitary S-matrix but it is not how clear how many M-matrices this allows. In any case M-matrices would depend on the triplet $(\mathcal{N}, \mathcal{M}_+, \mathcal{M}_-)$ and this would correspond to p-adic length scale evolution giving replacing coupling constant evolution in TGD framework. Thermodynamics would enter the fundamental quantum theory via the square root of density matrix.

3. The defining condition for the variant of the Connes tensor product proposed here has the following equivalent forms

$$MN = N^*M \quad , \quad N = M^{-1}N^*M \quad , \quad N^* = MNM^{-1} \quad . \tag{8}$$

   If M_1 and M_2 are two M-matrices satisfying the conditions then the matrix $M_{12} = M_1M_2^{-1}$ satisfies the following equivalent conditions

$$N = M_{12}NM_{12}^{-1} \quad , \quad [N, M_{12}] = 0 \quad . \tag{9}$$

Jones inclusions with $\mathcal{M} : \mathcal{N} \leq 4$ are irreducible which means that the operators commuting with $\mathcal{N}$ consist of complex multiples of identity. Hence one must have $M_{12} = 1$ so that M-matrix is unique in this case. For $\mathcal{M} : \mathcal{N} > 4$ the complex dimension of commutator algebra of $\mathcal{N}$ is 2 so that M-matrix depends should depend on single complex parameter. The dimension of the commutator algebra associated with the inclusion gives the number of parameters appearing in the M-matrix in the general case.

When the commutator has complex dimension $d > 1$, the representation of $\mathcal{N}$ in $\mathcal{M}$ is reducible: the matrix analogy is the representation of elements of $\mathcal{N}$ as direct sums of d representation matrices. M-matrix is a direct sum of form $M = a_1M_1 \oplus a_2M_2 \oplus ...$, where M_i are unique. The condition $\sum_i; |a_i|^2 = 1$ is satisfied and *-commutativity holds in each summand separately.

There are several questions. Could M_i define unique universal unitary S-matrices in their own blocks? Could the direct sum define a counterpart of a statistical ensemble? Could irreducible inclusions correspond to pure states and reducible inclusions to mixed states? Could different values of energy in thermodynamics and of the scaling generator L_0 in p-adic thermodynamics define direct summands of the inclusion? The values of conserved quantum numbers for the positive energy part of the state indeed naturally define this kind of direct direct summands.

It must be of course noticed that reducibility and thermodynamics emerge naturally also in another sense since a direct sum of HFFs of type II_1 is what one expects. The radial conformal weights associated light-cone boundary and X^3_l would indeed naturally label the factors in the direct sum.

4. Zero energy ontology is a key element of this picture and the most compelling argument for zero energy ontology is the possibility of describing coherent states of Cooper pairs without giving up fermion number, charge, etc. conservation and automatic emerges of length scale dependent notion of quantum numbers (quantum numbers identified as those associated with positive energy factor).

To sum up, interactions would be an outcome of a finite measurement resolution and at the never-achievable limit of infinite measurement resolution the theory would be free: this would be the counterpart of asymptotic freedom.

4.4 How p-adic coupling constant evolution and p-adic length scale hypothesis emerge from quantum TGD proper?

What p-adic coupling constant evolution really means has remained for a long time more or less open. The progress made in the understanding of the S-matrix of theory has however changed the situation dramatically.

4.4.1 M-matrix and coupling constant evolution

The final breakthrough in the understanding of p-adic coupling constant evolution came through the understanding of S-matrix, or actually M-matrix defining entanglement coefficients between positive and negative energy parts of zero energy states in zero energy ontology [C2]. M-matrix has interpretation as a "complex square root" of density matrix and thus provides a unification of thermodynamics and quantum theory. S-matrix is analogous to the phase of Schrödinger amplitude multiplying positive and real square root of density matrix analogous to modulus of Schrödinger amplitude.

The notion of finite measurement resolution realized in terms of inclusions of von Neumann algebras allows to demonstrate that the irreducible components of M-matrix are unique and possesses huge symmetries in the sense that the hermitian elements of included factor $\mathcal{N} \subset \mathcal{M}$ defining the measurement resolution act as symmetries of M-matrix, which suggests a connection with integrable quantum field theories.

It is also possible to understand coupling constant evolution as a discretized evolution associated with time scales T_n, which come as octaves of a fundamental time scale: $T_n = 2^n T_0$. Number theoretic universality requires that renormalized coupling constants are rational or at most algebraic numbers and this is achieved by this discretization since the logarithms of discretized mass scale appearing in the expressions of renormalized coupling constants reduce to the form $log(2^n) = nlog(2)$ and with a proper choice of the coefficient of logarithm log(2) dependence disappears so that rational number results.

4.4.2 p-Adic coupling constant evolution

One can wonder how this picture relates to the earlier hypothesis that p-adic length coupling constant evolution is coded to the hypothesized $log(p)$ normalization of the eigenvalues of the modified Dirac operator D. There are objections against this normalization. $log(p)$ factors are not number theoretically favored and one could consider also other dependencies on p. Since the eigenvalue spectrum of D corresponds to the values of Higgs expectation at points of partonic 2-surface defining number theoretic braids, Higgs expectation would have $log(p)$ multiplicative dependence on p-adic length scale, which does not look attractive.

Is there really any need to assume this kind of normalization? Could the coupling constant evolution in powers of 2 implying time scale hierarchy $T_n = 2^n T_0$ induce p-adic coupling constant evolution and explain why p-adic length scales correspond to $L_p \propto \sqrt{p}R$, $p \simeq 2^k$, R CP_2 length scale? This looks attractive but there is a problem. p-Adic length scales come as powers of $\sqrt{2}$ rather than 2 and the strongly favored values of k are primes and thus odd so that $n = k/2$ would be half odd integer. This problem can be solved.

1. The observation that the distance traveled by a Brownian particle during time t satisfies $r^2 = Dt$ suggests a solution to the problem. p-Adic thermodynamics applies because the partonic 3-surfaces X^2 are as 2-D dynamical systems random apart from light-likeness of their orbit. For CP_2 type vacuum extremals the situation reduces to that for a one-dimensional random light-like curve in M^4. The orbits of Brownian particle would now correspond to light-like geodesics γ_3 at X^3. The projection of γ_3 to a time=constant section $X^2 \subset X^3$ would define the 2-D path γ_2 of the Brownian particle. The M^4 distance r between the end points of γ_2 would be given $r^2 = Dt$. The favored values of t would correspond to $T_n = 2^n T_0$ (the full light-like geodesic). p-Adic length scales would result as $L^2(k) = DT(k) = D2^k T_0$ for $D = R^2/T_0$. Since only CP_2 scale is available as a fundamental scale, one would have $T_0 = R$ and $D = R$ and $L^2(k) = T(k)R$.

2. p-Adic primes near powers of 2 would be in preferred position. p-Adic time scale would not relate to the p-adic length scale via $T_p = L_p/c$ as assumed implicitly earlier but via $T_p = L_p^2/R_0 = \sqrt{p}L_p$, which corresponds to secondary p-adic length scale. For instance, in the case of electron with $p = M_{127}$ one would have $T_{127} = .1$ second which defines a fundamental biological rhythm. Neutrinos with mass around .1 eV would correspond to $L(169) \simeq 5\ \mu$m (size of a small cell) and $T(169) \simeq 1. \times 10^4$ years. A deep connection between elementary particle physics and biology becomes highly suggestive.

3. In the proposed picture the p-adic prime $p \simeq 2^k$ would characterize the thermodynamics of the random motion of light-like geodesics of X^3 so that p-adic prime p would indeed be an inherent property of X^3.

4. The fundamental role of 2-adicity suggests that the fundamental coupling constant evolution and p-adic mass calculations could be formulated also in terms of 2-adic thermodynamics. With a suitable definition of the canonical identification used to map 2-adic mass squared values to real numbers this is possible, and the differences between 2-adic and p-adic thermodynamics are extremely small for large values of for $p \simeq 2^k$. 2-adic temperature must be chosen to be $T_2 = 1/k$ whereas p-adic temperature is $T_p = 1$ for fermions. If the canonical identification is defined as

$$\sum_{n\geq 0} b_n 2^n \rightarrow \sum_{m\geq 1} 2^{-m+1} \sum_{(k-1)m\leq n<km} b_n 2^n \ ,$$

it maps all 2-adic integers $n < 2^k$ to themselves and the predictions are essentially same as for p-adic thermodynamics. For large values of $p \simeq 2^k$ 2-adic real thermodynamics with $T_R = 1/k$ gives essentially the same results as the 2-adic one in the lowest order so that the interpretation in terms of effective 2-adic/p-adic topology is possible.

5 Number theoretic braids and S-matrix

Number theoretical braids assignable to the partonic 2-surfaces define representations of Galois groups assignable to them in HFFs of type II_1 so that number theoretical quantum numbers become part of physics. This leads to a bundle of ideas about the realization of Langlands program in TGD framework [27]. The most unexpected implication is that topological quantum computation including copying of information by braid replication and its transfer by particle exchange could be present already at elementary particle level.

5.1 Generalization of the notion of imbedding space

The hypothesis that Planck constant is quantized having in principle all possible rational values but with some preferred values implying algebraically simple quantum phases has been one of the main ideas of TGD during last years. The mathematical realization of this idea leads to a profound generalization of the notion of imbedding space obtained by gluing together infinite number of copies of imbedding space along common 4-dimensional intersection. The hope was that this generalization could explain charge fractionization but this does not seem to be the case. This problem led to a further generalization of the imbedding space and this is what I want to discussed below.

5.1.1 The original view about generalized imbedding space

The original generalization of imbedding space was basically following. Take imbedding space $H = M^4 \times CP_2$. Choose submanifold $M^2 \times S^2$, where S^2 is homologically non-trivial geodesic sub-manifold of CP_2. The motivation is that for a given choice of Cartan algebra of Poincare algebra (translations in time direction and spin quantization axis plus rotations in plane orthogonal to this plane plus color hypercharge and isospin) this sub-manifold remains invariant under the transformations leaving the quantization axes invariant.

Form spaces $\hat{M}^4 = M^4 \backslash M^2$ and $\hat{CP}_2 = CP_2 \backslash S^2$ and their Cartesian product. Both spaces have a hole of co-dimension 2 so that the first homotopy group is Z. From these spaces one can construct an infinite hierarchy of factor spaces $\hat{M}^4/G_a$ and $\hat{CP}_2/G_b$, where G_a is a discrete group of SU(2) leaving quantization axis invariant. In case of Minkowski factor this means that the group in question acts essentially as a combination reflection and to rotations around quantization axes of angular momentum. The generalized imbedding space is obtained by gluing all these spaces together along $M^2 \times S^2$.

The hypothesis is that Planck constant is given by the ratio $\hbar/hbar_0 = (n_/n_b)$, where n_i is the order of maximal cyclic subgroups of G_i. The hypothesis states also that the covariant metric of the Minkowski factor is scaled by the factor $(n_a/n_b)^2$. One must take care of this in the gluing procedure. One can assign to the field bodies describing both self interactions and interactions between physical systems definite sector of generalized imbedding space characterized partially by the Planck constant. The phase transitions changing Planck constant correspond to tunnelling between different sectors of the imbedding space.

5.1.2 Fractionization of quantum numbers is not possible if only factor spaces are allowed

The original idea was that the proposed modification of the imbedding space could explain naturally phenomena like quantum Hall effect involving fractionization of quantum numbers like spin and charge. This does not however seem to be the case. $G_a \times G_b$ implies just the opposite if these quantum numbers are assigned with the symmetries of the imbedding space. For instance, quantization unit for orbital angular momentum becomes n_a where Z_{n_a} is the maximal cyclic subgroup of G_a.

One can however imagine of obtaining fractionization at the level of imbedding space for space-time sheets, which are analogous to multi-sheeted Riemann surfaces (say Riemann surfaces associated with $z^{1/n}$ since the rotation by 2π understood as a homotopy of M^4 lifted to the space-time sheet is a non-closed curve. Continuity requirement indeed allows fractionization of the orbital quantum numbers and color in this kind of situation.

5.1.3 Both covering spaces and factor spaces are possible

The observation above stimulates the question whether it might be possible in some sense to replace H or its factors by their multiple coverings.

This is certainly not possible for M^4, CP_2, or H since their fundamental groups are trivial. On the other hand, the fixing of quantization axes implies a selection of the sub-space $H_4 = M^2 \times S^2 \subset M^4 \times CP_2$, where S^2 is a geodesic sphere of CP_2. $\hat{M}^4 = M^4 \backslash M^2$ and $\hat{CP}_2 = CP_2 \backslash S^2$ have fundamental group Z since the codimension of the excluded sub-manifold is equal to two and homotopically the situation is like that for a punctured plane. The exclusion of these sub-manifolds defined by the choice of quantization axes could naturally give rise to the desired situation.

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2. There are two geodesic spheres in CP_2. Which one should choose or are both possible?

   i) For the homologically non-trivial one corresponding to cosmic strings, the isometry group is $SU(2) \subset SU(3)$. The homologically trivial one S^2 corresponds to vacuum extremals and has isometry group $SO(3) \subset SU(3)$. The natural question is which one should choose. At quantum criticality the value of Planck constant is undetermined. The vacuum extremal would be a natural choice from the point of view of quantum criticality since in this case the value of Planck constant does not matter at all and one would obtain a direct connection with the vacuum degeneracy.

ii) The choice of the homologically non-trivial geodesic sphere as a quantum critical sub-manifold would conform with the previous guess that $\mathcal{M} : \mathcal{N} = 4$ corresponds to cosmic strings. It is however questionable whether the ill-definedness of the Planck constant is consistent with the non-vacuum extremal property of cosmic strings unless one assumes that for partonic 3-surfaces $X^3 \subset M^2 \times S^2$ the effective degrees of freedom reduce to mere topological ones.

3. The covering spaces in question would correspond to the Cartesian products $\hat{M}^4{}_{n_a} \times \hat{CP}_{2n_b}$ of the covering spaces of $\hat{M}^4$ and $\hat{CP}_2$ by Z_{n_a} and Z_{n_b} with fundamental group is $Z_{n_a} \times Z_{n_b}$. One can also consider extension by replacing M^2 and S^2 with its orbit under G_a (say tedrahedral, octahedral, or icosahedral group). The resulting space will be denoted by $\hat{M}^4 \hat{\times} G_a$ *resp.* $\hat{CP}_2 \hat{\times} G_b$.

4. One expects the discrete subgroups of $SU(2)$ emerge naturally in this framework if one allows the action of these groups on the singular sub-manifolds M^2 or S^2. This would replace the singular manifold with a set of its rotated copies in the case that the subgroups have genuinely 3-dimensional action (the subgroups which corresponds to exceptional groups in the ADE correspondence). For instance, in the case of M^2 the quantization axes for angular momentum would be replaced by the set of quantization axes going through the vertices of tedrahedron, octahedron, or icosahedron. This would bring non-commutative homotopy groups into the picture in a natural manner.

5. Also the orbifolds $\hat{M}^4/G_a \times \hat{CP}_2/G_b$ can be allowed as also the spaces $\hat{M}^4/G_a \times (\hat{CP}_2 \hat{\times} G_b)$ and $(\hat{M}^4 \hat{\times} G_a) \times \hat{CP}_2/G_b$. Hence the previous framework would generalize considerably by the allowance of both coset spaces and covering spaces.

There are several non-trivial questions related to the details of the gluing procedure and phase transition as motion of partonic 2-surface from one sector of the imbedding space to another one.

1. How the gluing of copies of imbedding space at $M^2 \times CP_2$ takes place? It would seem that the covariant metric of M^4 factor proportional to $\hbar^2$ must be discontinuous at the singular manifold since only in this manner the idea about different scaling factor of M^4 metric can make sense. This is consistent with the identical vanishing of Chern-Simons action in $M^2 \times S^2$.

2. One might worry whether the phase transition changing Planck constant means an instantaneous change of the size of partonic 2-surface in M^4 degrees of freedom. This is not the case. Light-likeness in $M^2 \times S^2$ makes sense only for surfaces $X^1 \times D^2 \subset M^2 \times S^2$, where X^1 is light-like geodesic. The requirement that the partonic 2-surface X^2 moving from one sector of H to another one is light-like at $M^2 \times S^2$ irrespective of the value of Planck constant requires that X^2 has single point of M^2 as M^2 projection. Hence no sudden change of the size X^2 occurs.

3. A natural question is whether the phase transition changing the value of Planck constant can occur purely classically or whether it is analogous to quantum tunnelling. Classical non-vacuum extremals of Chern-Simons action have two-dimensional CP_2 projection to homologically non-trivial geodesic sphere S^2_I. The deformation of the entire S^2_I to homologically trivial geodesic sphere S^2_{II} is not possible so that only combinations of partonic 2-surfaces with vanishing total homology charge (Kähler magnetic charge) can in principle move from

sector to another one, and this process involves fusion of these 2-surfaces such that CP_2 projection becomes single homologically trivial 2-surface. A piece of a non-trivial geodesic sphere S^2_I of CP_2 can be deformed to that of S^2_{II} using 2-dimensional homotopy flattening the piece of S^2 to curve. If this homotopy cannot be chosen to be light-like, the phase transitions changing Planck constant take place only via quantum tunnelling. Obviously the notions of light-like homotopies (cobordisms) and classical light-like homotopies (cobordisms) are very relevant for the understanding of phase transitions changing Planck constant.

5.1.4 Do factor spaces and coverings correspond to the two kinds of Jones inclusions?

What could be the interpretation of these two kinds of spaces?

1. Jones inclusions appear in two varieties corresponding to $\mathcal{M}:\mathcal{N}<4$ and $\mathcal{M}:\mathcal{N}=4$ and one can assign a hierarchy of subgroups of $SU(2)$ with both of them. In particular, their maximal Abelian subgroups Z_n label these inclusions. The interpretation of Z_n as invariance group is natural for $\mathcal{M}:\mathcal{N}<4$ and it naturally corresponds to the coset spaces. For $\mathcal{M}:\mathcal{N}=4$ the interpretation of Z_n has remained open. Obviously the interpretation of Z_n as the homology group defining covering would be natural.

2. $\mathcal{M}:\mathcal{N}=4$ should correspond to the allowance of cosmic strings and other analogous objects. Does the introduction of the covering spaces bring in cosmic strings in some controlled manner? Formally the subgroup of $SU(2)$ defining the inclusion is $SU(2)$ would mean that states are $SU(2)$ singlets which is something non-physical. For covering spaces one would however obtain the degrees of freedom associated with the discrete fiber and the degrees of freedom in question would not disappear completely and would be characterized by the discrete subgroup of $SU(2)$.

   For anyons the non-trivial homotopy of plane brings in non-trivial connection with a flat curvature and the non-trivial dynamics of topological QFTs. Also now one might expect similar non-trivial contribution to appear in the spinor connection of $\hat{M}^2\hat{\times}G_a$ and $\hat{CP}_2\hat{\times}G_b$. In conformal field theory models non-trivial monodromy would correspond to the presence of punctures in plane.

3. For factor spaces the unit for quantum numbers like orbital angular momentum is multiplied by n_a *resp.* n_b and for coverings it is divided by this number. These two kind of spaces are in a well defined sense obtained by multiplying and dividing the factors of $\hat{H}$ by G_a *resp.* G_b and multiplication and division are expected to relate to Jones inclusions with $\mathcal{M}:\mathcal{N}<4$ and $\mathcal{M}:\mathcal{N}=4$, which both are labelled by a subset of discrete subgroups of SU(2).

4. The discrete subgroups of $SU(2)$ with fixed quantization axes possess a well defined multiplication with product defined as the group generated by forming all possible products of group elements as elements of $SU(2)$. This product is commutative and all elements are idempotent and thus analogous to projectors. Trivial group G_1, two-element group G_2 consisting of reflection and identity, the cyclic groups Z_p, p prime, and tedrahedral, octahedral, and icosahedral groups are the generators of this algebra.

   By commutativity one can regard this algebra as an 11-dimensional module having natural numbers as coefficients ("rig"). The trivial group G_1, two-element group G_2¡ generated by reflection, and tedrahedral, octahedral, and icosahedral groups define 5 generating elements for this algebra. The products of groups other than trivial group define 10 units for this algebra so that there are 11 units altogether. The groups Z_p generate a structure analogous to natural numbers acting as analog of coefficients of this structure. Clearly, one has effectively 11-dimensional commutative algebra in 1-1 correspondence with the 11-dimensional "half-lattice" N^{11} (N denotes natural numbers). Leaving away reflections, one obtains N^7. The projector representation suggests a connection with Jones inclusions. An interesting question concerns the possible Jones inclusions assignable to the subgroups containing infinitely manner elements. Reader has of course already asked whether dimensions 11, 7 and their difference 4 might relate somehow to the mathematical structures of M-theory with 7 compactified dimensions. One could introduce generalized configuration space spinor fields in the configuration space labelled by sectors of H with given quantization axes. By introducing Fourier transform in N^{11} one would formally obtain an infinite-component field in 11-D space.

5. How do the Planck constants associated with factors and coverings relate? One might argue that Planck constant defines a homomorphism respecting the multiplication and division (when possible) by G_i. If so, then Planck constant in units of $\hbar_0$ would be equal to n_a/n_b for $\hat{H}/G_a \times G_b$ option and n_b/n_a for $\hat{H}\hat{\times}(G_a \times G_b)$ with obvious formulas for hybrid cases. This option would put M^4 and CP_2 in a very symmetric role and allow much more flexibility in the identification of symmetries associated with large Planck constant phases.

5.1.5 Is factorizable QFT in M^2 associated with quantum criticality?

2-D QFT:s in M^2 are almost trivial and generalize topological QFT:s associated with braids. Planck constant depends on the sector of generalized imbedding space and is ill-defined in $M^2 \times S^2_{II}$ which thus represents quantum critical sub-manifold and must be a vacuum extremal. TGD should reduce to a pure topological QFT for partons moving in this sub-manifold. Since partons are 2-dimensional, one would have essentially light-like geodesics as allowed solutions of field equations and thus classical theory of free massless particles. Hence factorizing QFT would be a natural description for the quantum critical dynamics at quantum criticality. This conforms also with the idea that intentional action takes place at quantum criticality.

5.2 Physical representations of Galois groups

It would be highly desirable to have concrete physical realizations for the action of finite Galois groups. TGD indeed provides two kinds of realizations of this kind. For both options there are good hopes about the unification of number theoretical and geometric Galois programs obtained by replacing permutations with braiding homotopies and by a discretization of the continuous situation to a finite number theoretic braids having finite Galois groups as automorphisms.

5.2.1 Number theoretical braids and the representations of finite Galois groups as outer automorphisms of braid group algebra

Number theoretical braids [E1, C1, C2] are in a central role in the formulation of quantum TGD based on general philosophical ideas which might apply to both physics and mathematical cognition and, one might hope, also to a good mathematics.

An attractive idea inspired by the notion of the number theoretical braid is that the symmetric group S_n might act on roots of a polynomial represented by the strands of braid and could thus be replaced by braid group.

The basic philosophy underlying quantum TGD is the notion of finite resolution, both the finite resolution of quantum measurement and finite cognitive resolution [C1, C2]. The basic implication is discretization at space-time level and finite-dimensionality of all mathematical structures which can be represented in the physical world. At space-time level the discretization means that the data involved with the definition of S-matrix comes from a subset of a discrete set of points in the intersection of real and p-adic variants of partonic 2-surface obeying same algebraic equations. Note that a finite number of braids could be enough to code for the information needed to reconstruct the entire partonic 2-surface if it is given by polynomial or rational function having coefficients as algebraic numbers. Entire configuration space of 3-surfaces would be discretized in this picture. Also the reduction of the infinite braid to a finite one would conform with the spontaneous symmetry breaking S_∞ to diagonally imbedded finite Galois group imbedded diagonally.

1. Two objections

Langlands correspondence assumes the existence of finite-dimensional representations of $Gal(\overline{Q}/Q)$. In the recent situation this encourages the idea that the restrictions of mathematical cognition allow to realize only the representations of $Gal(\overline{Q}/Q)$ reducing in some sense to representations for finite Galois groups. There are two counter arguments against the idea.

1. It is good to start from a simple abelian situation. The abelianization of $G(\overline{A}/Q)$ must give rise to multiplicative group of adeles defined as $\hat{Z} = \prod_p Z_p^\times$ where $Z_p^\times$ corresponds to the multiplicative group of invertible p-adic integers consisting of p-adic integers having p-adic norm equal to one. This group results as the inverse limit containing the information about subgroup inclusion hierarchies resulting as sequences $Z^\times/(1+pZ)^\times \subset Z^\times/(1+p^2Z)^\times \subset ..$ and expressed in terms factor groups of multiplicative group of invertible p-adic integers. Z_∞/A_∞ must give the group $\prod_p Z_p^\times$ as maximal abelian subgroup of Galois group. All smaller abelian subgroups of S_∞ would correspond to the products of subgroups of $\hat{Z}^\times$ coming as $Z_p^\times/(1+p^nZ)^\times$. Representations of finite cyclic Galois groups would be obtained by representing trivially the product of a commutator group with a subgroup of $\hat{Z}$. Thus one would obtain finite subgroups of the maximal abelian Galois group at the level of representations as effective Galois groups. The representations would be of course one-dimensional.

   One might hope that the representations of finite Galois groups could result by a reduction of the representations of S_∞ to $G = S_\infty/H$ where H is normal subgroup of S_∞. Schreier-Ulam theorem [55] however implies that the only normal subgroup of S_∞ is the alternating subgroup A_∞. Since the braid group B_∞ as a special case reduces to S_∞ there is no hope of obtaining finite-dimensional representations except abelian ones.

2. The identification of $Gal(\overline{Q}/Q) = S_\infty$ is not consistent with the finite-dimensionality in the case of complex representations. The irreducible unitary representations of S_n are in one-one correspondence with partitions of n objects. The direct numerical inspection based on the formula for the dimension of the irreducible representation of S_n in terms of Yang tableau [56] suggests that the partitions for which the number r of summands differs from $r = 1$ or $r = n$ (1-dimensional representations) quite generally have dimensions which are at least of order n. If d-dimensional representations corresponds to representations in $GL(d, C)$, this means that important representations correspond to dimensions $d \to \infty$ for S_∞.

Both these arguments would suggest that Langlands program is consistent with the identification $Gal(\overline{F}, F) = S_\infty$ only if the representations of $Gal(\overline{Q}, Q)$ reduce to those for finite Galois subgroups via some kind of symmetry breaking.

2. Diagonal imbedding of finite Galois group to S_∞ as a solution of problems

The idea is to imbed the Galois group acting as inner automorphisms diagonally to the m-fold Cartesian power of S_n imbedded to S_∞. The limit $m \to \infty$ gives rise to outer automorphic action since the resulting group would not be contained in S_∞. Physicist might prefer to speak about number theoretic symmetry breaking $Gal(\overline{Q}/Q) \to G$ implying that the representations are irreducible only in finite Galois subgroups of $Gal(\overline{Q}/Q)$. The action of finite Galois group G is indeed analogous to that of global gauge transformation group which belongs to the completion of the group of local gauge transformations. Note that G is necessarily finite.

5.2.2 Representation of finite Galois groups as outer automorphism groups of HFFs

Any finite group G has a representation as outer automorphisms of a hyper-finite factor of type II_1 (briefly HFF in the sequel) and this automorphism defines sub-factor $\mathcal{N} \subset \mathcal{M}$ with a finite value of index $\mathcal{M} : \mathcal{N}$ [60]. Hence a promising idea is that finite Galois groups act as outer automorphisms of the associated hyper-finite factor of type II_1.

More precisely, sub-factors (containing Jones inclusions as a special case) $\mathcal{N} \subset \mathcal{M}$ are characterized by finite groups G acting on elements of $\mathcal{M}$ as outer automorphisms and leave the elements of $\mathcal{N}$ invariant whereas finite Galois group associated with the field extension K/L act as automorphisms of K and leave elements of L invariant. For finite groups the action as outer automorphisms is unique apart from a conjugation in von Neumann algebra. Hence the natural idea is that the finite subgroups of $Gal(\overline{Q}/Q)$ have outer automorphism action in group algebra of $Gal(\overline{Q}/Q)$ and that the hierarchies of inclusions provide a representation for the hierarchies of algebraic extensions. Amusingly, the notion of Jones inclusion was originally inspired by the analogy with field extensions [60]!

It must be emphasized that the groups defining sub-factors can be extremely general and can represent much more than number theoretical information understood in the narrow sense of the word. Even if one requires that the inclusion is determined by outer automorphism action of group G uniquely, one finds that any amenable, in particular compact [59], group defines a unique sub-factor by outer action [60]. It seems that practically any group works if uniqueness condition is given up.

The TGD inspired physical interpretation is that compact groups would serve as effective gauge groups defining measurement resolution by determining the measured quantum numbers. Hence the physical states differing by the action of $\mathcal{N}$ elements which are G singlets would not be indistinguishable from each other in the resolution used. The physical states would transform according to the finite-dimensional representations in the resolution defined by G.

The possibility of Lie groups as groups defining inclusions raises the question whether hyper-finite factors of type II_1 could mimic any gauge theory and one might think of interpreting gauge groups as Galois groups of the algebraic structure of this kind of theories. Also Kac-Moody algebras emerge naturally in this framework as will be discussed, and could also have an interpretation as Galois algebras for number theoretical dynamical systems obeying dynamics dictated by conformal field theory. The infinite hierarchy of infinite rationals in turn suggests a hierarchy of groups S_∞ so that even algebraic variants of Lie groups could be interpreted as Galois groups. These arguments would suggest that HFFs might be kind of Universal Math Machines able to mimic any respectable mathematical structure.

5.2.3 Number theoretic braids and unification of geometric and number theoretic Langlands programs

The notion of number theoretic braid has become central in the attempts to fuse real physics and p-adic physics to single coherent whole. Number theoretic braid leads to the discretization of quantum physics by replacing the stringy amplitudes defined over curves of partonic 2-surface with amplitudes involving only data coded by points of number theoretic braid. The discretization of quantum physics could have counterpart at the level of geometric Langlands program [28, 37], whose discrete version would correspond to number theoretic Galois groups associated with the points of number theoretic braid. The extension to braid group would mean that the global homotopic information is not lost.

1. Number theoretic braids belong to the intersection of real and p-adic partonic surface

The points of number theoretic braid belong to the intersection of the real and p-adic variant of partonic 2-surface consisting of rationals and algebraic points in the extension used for p-adic numbers. The points of braid have same projection on an algebraic point of the geodesic sphere of $S^2 \subset CP_2$ belonging to the algebraic extension of rationals considered (the reader willing to understand the details can consult [C1]).

There are two different geodesic spheres in CP_2 and the homologically trivial geodesic sphere S^2_{II} is the most natural choice from the point of view of the generalized imbedding space since $M^2 \times S^2_{II}$, which defines the intersection of all sectors of H, is vacuum extremal so that ill-definedness of Planck constant does not matter. Note that also the M^4 part of the metric is discontinuous at $M^2 \times S^2_{II}$.

One can argue that algebraicity condition is not strong enough and gives too many points unless one introduces a cutoff in some manner. Since TQFT like theory can naturally assigned with the partonic 2-surfaces in $M^2 \times S^2_{II}$, the natural identification of the intersection points of number theoretical braids with $\delta M^4_\pm \times CP_2$ would be as the intersection of the 2-D CP_2 projection of the partonic 2-surface in $\delta M^4_\pm \times CP_2$ with S^2_{II}. In the generic case the intersection would consist of discrete points and for non-vacuum extremals this would certainly be the case. The intersection should consist of algebraic points allowing also p-adic interpretation: the condition that CP_2 projection is an algebraic surface is a necessary condition for this.

The points of braid are obtained as solutions of polynomial equation and thus one can assign to them a Galois group permuting the points of the braid. In this case finite Galois group could be realized as left or right translation or conjugation in S_∞ or in braid group.

To make the notion of number theoretic braid more concrete, suppose that the complex coordinate w of $\delta M^4_\pm$ is expressible as a polynomial of the complex coordinate z of CP_2 geodesic sphere and the radial light-like coordinate r of $\delta M^4_\pm$ is obtained as a solution of polynomial equation $P(r, z, w) = 0$. By substituting w as a polynomial $w = Q(z, r)$ of z and r this gives polynomial equation $P(r, z, Q(z, r)) = 0$ for r for a given value of z. Only real roots can be accepted. Local Galois group (in a sense different as it is used normally in literature) associated with the algebraic point of S^2 defining the number theoretical braid is thus well defined.

If the partonic 2-surface involves all roots of an irreducible polynomial, one indeed obtains a braid for each point of the geodesic sphere $S^2 \subset CP_2$. In this case the action of Galois group is naturally a braid group action realized as the action on induced spinor fields and configuration space spinors.

The choice of the points of braid as points common to the real and p-adic partonic 2-surfaces would be unique so that the obstacle created by the fact that the finite Galois group as function of point of S^2 fluctuates wildly (when some roots become rational Galois group changes dramatically: the simplest example is provided by $y - x^2 = 0$ for which Galois group is Z_2 when y is not a square of rational and trivial group if y is rational).

2. Modified Dirac operator assigns to partonic 2-surface a unique prime p which could define l-adic representations of Galois group

The overall scaling of the eigenvalue spectrum of the modified Dirac operator assigns to the partonic surface a unique p-adic prime p which physically corresponds to the p-adic length scale which appears in the discrete coupling constant evolution [C1, C5]. One can solve the roots of the the resulting polynomial also in the p-adic number field associated with the partonic 2-surface by the modified Dirac equation and find the Galois group of the extension involved. The p-adic Galois group, known as local Galois group in literature, could be assigned to the p-adic variant of partonic surface and would have naturally l-adic representation, most naturally in the p-adic variant of the group algebra of S_∞ or B_∞ or equivalently in the p-adic variant of infinite-dimensional Clifford algebra. There are however physical reasons to believe that infinite-dimensional Clifford algebra does not depend on number field. Restriction to an algebraic number based group algebra therefore suggests itself. Hence, if one requires that the representations involve only algebraic numbers, these representation spaces might be regarded as equivalent.

3. Problems

There are however problems.

1. The triviality of the action of Galois group on the entire partonic 2-surface seems to destroy the hopes about genuine representations of Galois group.

2. For a given partonic 2-surface there are several number theoretic braids since there are several algebraic points of geodesic sphere S^2 at which braids are projected. What happens if the Galois groups are different? What Galois group should one choose?

A possible solution to both problems is to assign to each braid its own piece X_k^2 of the partonic 2-surface X^2 such that the deformations X^2 can be non-trivial only in X_k^2. This means separation of modular degrees of freedom to those assignable to X_k^2 and to "center of mass" modular degrees of freedom assignable to the boundaries between X_k^2. Only the piece X_k^2 associated with the k^{th} braid would be affected non-trivially by the Galois group of braid. The modular invariance of the conformal field theory however requires that the entire quantum state is modular invariant under the modular group of X^2. The analog of color confinement would take place in modular degrees of freedom. Note that the region containing braid must contain single handle at least in order to allow representations of $SL(2,C)$ (or $Sp(2g,Z)$ for genus g).

As already explained, in the general case only the invariance under the subgroup $\Gamma_0(N)$ [35] of the modular group $SL(2,Z)$ can be assumed for automorphic representations of $GL(2,R)$ [30, 28, 26]. This is due to the fact that there is a finite set of primes (prime ideals in the algebra of integers), which are ramified [30]. Ramification means that their decomposition to a product of prime ideals of the algebraic extension of Q contains higher powers of these prime ideals: $p \rightarrow (\prod_k P_k)^e$ with $e > 1$. The congruence group is fixed by the integer $N = \prod_k p^{n_k}$ known as conductor coding the set of exceptional primes which are ramified.

The construction of modular forms in terms of representations of $SL(2,R)$ suggests that it is possible to replace $\Gamma_0(N)$ by the congruence subgroup $\Gamma(N)$, which is normal subgroup of $SL(2,R)$ so that $G_1 = SL(2,Z)/\Gamma$ is group. This would allow to assign to individual braid regions carrying single handle well-defined G_1 quantum numbers in such a manner that entire state would be G_1 singlet.

Physically this means that the separate regions of the partonic 2-surface each containing one braid strand cannot correspond to quantum states with full modular invariance. Elementary particle vacuum functionals [F1] defined in the moduli space of conformal equivalence classes of partonic 2-surface must however be modular invariant, and the analog of color confinement in modular degrees of freedom would take place.

5.3 Galois groups and definition of vertices

The idea about reducing the construction of S-matrix to the level of number theoretic braids means a deviation from stringy picture. One could imagine that strings are replaced by number theoretic braids in the sense that the finite measurement resolution represented by the inclusion implies that the complex coordinate z of the geodesic sphere of CP_2 becomes effectively non-commutative variable and the modes of the induced spinor field depending on z commute only at the points defining the number theoretic braid [C1, C2, A9].

The construction of S-matrix in braid picture involves two pieces.

1. The first piece of S-matrix should corresponds to an S-matrix characterizing braiding and would be assigned with each number theoretic braid associated with incoming and outgoing partons.

2. The challenge is to understand what happens in generalized vertices identifiable as partonic 2-surfaces at which incoming and outgoing partonic 3-surfaces meet along their ends. The unitary isomorphism between HFFs of type II_1 already discussed is the first principle answer to this question whereas braid picture should allow to gain detailed insights about the vertex.

5.3.1 The time evolution of number theoretic braids

Consider first on the general level what can happen to the number theoretic braids in the dynamics defined by light-like randomness of partonic 2-surfaces.

1. Assume that the number theoretic braid at $\delta M^4_\pm \times CP_2$ is defined in terms of zeros of a polynomial defining the light-like radial coordinate associated with the partonic 2-surface at a given algebraic point of the geodesic sphere of CP_2 belonging to intersection of real and p-adic partonic 2-surface. This polynomial is assumed to be irreducible.

2. Light-likeness of the partonic 3-surfaces is the only constraint to the time evolution of the partonic 2-surface. It is however not quite clear what this really means.

   i) First of all, it is not clear how to define the dynamics of braiding if braid consists of algebraic points. Continuous motion for braid points on S^2 does not respect the algebraic character of these points unless on assumes that the preferred coordinates of CP_2 suffer a suitable color rotation during braiding guaranteing that coordinates remain constant. Also the continuous motion of the strands of braid seems to be inconsistent with the property of being number theoretic braid. It would seem that number theoretic picture forces to replace continuous homotopy behind braiding with a discrete one, kind of series of discrete snapshot about the orbit of braid points defined by the algebraic intersection points with p-adic parton orbit. This of course is consistent with the fact that braid group is discrete.

   ii) This means that during the time evolution from $\delta M^4_\pm$ to the partonic 2-surface defining the vertex the partonic 2-surface could evolve in such a manner that the number of braid strands is not preserved. Even if the polynomial representation is preserved by the time evolution it could happen that the pair of real roots coincides and transforms to a complex pair of roots so that Galois group changes. Polynomial could also split to a product of irreducible polynomials. It could even occur that the expressibility of the light-like radial coordinate r as a polynomial is not preserved in intermediate states. This would mean disappearance or appearance of braid strands. There would be however an upper bound for the number of strands. This picture would be consistent with the replacement of braid with tangle.

   iii) Even more dramatic changes can happen if polynomial is given up in the "virtual" intermediate states during the travel of partonic 2-surface. If this is the case then the degree of polynomial could increase during the propagation. Hence the number of strands of braid could change. This picture is not favored by the very attractive idea that even the configuration space (the world of classical worlds) reduces to a discrete space by the assumption that surfaces are algebraic for a suitable choice of coordinates with coefficients of rational functions assumed to be rational numbers.

5.3.2 Does DNA replication have counterpart at the level of fundamental physics?

The fundamental question is what happens in the vertices represented by the partonic 2-surface? The study of the 3-vertex which might well represent the generic situation makes it clear that the incoming braid is replicated in a manner very much analogous to the replication of DNA. Braid replication would make it possible to make copies of classical representations of number theoretic information. Quantum representation of information by irreducible representations of Galois group would not be replicable since each incoming braid would correspond to its own irreducible representation and the choice of these representations would not be a fully deterministic process.

This process would mean that partons/elementary particles might be much more complex objects than they are though to be and would in some sense possess the analog of genome. In [E9] DNA has been proposed to act as topological quantum computer using braids. Also elementary particles could be seen as a kind of quantum computers and their "genome" would code at least the initial data for for the topological quantum computation program. Information processing involves besides computation also copying of data and its transfer. Particle interaction vertices would realize the copying of data and particle exchanges its communication whereas quantum computation would be carried by parton with quantum program identified with its execution (light-like 3-surfaces can be regarded either as states or processes).

5.3.3 Connection with TGD inspired model of genetic code

One of the TGD inspired models of genetic code suggests that the connection with genetic code and DNA replication might be much deeper than one might think first.

1. The model for genetic code discussed in [L4] assumes that codons are labelled by 64 5-adic integers with no vanishing 5-ary digit (these integers n are in the range $[31, 124]$. The crucial observation is that the number of primes in this range is 20, the number of aminoacids.

2. An assumption inspired by the symmetries of the code is that 5-adic thermodynamics dynamics for the partitions of the 16 2-letter codons (two first nucleotides) labelled by integers $n_{2)}$ in the interval $[6, 24]$ determines the p-adic prime labelling corresponding amino-adic as the prime for which the number theoretic entropy associated with the partition function for which Hamiltonian depends only on the number of summands in the partition of $n_{2)} = \sum n_k$ is minimal (and negative!).

3. The model of code actually combines a couple of other approaches to the genetic code so that very strong constraints result. In particular, the map of codons to aminoacids is determined to a rather high degree. The numerical scanning of 20 per cent of a huge number of available candidates allows two solutions to the conditions for genetic code.

4. The basic question concerns the interpretation of the thermodynamics for the partitions of $n_{2)}$. One possible interpretation is based on the observation is that the irreducible representations of symmetric group S_n are in one-one correspondence with the partitions of n objects. Hence the thermodynamics could correspond to a thermodynamics for irreducible representations of $S_{n_{2)}}$ associated with a number theoretic braid with $n_{2)}$ strands and corresponding to the generic Galois group $S_{n_{2)}})$.

5.3.4 Fusion rules number theoretically

The idea that partonic 2-surfaces decompose into regions, one for each number theoretic braid, and that the number theoretic braids define representations of Galois groups permuting the strands of the braid as automorphisms in HFFs of type II_1 suggests a fresh approach to the understanding of vertices. Kind of fusion rules would certainly be in question and the the interpretation as representations of Galois groups might allow to deduce information about the fusion rules using symmetry arguments.

The first thing to notice is that in the vertex the number theoretic braids coincide so that the Galois groups G associated with incoming and outgoing braids are identical. Only in the situation in which polynomial defining G becomes reducible it might occur that some of incoming lines corresponds to a group which is product of subgroups of G but this situation is not expected to be generic.

Suppose that the number theoretic braids define irreducible projective representations of the Galois group G associated with the braid in HFF of type II_1 as outer automorphisms via diagonal imbedding of G. In vertex one expects that fusion rules for these representations mean extraction of singlet from the tensor product of these representations. This suggest a picture very similar to the fusion of representations of $SU(2)_q$ in the fusion rules of WZW theory which also can be understood in terms of braiding. If one accepts generalized McKay correspondence [27], then the fusion rules for Galois group could have representation in terms of fusion groups for Lie group associated with it by generalized McKay correspondence.

5.4 Could McKay correspondence and Jones inclusions relate to each other?

The understanding of Langlands correspondence for general reductive Lie groups in TGD framework seems to require some physical mechanism allowing the emergence of these groups in TGD based physics. The physical idea would be that quantum dynamics of TGD is able to emulate the dynamics of any gauge theory or even stringy dynamics of conformal field theory having Kac-Moody type symmetry and that this emulation relies on quantum deformations induced by finite measurement resolution described in terms of Jones inclusions of sub-factors characterized by group G leaving elements of sub-factor invariant. Finite measurement resolution would would result simply from the fact that only quantum numbers defined by the Cartan algebra of G are measured.

There are good reasons to expect that infinite Clifford algebra has the capacity needed to realize representations of an arbitrary Lie group. It is indeed known that that any quantum group characterized by quantum parameter which is root of unity or positive real number can be assigned to Jones inclusion [60]. For $q = 1$ this would gives ordinary Lie groups. In fact, all amenable groups define unique sub-factor and compact Lie groups are amenable ones.

It was so called McKay correspondence [64] which originally stimulated the idea about TGD as an analog of Universal Turing machine able to mimic both ADE type gauge theories and theories with ADE type Kac-Moody symmetry algebra. This correspondence and its generalization might also provide understanding about how general reductive groups emerge. In the following I try to cheat the reader to believe that the tensor product of representations of SU(2) Lie algebras for Connes tensor powers of $\mathcal{M}$ could induce ADE type Lie algebras as quantum deformations for the direct sum of n copies of $SU(2)$ algebras This argument generalizes also to the case of other compact Lie groups.

5.4.1 About McKay correspondence

McKay correspondence [64] relates discrete finite subgroups of $SU(2)$ ADE groups. A simple description of the correspondences is as follows [64].

1. Consider the irreps of a discrete subgroup $G \subset SU(2)$ which correspond to irreps of G and can be obtained by restricting irreducible representations of $SU(2)$ to those of G. The irreducible representations of $SU(2)$ define the nodes of the graph.

2. Define the lines of graph by forming a tensor product of any of the representations appearing in the diagram with a doublet representation which is always present unless the subgroup is 2-element group. The tensor product regarded as that for $SU(2)$ representations gives representations $j - 1/2$, and $j + 1/2$ which one can decompose to irreducibles of G so that a branching of the graph can occur. Only branching to two branches occurs for subgroups yielding extended ADE diagrams. For the linear portions of the diagram the spins of corresponding SU(2) representations increase linearly as $.., j, j + 1/2, j + 1, ...$

One obtains extended Dynkin diagrams of ADE series representing also Kac-Moody algebras giving A_n, D_n, E_6, E_7, E_8. Also A_∞ and $A_{-\infty,\infty}$ are obtained in case that subgroups are infinite. The Dynkin diagrams of non-simply laced groups B_n ($SO(2n+1)$), C_n (symplectic group $Sp(2n)$ and quaternionic group $Sp(n)$), and exceptional groups G_2 and F_4 are not obtained.

ADE Dynkin diagrams labelling Lie groups instead of Kac-Moody algebras and having one node less, do not appear in this context but appear in the classification of Jones inclusions for $\mathcal{M} : \mathcal{N} < 4$. As a matter fact, ADE type Dynkin diagrams appear in very many contexts as one can learn from John Baez's This Week's Finds [65].

1. The classification of integral lattices in Rn having a basis of vectors whose length squared equals 2
2. The classification of simply laced semisimple Lie groups.
3. The classification of finite sub-groups of the 3-dimensional rotation group.
4. The classification of simple singularities . In TGD framework these singularities could be assigned to origin for orbifold CP_2/G, $G \subset SU(2)$.
5. The classification of tame quivers.

5.4.2 Principal graphs for Connes tensor powers $\mathcal{M}$

The thought provoking findings are following.

1. The so called principal graphs characterizing $\mathcal{M} : \mathcal{N} = 4$ Jones inclusions for $G = SU(2)$ are extended Dynkin diagrams characterizing ADE type affine (Kac-Moody) algebras. D_n is possible only for $n \geq 4$.
2. $\mathcal{M} : \mathcal{N} < 4$ Jones inclusions correspond to ordinary ADE type diagrams for a subset of simply laced Lie groups (all roots have same length) A_n ($SU(n)$), D_{2n} ($SO(2n)$), and E_6 and E_8. Thus D_{2n+1} ($SO(2n+2)$) and E_7 are not allowed. For instance, for $G = S_3$ the principal graph is not D_3 Dynkin diagram.

The conceptual background behind principal diagram is necessary if one wants to understand the relationship with McKay correspondence.

1. The hierarchy of higher commutations defines an invariant of Jones inclusion $\mathcal{N} \subset \mathcal{M}$. Denoting by $\mathcal{N}'$ the commutant of $\mathcal{N}$ one has sequences of horizontal inclusions defined as $C = \mathcal{N}' \cap \mathcal{N} \subset \mathcal{N}' \cap \mathcal{M} \subset \mathcal{N}' \cap \mathcal{M}^1 \subset ...$ and $C = \mathcal{M}' \cap \mathcal{M} \subset \mathcal{M}' \cap \mathcal{M}^1 \subset ....$ There is also a sequence of vertical inclusions $\mathcal{M}' \cap \mathcal{M}^k \subset \mathcal{N}' \cap \mathcal{M}^k$. This hierarchy defines a hierarchy of Temperley-Lieb algebras [62] assignable to a finite hierarchy of braids. The commutants in the hierarchy are direct sums of finite-dimensional matrix algebras (irreducible representations) and the inclusion hierarchy can be described in terms of decomposition of irreps of k^{th} level to irreps of $(k-1)^{th}$ level irreps. These decomposition can be described in terms of Bratteli diagrams [64, 63].
2. The information provided by infinite Bratteli diagram can be coded by a much simpler bi-partite diagram having a preferred vertex. For instance, the number of $2k$-loops starting from it tells the dimension of k^{th} level algebra. This diagram is known as principal graph.

Principal graph emerges also as a concise description of the fusion rules for Connes tensor powers of $\mathcal{M}$.

1. It is natural to decompose the Connes tensor powers [64] $\mathcal{M}_k = \mathcal{M} \otimes_{\mathcal{N}} ... \otimes_{\mathcal{N}} \mathcal{M}$ to irreducible $\mathcal{M} - \mathcal{M}$, $\mathcal{N} - \mathcal{M}$, $\mathcal{M} - \mathcal{N}$, or $\mathcal{N} - \mathcal{N}$ bi-modules. If $\mathcal{M} : \mathcal{N}$ is finite this decomposition involves only finite number of terms. The graphical representation of these decompositions gives rise to Bratteli diagram.

2. If $\mathcal{N}$ has finite depth the information provided by Bratteli diagram can be represented in nutshell using principal graph. The edges of this bipartite graph connect $\mathcal{M} - \mathcal{N}$ vertices to vertices describing irreducible $\mathcal{N} - \mathcal{N}$ representations resulting in the decomposition of $\mathcal{M} - \mathcal{N}$ irreducibles. If this graph is finite, $\mathcal{N}$ is said to have finite depth.

5.4.3 A mechanism assigning to tensor powers Jones inclusions ADE type gauge groups and Kac-Moody algebras

The proposal made for the first time in [A9] is that in $\mathcal{M} : \mathcal{N} < 4$ case it is possible to construct ADE representations of gauge groups or quantum groups and in $\mathcal{M} : \mathcal{N} = 4$ using the additional degeneracy of states implied by the multiple-sheeted cover $H \rightarrow H/G_a \times G_b$ associated with space-time correlates of Jones inclusions. Either G_a or G_b would correspond to G. In the following this mechanism is articulated in a more refined manner by utilizing the general properties of generators of Lie-algebras understood now as a minimal set of elements of algebra from which the entire algebra can be obtained by repeated commutation operator (I have often used " Lie algebra generator" as an synonym for "Lie algebra element"). This set is finite also for Kac-Moody algebras.

1. Two observations

The explanation to be discussed relies on two observations.

1. McKay correspondence for subgroups of G ($\mathcal{M} : \mathcal{N} = 4$) *resp.* its variants ($\mathcal{M} : \mathcal{N} < 4$) and its counterpart for Jones inclusions means that finite-dimensional irreducible representations of allowed $G \subset SU(2)$ label both the Cartan algebra generators and the Lie (Kac-Moody) algebra generators of t_+ and t_- in the decomposition $g = h \oplus t_+ \oplus t_-$, where h is the Lie algebra of maximal compact subgroup.

2. Second observation is related to the generators of Lie-algebras and their quantum counterparts (see Appendix for the explicit formulas for the generators of various algebras considered). The observation is that each Cartan algebra generator of Lie- and quantum group algebras, corresponds to a triplet of generators defining an SU(2) sub-algebra. The Cartan algebra of affine algebra contains besides Lie group Cartan algebra also a derivation d identifiable as an infinitesimal scaling operator L_0 measuring the conformal weight of the Kac-Moody generators. d is exceptional in that it does not give rise to a triplet. It corresponds to the preferred node added to the Dynkin diagram to get the extended Dynkin diagram.

2. Is ADE algebra generated as a quantum deformation of tensor powers of $SU(2)$ Lie algebras representations?

The ADE type symmetry groups could result as an effect of finite quantum resolution described by inclusions of HFFs in TGD inspired quantum measurement theory.

1. The description of finite resolution typically leads to quantization since complex rays of state space are replaced as $\mathcal{N}$ rays. Hence operators, which would commute for an ideal resolution cease to do so. Therefore the algebra $SU(2) \otimes ... \otimes SU(2)$ characterized by n mutually commuting triplets, where n is the number of copies of $SU(2)$ algebra in the original situation and identifiable as quantum algebra appearing in $\mathcal{M}$ tensor powers with $\mathcal{M}$ interpreted as $\mathcal{N}$

module, could suffer quantum deformation to a simple Lie algebra with $3n$ Cartan algebra generators. Also a deformation to a quantum group could occur as a consequence.

2. This argument makes sense also for discrete groups $G \subset SU(2)$ since the representations of G realized in terms of configuration space spinors extend to the representations of SU(2) naturally.

3. Arbitrarily high tensor powers of $\mathcal{M}$ are possible and one can wonder why only finite-dimensional Lie algebra results. The fact that $\mathcal{N}$ has finite depth as a sub-factor means that the tensor products in tensor powers of $\mathcal{N}$ are representable by a finite Dynkin diagram. Finite depth could thus mean that there is a periodicity involved: the kn tensor powers decomposes to representations of a Lie algebra with $3n$ Cartan algebra generators. Thus the additional requirement would be that the number of tensor powers of $\mathcal{M}$ is multiple of n.

3. Space-time correlate for the tensor powers $\mathcal{M} \otimes_{\mathcal{N}} ... \otimes_{\mathcal{N}} \mathcal{M}$

By quantum classical correspondence there should exist space-time correlate for the formation of tensor powers of $\mathcal{M}$ regarded as $\mathcal{N}$ module. A concrete space-time realization for this kind of situation in TGD would be based on n-fold cyclic covering of H implied by the $H \to H/G_a \times G_b$ bundle structure in the case of say G_b. The sheets of the cyclic covering would correspond to various factors in the n-fold tensor power of $SU(2)$ and one would obtain a Lie algebra, affine algebra or its quantum counterpart with n Cartan algebra generators in the process naturally. The number n for space-time sheets would be also a space-time correlate for the finite depth of $\mathcal{N}$ as a factor.

Configuration space spinors could provide fermionic representations of $G \subset SU(2)$. The Dynkin diagram characterizing tensor products of representations of $G \subset SU(2)$ with doublet representation suggests that tensor products of doublet representations associated with n sheets of the covering could realize the Dynkin diagram.

Singlet representation in the Dynkin diagram associated with irreps of G would not give rise to an SU(2) sub-algebra in ADE Lie algebra and would correspond to the scaling generator. For ordinary Dynkin diagram representing gauge group algebra scaling operator would be absent and therefore also the exceptional node. Thus the difference between $(\mathcal{M} : \mathcal{N} = 4)$ and $(\mathcal{M} : \mathcal{N} < 4)$ cases would be that in the Kac-Moody group would reduce to gauge group $\mathcal{M} : \mathcal{N} < 4$ because Kac-Moody central charge k and therefore also Virasoro central charge resulting in Sugawara construction would vanish.

4. Do finite subgroups of $SU(2)$ play some role also in $\mathcal{M} : \mathcal{N} = 4$ case?

One can ask wonder the possible interpretation for the appearance of extended Dynkin diagrams in $(\mathcal{M} : \mathcal{N} = 4)$ case. Do finite subgroups $G \subset SU(2)$ associated with extended Dynkin diagrams appear also in this case. The formal analog for $H \to G_a \times G_b$ bundle structure would be $H \to H/G_a \times SU(2)$. This would mean that the geodesic sphere of CP_2 would define the fiber. The notion of number theoretic braid meaning a selection of a discrete subset of algebraic points of the geodesic sphere of CP_2 suggests that $SU(2)$ actually reduces to its subgroup G also in this case.

5. Why Kac-Moody central charge can be non-vanishing only for $\mathcal{M} : \mathcal{N} = 4$?

From the physical point of view the vanishing of Kac-Moody central charge for $\mathcal{M} : \mathcal{N} < 4$ is easy to understand. If parton corresponds to a homologically non-trivial geodesic sphere, space-time surface typically represents a string like object so that the generation of Kac-Moody central extension would relate directly to the homological non-triviality of partons. For instance, cosmic strings are string like objects of form $X^2 \times Y^2$, where X^2 is minimal surface of M^2 and Y^2 is a holomorphic sub-manifold of CP_2 reducing to a homologically non-trivial geodesic sphere in the simplest situation. A conjecture that deserves to be shown wrong is that central charge k is proportional/equal to the absolute value of the homology (Kähler magnetic) charge h.

6. More general situation

McKay correspondence generalizes also to the case of subgroups of higher-dimensional Lie groups [64]. The argument above makes sense also for discrete subgroups of more general compact Lie groups H since also they define unique sub-factors. In this case, algebras having Cartan algebra with nk generators, where n is the dimension of Cartan algebra of H, would emerge in the process. Thus there are reasons to believe that TGD could emulate practically any dynamics having gauge group or Kac-Moody type symmetry. An interesting question concerns the interpretation of non-ADE type principal graphs associated with subgroups of SU(2).

7. Flavor groups of hadron physics as a support for HFF?

The deformation assigning to an n-fold tensor power of representations of Lie group G with k-dimensional Cartan algebra a representation of a Lie group with nk-dimensional Cartan algebra could be also seen as a dynamically generated symmetry. If quantum measurement is characterized by the choice of Lie group G defining measured quantum numbers and defining Jones inclusion characterizing the measurement resolution, the measurement process itself would generate these dynamical symmetries. Interestingly, the flavor symmetry groups of hadron physics cannot be justified from the structure of the standard model having only electro-weak and color group as fundamental symmetries. In TGD framework flavor group $SU(n)$ could emerge naturally as a fusion of n quark doublets to form a representation of $SU(n)$.

5.5 Farey sequences, Riemann hypothesis, tangles, and TGD

Farey sequences allow an alternative formulation of Riemann Hypothesis and subsequent pairs in Farey sequence characterize so called rational 2-tangles. In TGD framework Farey sequences relate very closely to dark matter hierarchy, which inspires *"Platonia as the best possible world in the sense that cognitive representations are optimal"* as the basic variational principle of mathematics. This variational principle supports RH.

Possible TGD realizations of tangles, which are considerably more general objects than braids, are considered. One can assign to a given rational tangle a rational number a/b and the tangles labelled by a/b and c/d are equivalent if $ad - bc = \pm 1$ holds true. This means that the rationals in question are neighboring members of Farey sequence. Very light-hearted guesses about possible generalization of these invariants to the case of general N-tangles are made.

5.5.1 Farey sequences

Some basic facts about Farey sequences [75] demonstrate that they are very interesting also from TGD point of view.

1. Farey sequence F_N is defined as the set of rationals $0 \leq q = m/n \leq 1$ satisfying the conditions $n \leq N$ ordered in an increasing sequence.

2. Two subsequent terms a/b and c/d in F_N satisfy the condition $ad - bc = 1$ and thus define and element of the modular group $SL(2, Z)$.

3. The number $|F(N)|$ of terms in Farey sequence is given by

$$|F(N)| = |F(N-1)| + \phi(N-1) \ . \tag{10}$$

Here $\phi(n)$ is Euler's totient function giving the number of divisors of n. For primes one has $\phi(p) = 1$ so that in the transition from p to $p+1$ the length of Farey sequence increases by one unit by the addition of $q = 1/(p+1)$ to the sequence.

The members of Farey sequence F_N are in one-one correspondence with the set of quantum phases $q_n = exp(i2\pi/n)$, $0 \leq n \leq N$. This suggests a close connection with the hierarchy of Jones inclusions, quantum groups, and in TGD context with quantum measurement theory with finite measurement resolution and the hierarchy of Planck constants involving the generalization of the imbedding space. Also the recent TGD inspired ideas about the hierarchy of subgroups of the rational modular group with subgroups labelled by integers N and in direct correspondence with the hierarchy of quantum critical phases [C1] would naturally relate to the Farey sequence.

5.5.2 Riemann Hypothesis and Farey sequences

Farey sequences are used in two equivalent formulations of the Riemann hypothesis. Suppose the terms of F_N are $a_{n,N}$, $0 < n \leq |F_N|$. Define

$$d_{n,N} = a_{n,N} - \frac{n}{|F_N|} \quad .$$

In other words, $d_{n,N}$ is the difference between the n:th term of the N:th Farey sequence, and the n:th member of a set of the same number of points, distributed evenly on the unit interval. Franel and Landau proved that both of the following statements

$$\begin{aligned} \sum_{n=1,\ldots,|F_N|} |d_{n,N}| &= O(N^r) \text{ for any } r > 1/2 \ , \\ \sum_{n=1,\ldots,|F_N|} d_{n,N}^2 &= O(N^r) \text{ for any } r > 1 \ . \end{aligned} \tag{11}$$

are equivalent with Riemann hypothesis.

One could say that RH would guarantee that the numbers of Farey sequence provide the best possible approximate representation for the evenly distributed rational numbers $n/|F_N|$.

5.5.3 Farey sequences and TGD

Farey sequences seem to relate very closely to TGD.

1. The rationals in the Farey sequence can be mapped to the roots of unity by the map $q \rightarrow exp(i2\pi q)$. The numbers $1/|F_N|$ are in turn mapped to the numbers $exp(i2\pi/|F_N|)$, which are also roots of unity. The statement would be that the algebraic phases defined by Farey sequence give the best possible approximate representation for the phases $exp(in2\pi/|F_N|)$ with evenly distributed phase angle.

2. In TGD framework the phase factors defined by F_N corresponds to the set of quantum phases corresponding to Jones inclusions labelled by $q = exp(i2\pi/n)$, $n \leq N$, and thus to the N lowest levels of dark matter hierarchy. There are actually two hierarchies corresponding to M^4 and CP_2 degrees of freedom and the Planck constant appearing in Schrödinger equation corresponds to the ratio n_a/n_b defining quantum phases in these degrees of freedom. $Z_{n_a \times n_b}$ appears as a conformal symmetry of "dark" partonic 2-surfaces and with very general assumptions this implies that there are only in TGD Universe [F1, C1].

3. The fusion of physics associated with various number fields to single coherent whole requires algebraic universality. In particular, the roots of unity, which are complex algebraic numbers, should define approximations to continuum of phase factors. At least the S-matrix associated with p-adic-to-real transitions and more generally $p_1 \rightarrow p_2$ transitions between states for which the partonic space-time sheets are p_1- resp. p_2-adic can involve only this kind of algebraic phases. One can also say that cognitive representations can involve only algebraic phases and algebraic numbers in general. For real-to-real transitions and real-to-padic transitions U-matrix might be non-algebraic or obtained by analytic continuation of algebraic U-matrix. S-matrix is by definition diagonal with respect to number field and similar continuation principle might apply also in this case.

4. The subgroups of the hierarchy of subgroups of the modular group with rational matrix elements are labelled by integer N and relate naturally to the hierarchy of Farey sequences. The hierarchy of quantum critical phases is labelled by integers N with quantum phase transitions occurring only between phases for which the smaller integer divides the larger one [C1].

5.5.4 Interpretation of RH in TGD framework

Number theoretic universality of physics suggests an interpretation for the Riemann hypothesis in TGD framework. RH would be equivalent to the statement that the Farey numbers provide best possible approximation to the set of rationals $k/|F_N|$ or to the statement that the roots of unity contained by F_N define the best possible approximation for the roots of unity defined as $exp(ik2\pi/|F_N|)$ with evenly spaced phase angles. The roots of unity allowed by the lowest N levels of the dark matter hierarchy allows the best possible approximate representation for algebraic phases represented exactly at $|F_N|$:th level of hierarchy.

A stronger statement would be that the Platonia, where RH holds true would be the best possible world in the sense that algebraic physics behind the cognitive representations would allow the best possible approximation hierarchy for the continuum physics (both for numbers in unit interval and for phases on unit circle). Platonia with RH would be cognitive paradise.

One could see this also from different view point. "Platonia as the cognitively best possible world" could be taken as the "axiom of all axioms": a kind of fundamental variational principle of mathematics. Among other things it would allow to conclude that RH is true: RH must hold true either as a theorem following from some axiomatics or as an axiom in itself.

5.5.5 Could rational N-tangles exist in some sense?

The article of Kauffman and Lambropoulou [76] about rational 2-tangles having commutative sum and product allowing to map them to rationals is very interesting from TGD point of view. The illustrations of the article are beautiful and make it easy to get the gist of various ideas. The theorem of the article states that equivalent rational tangles giving trivial tangle in the product correspond to subsequent Farey numbers a/b and c/d satisfying $ad - bc = \pm 1$ so that the pair defines element of the modular group SL(2,Z).

1. Rational 2-tangles

1. The basic observation is that 2-tangles are 2-tangles in both "s- and t-channels". Product and sum can be defined for all tangles but only in the case of 2-tangles the sum, which in this case reduces to product in t-channel obtained by putting tangles in series, gives 2-tangle. The so called rational tangles are 2-tangles constructible by using addition of $\pm[1]$ on left or right of tangle and multiplication by $\pm[1]$ on top or bottom. Product and sum are commutative for rational 2-tangles but the outcome is not a rational 2-tangle in the general case. One can

also assign to rational 2-tangle its negative and inverse. One can map 2-tangle to a number which is rational for rational tangles. The tangles [0], [∞], $\pm$[1], $\pm 1/[1]$, $\pm$[2], $\pm[1/2]$ define so called elementary rational 2-tangles.

2. In the general case the sum of $M-$ and $N-$tangles is $M+N-2$-tangle and combines various $N-$tangles to a monoidal structure. Tensor product like operation giving $M+N$-tangle looks to me physically more natural than the sum.

3. The reason why general 2-tangles are non-commutative although 2-braids obviously commute is that 2-tangles can be regarded as sequences of $N-$tangles with 2-tangles appearing only as the initial and final state: N is actually even for intermediate states. Since $N>2$-braid groups are non-commutative, non-commutativity results. It would be interesting to know whether braid group representations have been used to construct representations of $N-$tangles.

2. Does generalization to $N >> 2$ case exist?

One can wonder whether the notion of rational tangle and the basic result of the article about equivalence of tangles might somehow generalize to the $N>2$ case.

1. Could the commutativity of tangle product allow to characterize the $N>2$ generalizations of rational 2-tangles. The commutativity of product would be a space-time correlate for the commutativity of the S-matrices defining time like entanglement between the initial and final quantum states assignable to the N-tangle. For 2-tangles commutativity of the sum would have an analogous interpretation. Sum is not a very natural operation for N-tangles for $N>2$. Commutativity means that the representation matrices defined as products of braid group actions associated with the various intermediate states and acting in the same representation space commute. Only in very special cases one can expect commutativity for tangles since commutativity is lost already for braids.

2. The representations of 2-tangles should involve the subgroups of N-braid groups of intermediate braids identifiable as Galois groups of N:th order polynomials in the realization as number theoretic tangles. Could non-commutative 2-tangles be characterized by algebraic numbers in the extensions to which the Galois groups are associated? Could the non-commutativity reflect directly the non-commutativity of Galois groups involved? Quite generally one can ask whether the invariants should be expressible using algebraic numbers in the extensions of rationals associated with the intermediate braids.

3. Rational 2-tangles can be characterized by a rational number obtained by a projective identification $[a,b]^T \rightarrow a/b$ from a rational 2-spinor $[a,b]^T$ to which SL(2(N-1),Z) acts. Equivalence means that the columns $[a,b]^T$ and $[c,d]^T$ combine to form element of SL(2,Z) and thus defining a modular transformation. Could more general 2-tangles have a similar representation but in terms of algebraic integers?

4. Could N-tangles be characterized by $N-1$ $2(N-1)$-component projective column-spinors $[a_i^1, a_i^2, .., a_i^{2(N-1)}]^T$, $i=1,...N-1$ so that only the ratios $a_i^k/a_i^{2(N-1)} \leq 1$ matter? Could equivalence for them mean that the $N-1$ spinors combine to form $N-1+N-1$ columns of $SL(2(N-1),Z)$ matrix. Could N-tangles quite generally correspond to collections of projective $N-1$ spinors having as components algebraic integers and could $ad-bc=\pm 1$ criterion generalize? Note that the modular group for surfaces of genus g is SL(2g,Z) so that $N-1$ would be analogous to g and $1 \leq N \geq 3$- braids would correspond to $g \leq 2$ Riemann surfaces.

5. Dark matter hierarchy leads naturally to a hierarchy of modular sub-groups of $SL(2,Q)$ labelled by N (the generator $\tau \rightarrow \tau + 2$ of modular group is replaced with $\tau \rightarrow \tau + 2/N$). What might be the role of these subgroups and corresponding subgroups of $SL(2(N-1),Q)$. Could they arise in "anyonization" when one considers quantum group representations of 2-tangles with twist operation represented by an N:th root of unity instead of phase U satisfying $U^2 = 1$?

5.5.6 How tangles could be realized in TGD Universe?

The article of Kauffman and Lambropoulou stimulated the question in what senses N-tangles could be be realized in TGD Universe as fundamental structures.

1. Tangles as number theoretic braids?

The strands of number theoretical $N-$braids correspond to roots of N:th order polynomial and if one allows time evolutions of partonic 2-surface leading to the disappearance or appearance of real roots $N-$tangles become possible. This however means continuous evolution of roots so that the coefficients of polynomials defining the partonic 2-surface can be rational only in initial and final state but not in all intermediate "virtual" states.

2. Tangles as tangled partonic 2-surfaces?

Tangles could appear in TGD also in second manner.

1. Partonic 2-surfaces are sub-manifolds of a 3-D section of space-time surface. If partonic 2-surfaces have genus $g > 0$ the handles can become knotted and linked and one obtains besides ordinary knots and links more general knots and links in which circle is replaced by figure eight and its generalizations obtained by adding more circles (eyeglasses for $N-$eyed creatures).

2. Since these 2-surfaces are space-like, the resulting structures are indeed tangles rather than only braids. Tangles made of strands with fixed ends would result by allowing spherical partons elongate to long strands with fixed ends. DNA tangles would the basic example, and are discussed also in the article. DNA sequences to which I have speculatively assigned invisible (dark) braid structures might be seen in this context as space-like "written language representations" of genetic programs represented as number theoretic braids.

6 Appendix

6.1 Hecke algebra and Temperley-Lieb algebra

Braid group is accompanied by several algebras. For Hecke algebra, which is particular case of braid algebra, one has

$$\begin{aligned} e_{n+1}e_ne_{n+1} &= e_ne_{n+1}e_n \ , \\ e_n^2 &= (t-1)e_n + t \ . \end{aligned} \tag{12}$$

The algebra reduces to that for symmetric group for $t = 1$.

Hecke algebra can be regarded as a discrete analog of Kac Moody algebra or loop algebra with G replaced by S_n. This suggests a connection with Kac-Moody algebras and imbedding of Galois groups to Kac-Moody group. $t = p^n$ corresponds to a finite field. Fractal dimension $t = \mathcal{M} : \mathcal{N}$ relates naturally to braid group representations: fractal dimension of quantum quaternions might be appropriate interpretation. t=1 gives symmetric group. Infinite braid group could be seen as a quantum variant of Galois group for algebraic closure of rationals.

Temperley-Lieb algebra assignable with Jones inclusions of hyper-finite factors of type II_1 with $\mathcal{M}:\mathcal{N}<4$ is given by the relations

$$\begin{aligned} e_{n+1}e_nen+1 &= e_{n+1} \\ e_ne_{n+1}e_n &= e_n \ , \\ e_n^2 &= te_n \ , \ \ ,t=-\sqrt{\mathcal{M}:\mathcal{N}}=-2cos(\pi/n) \ \ ,n=3,4,... \end{aligned} \tag{13}$$

The conditions involving three generators differ from those for braid group algebra since e_n are now proportional to projection operators. An alternative form of this algebra is given by

$$\begin{aligned} e_{n+1}e_nen+1 &= te_{n+1} \\ e_ne_{n+1}e_n &= te_n \ , \\ e_n^2 &= e_n=e_n^* \ , \ \ ,t=-\sqrt{\mathcal{M}:\mathcal{N}}=-2cos(\pi/n) \ \ ,n=3,4,... \end{aligned} \tag{14}$$

This representation reduces to that for Temperley-Lieb algebra with obvious normalization of projection operators. These algebras are somewhat analogous to function fields but the value of coordinate is fixed to some particular values. An analogous discretization for function fields corresponds to a formation of number theoretical braids.

6.2 Some examples of bi-algebras and quantum groups

The appendix summarizes briefly the simplest bi- and Hopf algebras and some basic constructions related to quantum groups.

6.2.1 Simplest bi-algebras

Let $k(x_1,..,x_n)$ denote the free algebra of polynomials in variables x_i with coefficients in field k. x_i can be regarded as points of a set. The algebra $Hom(k(x_1,...,x_n),A)$ of algebra homomorphisms $k(x_1,...,x_n)\rightarrow A$ can be identified as A^n since by the homomorphism property the images $f(x_i)$ of the generators $x_1,...x_n$ determined the homomorphism completely. Any commutative algebra A can be identified as the $Hom(k[x],A)$ with a particular homomorphism corresponding to a line in A determined uniquely by an element of A.

The matrix algebra $M(2)$ can be defined as the polynomial algebra $k(a,b,c,d)$. Matrix multiplication can be represented universally as an algebra morphism Δ from from $M_2=k(a,b,c,d)$ to $M_2^{\otimes 2}=k(a',a'',b',b'',c',c'',d',d'')$ to $k(a,b,c,d)$ in matrix form as

$$\Delta\left(\begin{array}{cc} a & b \\ c & d \end{array}\right)=\left(\begin{array}{cc} a' & b' \\ c' & d' \end{array}\right)\left(\begin{array}{cc} a'' & b'' \\ c'' & d'' \end{array}\right) \ .$$

This morphism induces algebra multiplication in the matrix algebra $M_2(A)$ for any commutative algebra A.

$M(2)$, $GL(2)$ and $SL(2)$ provide standard examples about bi-algebras. $SL(2)$ can be defined as a commutative algebra by dividing free polynomial algebra $k(a,b,c,d)$ spanned by the generators a,b,c,d by the ideal $det-1=ad-bc-1=0$ expressing that the determinant of the matrix is one. In the matrix representation μ and η are defined in obvious manner and μ gives powers of the matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad .$$

Δ, counit ϵ, and antipode S can be written in case of $SL(2)$ as

$$\begin{pmatrix} \Delta(a) & \Delta(b) \\ \Delta(c) & \Delta(d) \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad ,$$

$$\begin{pmatrix} \epsilon(a) & \epsilon(b) \\ \epsilon(c) & \epsilon(d) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad .$$

$$S \begin{pmatrix} a & b \\ c & d \end{pmatrix} = (ad - bc)^{-1} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \quad .$$

Note that matrix representation is only an economical manner to summarize the action of Δ on the generators a, b, c, d of the algebra. For instance, one has $\Delta(a) = a \to a \otimes a + b \otimes c$. The resulting algebra is both commutative and co-commutative.

$SL(2)_q$ can be defined as a Hopf algebra by dividing the free algebra generated by elements a, b, c, d by the relations

$$\begin{array}{rr} ba = qab \ , & db = qbd \ , \\ ca = qac \ , & dc = qcd \ , \\ bc = cb \ , & ad - da = (q^{-1} - 1)bc \ , \end{array}$$

and the relation

$$det_q = ad - q^{-1}bc = 1$$

stating that the quantum determinant of $SL(2)_q$ matrix is one.

$\mu, \eta, \Delta, \epsilon$ are defined as in the case of $SL(2)$. Antipode S is defined by

$$S \begin{pmatrix} a & b \\ c & d \end{pmatrix} = det_q^{-1} \begin{pmatrix} d & -qb \\ -q^{-1}c & a \end{pmatrix} \quad .$$

The relations above guarantee that it defines quantum inverse of A. For q an n^{th} root of unity, $S^{2n} = id$ holds true which signals that these parameter values are somehow exceptional. This result is completely general.

Given an algebra, the R point of $SL_q(2)$ is defined as a four-tuple (A, B, C, D) in R^4 satisfying the relations defining the point of $SL_q(2)$. One can say that R-points provide representations of the universal quantum algebra $SL_q(2)$.

6.2.2 Quantum group $U_q(sl(2))$

Quantum group $U_q(sl(2))$ or rather, quantum enveloping algebra of $sl(2)$, can be constructed by applying Drinfeld's quantum double construction (to avoid confusion note that the quantum Hopf algebra associated with $SL(2)$ is the quantum analog of a commutative algebra generated by powers of a 2×2 matrix of unit determinant).

The commutation relations of $sl(2)$ read as

$$[X_+, X_-] = H \quad , \quad [H, X_\pm] = \pm 2X_\pm \quad . \tag{15}$$

$U_q(sl(2))$ allows co-algebra structure given by

$$\begin{aligned} &\Delta(J) = J\otimes 1 + 1\otimes J \ , \quad S(J) = -J \ , \quad \epsilon(J) = 0 \ , \quad J = X_{\pm}, H \ , \\ &S(1) = 1 \ , \qquad\qquad\qquad \epsilon(1) = 1 \ . \end{aligned} \tag{16}$$

The enveloping algebras of Borel algebras $U(B_{\pm})$ generated by $\{1, X_+, H\}$ $\{1, X_-, hH\}$ define the Hopf algebra H and its dual $H^\star$ in Drinfeld's construction. h could be called Planck's constant vanishes at the classical limit. Note that $H^\star$ reduces to $\{1, X_-\}$ at this limit. Quantum deformation parameter q is given by $exp(2h)$. The duality map $\star : H \rightarrow H^\star$ reads as

$$\begin{array}{l} a \rightarrow a^\star \ , \quad ab = (ab)^\star = b^\star a^\star \ , \\ 1 \rightarrow 1 \ , \qquad H \rightarrow H^\star = hH \ , \qquad X_+ \rightarrow (X_+)^\star = hX_- \ . \end{array} \tag{17}$$

The commutation relations of $U_q(sl(2)$ read as

$$[X_+, X_-] = \frac{q^H - q^{-H}}{q - q^{-1}} \ , \quad [H, X_\pm] = \pm 2X_\pm \ . \tag{18}$$

Co-product Δ, antipode S, and co-unit ϵ differ from those $U(sl(2))$ only in the case of $X_\pm$:

$$\begin{aligned} &\Delta(X_\pm) = X_\pm \otimes q^{H/2} + q^{-H/2}\otimes X_\pm \ , \\ &S(X_\pm) = -q^{\pm 1}X_\pm \ . \end{aligned} \tag{19}$$

When q is not a root of unity, the universal R-matrix is given by

$$R = q^{\frac{H\otimes H}{2}} \sum_{n=0}^{\infty} \frac{(1-q^{-2})^n}{[n]_q!} q^{\frac{n(1-n)}{2}} q^{\frac{nH}{2}} X_+^n \otimes q^{-\frac{nH}{2}} X_-^n \ . \tag{20}$$

When q is m:th root of unity the q-factorial $[n]_q!$ vanishes for $n \geq m$ and the expansion does not make sense.

For q not a root of unity the representation theory of quantum groups is essentially the same as of ordinary groups. When q is m^{th} root of unity, the situation changes. For $l = m = 2n$ n^{th} powers of generators span together with the Casimir operator a sub-algebra commuting with the whole algebra providing additional numbers characterizing the representations. For $l = m = 2n+1$ same happens for m^{th} powers of Lie-algebra generators. The generic representations are not fully reducible anymore. In the case of $U_q(sl(2))$ irreducibility occurs for spins $n < l$ only. Under certain conditions on q it is possible to decouple the higher representations from the theory. Physically the reduction of the number of representations to a finite number means a symmetry analogous to a gauge symmetry. The phenomenon resembles the occurrence of null vectors in the case of Virasoro and Kac Moody representations and there indeed is a deep connection between quantum groups and Kac-Moody algebras [52].

One can wonder what is the precise relationship between $U_q(sl(2)$ and $SL_q(2)$ which both are quantum groups using loose terminology. The relationship is duality. This means the existence of a morphism $x \rightarrow \Psi(x)$ $M_q(2) \rightarrow U_q^\star$ defined by a bilinear form $\langle u, x\rangle = \Psi(x)(u)$ on $U_q \times M_q(2)$, which is bi-algebra morphism. This means that the conditions

$$\begin{aligned} &\langle uv, x\rangle = \langle u\otimes v, \Delta(x)\rangle \ , \quad \langle u, xy\rangle = \langle \Delta(u), x\otimes y\rangle \ , \\ &\langle 1, x\rangle = \epsilon(x) \ , \qquad\qquad \langle u, 1\rangle = \epsilon(u) \end{aligned}$$

are satisfied. It is enough to find $\Psi(x)$ for the generators $x = A, B, C, D$ of $M_q(2)$ and show that the duality conditions are satisfied. The representation

$$\rho(E) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \ , \quad \rho(F) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \ , \quad \rho(K = q^H) = \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix} \ ,$$

extended to a representation

$$\rho(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}$$

of arbitrary element u of $U_q(sl(2)$ defines for elements in $U_q^\star$. It is easy to guess that $A(u), B(u), C(u), D(u)$, which can be regarded as elements of $U_q^\star$, can be regarded also as R points that is images of the generators a, b, c, d of $SL_q(2)$ under an algebra morphism $SL_q(2) \rightarrow U_q^\star$.

6.2.3 General semisimple quantum group

The Drinfeld's construction of quantum groups applies to arbitrary semi-simple Lie algebra and is discussed in detail in [52]. The construction relies on the use of Cartan matrix.

Quite generally, Cartan matrix $A = \{a_{ij}\}$ is $n \times n$ matrix satisfying the following conditions:
i) A is indecomposable, that is does not reduce to a direct sum of matrices.
ii) $a_{ij} \leq 0$ holds true for $i < j$.
iii) $a_{ij} = 0$ is equivalent with $a_{ij} = 0$.
A can be normalized so that the diagonal components satisfy $a_{ii} = 2$.
The generators e_i, f_i, k_i satisfying the commutations relations

$$\begin{array}{cc} k_i k_j = k_j k_i \ , & k_i e_j = q_i^{a_{ij}} e_j k_i \ , \\ k_i f_j = q_i^{-a_{ij}} e_j k_i \ , & e_i f_j - f_j e_i = \delta_{ij} \frac{k_i - k_i^{-1}}{q_i - q_i^{-1}} \ , \end{array} \tag{21}$$

and so called Serre relations

$$\begin{array}{l} \sum_{l=0}^{1-a_{ij}} (-1)^l \begin{bmatrix} 1 - a_{ij} \\ l \end{bmatrix}_{q_i} e_i^{1-a_{ij}-l} e_j e_i^l = 0, \quad i \neq j \ , \\ \sum_{l=0}^{1-a_{ij}} (-1)^l \begin{bmatrix} 1 - a_{ij} \\ l \end{bmatrix}_{q_i} f_i^{1-a_{ij}-l} f_j f_i^l = 0 \ , \quad i \neq j \ . \end{array} \tag{22}$$

Here $q_i = q^{D_i}$ where one has $D_i a_{ij} = a_{ij} D_i$. $D_i = 1$ is the simplest choice in this case.

Comultiplication is given by

$$\Delta(k_i) = k_i \otimes k_i \ , \tag{23}$$

$$\Delta(e_i) = e_i \otimes k_i + 1 \otimes e_i \ , \tag{24}$$

$$\Delta(f_i) = f_i \otimes 1 + k_i^{-1} \otimes 1 \ . \tag{25}$$

(26)

The action of antipode S is defined as

$$S(e_i) = -e_i k_i^{-1} \ , \quad S(f_i) = -k_i f_i \ , \quad S(k_i) = -k_i^{-1} \ . \tag{27}$$

6.2.4 Quantum affine algebras

The construction of Drinfeld and Jimbo generalizes also to the case of untwisted affine Lie algebras, which are in one-one correspondence with semisimple Lie algebras. The representations of quantum deformed affine algebras define corresponding deformations of Kac-Moody algebras. In the following only the basic formulas are summarized and the reader not familiar with the formalism can consult a more detailed treatment can be found in [52].

1. Affine algebras

The Cartan matrix A is said to be of affine type if the conditions $det(A) = 0$ and $a_{ij}a_{ji} \geq 4$ (no summation) hold true. There always exists a diagonal matrix D such that $B = DA$ is symmetric and defines symmetric bilinear degenerate metric on the affine Lie algebra.

The Dynkin diagrams of affine algebra of rank l have $l+1$ vertices (so that Cartan matrix has one null eigenvector). The diagrams of semisimple Lie-algebras are sub-diagrams of affine algebras. From the $(l+1)\times(l+1)$ Cartan matrix of an untwisted affine algebra $\hat{A}$ one can recover the $l \times l$ Cartan matrix of A by dropping away 0:th row and column.

For instance, the algebra A_1^1, which is affine counterpart of $SL(2)$, has Cartan matrix a_{ij}

$$A = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$$

with a vanishing determinant.

Quite generally, in untwisted case quantum algebra $U_q(\hat{G}_l)$ as $3(l+1)$ generators e_i, f_i, k_i $(i = 0, 1, .., l)$ satisfying the relations of Eq. 22 for Cartan matrix of $\mathcal{G}^{(1)}$. Affine quantum group is obtained by adding to $U_q(\hat{\mathcal{G}}_l)$ a derivation d satisfying the relations

$$[d, e_i] = \delta_{i0}e_i \ , \quad [d, f_i] = \delta_{i0}f_i, \quad [d, k_i] = 0 \ . \tag{28}$$

with comultiplication $\Delta(d) = d \otimes 1 + 1 \otimes d$.

2. Kac Moody algebras

The undeformed extension $\hat{\mathcal{G}}_l$ associated with the affine Cartan matrix $\mathcal{G}_l^{(1)}$ is the Kac Moody algebra associated with the group G obtained as the central extension of the corresponding loop algebra. The loop algebra is defined as

$$L(\mathcal{G}) = \mathcal{G} \otimes C\left[t, t^{-1}\right] \ , \tag{29}$$

where $C\left[t, t^{-1}\right]$ is the algebra of Laurent polynomials with complex coefficients. The Lie bracket is

$$[x \times P, y \otimes Q] = [x, y] \otimes PQ \ . \tag{30}$$

The non-degenerate bilinear symmetric form $(,)$ in $\mathcal{G}_l$ induces corresponding form in $L(\mathcal{G}_l)$ as $(x \otimes P, y \otimes Q) = (x, y)PQ$.

A two-cocycle on $L(\mathcal{G}_l)$ is defined as

$$\Psi(a, b) = Res(\frac{da}{dt}, b) \ , \tag{31}$$

where the residue of a Laurent is defined as $Res(\sum_n a_n t^n) = a_{-1}$. The two-cocycle satisfies the conditions

$$\Psi(a,b) = -\Psi(b,a) \ ,$$
$$\Psi([a,b],c) + \Psi([b,c],a) + \Psi([c,a],b) = 0 \ . \tag{32}$$

The two-cocycle defines the central extension of loop algebra $L(\mathcal{G}_l)$ to Kac Moody algebra $L(\mathcal{G}_l) \otimes Cc$, where c is a new central element commuting with the loop algebra. The new bracket is defined as $[,]+\Psi(,)c$. The algebra $\tilde{L}(\mathcal{G}_l)$ is defined by adding the derivation d which acts as td/dt measuring the conformal weight.

The standard basis for Kac Moody algebra and corresponding commutation relations are given by

$$\begin{aligned} J_n^x &= x \otimes t^n \ , \\ [J_n^x, J_m^y] &= J_{n+m}^{[x,y]} + n\delta_{m+n,0}c \ . \end{aligned} \tag{33}$$

The finite dimensional irreducible representations of G defined representations of Kac Moody algebra with a vanishing central extension $c = 0$. The highest weight representations are characterized by highest weight vector $|v\rangle$ such that

$$\begin{aligned} J_n^x|v\rangle &= 0, \ \ n > 0 \ , \\ c|v\rangle &= k|v\rangle \ . \end{aligned} \tag{34}$$

3. Quantum affine algebras

Drinfeld has constructed the quantum affine extension $U_q(\mathcal{G}_l)$ using quantum double construction. The construction of generators uses almost the same basic formulas as the construction of semi-simple algebras. The construction involves the automorphism $D_t : U_q(\tilde{\mathcal{G}}_l) \otimes C\left[t, t^{-1}\right] \rightarrow U_q(\tilde{\mathcal{G}}_l) \otimes C\left[t, t^{-1}\right]$ given by

$$\begin{array}{ll} D_t(e_i) = t^{\delta_{i0}} e_i \ , & D_t(f_i) = t^{\delta_{i0}} f_i \ , \\ D_t(k_i) = k_i & D_t(d) = d \ , \end{array} \tag{35}$$

and the co-product

$$\Delta_t(a) = (D_t \otimes 1)\Delta(a) \ , \quad \Delta_t^{op}(a) = (D_t \otimes 1)\Delta^{op}(a) \ , \tag{36}$$

where the $\Delta(a)$ is the co-product defined by the same general formula as applying in the case of semi-simple Lie algebras. The universal R-matrix is given by

$$\mathcal{R}(t) = (D_t \otimes 1)\mathcal{R} \ , \tag{37}$$

and satisfies the equations

$$\begin{aligned} &\mathcal{R}(t)\Delta_t(a) = \Delta_t^{op}(a)\mathcal{R} \ , \\ &(\Delta_z \otimes id)\mathcal{R}(u) = \mathcal{R}_{13}(zu)\mathcal{R}_{23}(u) \ , \\ &(id \otimes \Delta_u)\mathcal{R}(zu) = \mathcal{R}_{13}(z)\mathcal{R}_{12}(zu) \ , \\ &\mathcal{R}_{12}(t)\mathcal{R}_{13}(tw)\mathcal{R}_{23}(w) = \mathcal{R}_{23}(w)\mathcal{R}_{13}(tw)\mathcal{R}_{12}(t) \ . \end{aligned} \tag{38}$$

The infinite-dimensional representations of affine algebra give representations of Kac-Moody algebra when one restricts the consideration to generations e_i, f_i, k_i, $i > 0$.

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