Topological Quantum Computation in TGD Universe

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Abstract

Topological quantum computation (TQC) is one of the most promising approaches to quantum computation. The coding of logical qubits to the entanglement of topological quantum numbers promises to solve the de-coherence problem whereas the S-matrices of topological field theories (modular functors) providing unitary representations for braids provide a realization of quantum computer programs with gates represented as simple braiding operations. Because of their effective 2-dimensionality anyon systems are the best candidates for realizing the representations of braid groups.

TGD allows several new insights related to quantum computation. TGD predicts new information measures as number theoretical negative valued entanglement entropies defined for systems having extended rational entanglement and characterizes bound state entanglement as bound state entanglement. Negentropy Maximization Principle and p-adic length scale hierarchy of space-time sheets encourage to believe that Universe itself might do its best to resolve the de-coherence problem. The new view about quantum jump suggests strongly the notion of quantum parallel dissipation so that thermalization in shorter length scales would guarantee coherence in longer length scales. The possibility of negative energies and communications to geometric future in turn might trivialize the problems caused by long computation times: computation could be iterated again and again by turning the computer on in the geometric past and TGD inspired theory of consciousness predicts that something like this occurs routinely in living matter.

The absolute minimization of Kähler action is the basic variational principle of classical TGD and predicts extremely complex but non-chaotic magnetic flux tube structures, which can get knotted and linked. The dimension of CP_2 projection for these structures is $D = 3$. These structures are the corner stone of TGD inspired theory of living matter and provide the braid structures needed by TQC.

Anyons are the key actors of TQC and TGD leads to detailed model of anyons as systems consisting of track of a periodically moving charged particle realized as a flux tube containing the particle inside it. This track would be a space-time correlate for the outcome of dissipative processes producing the asymptotic self-organization pattern. These tracks in general carry vacuum Kähler charge which is topologized when the CP_2 projection of space-time sheet is $D = 3$. This explains charge fractionization predicted to occur also for other charged particles. When a system approaches chaos periodic orbits become slightly aperiodic and the correlate is flux tube which rotates N times before closing. This gives rise to Z_N valued topological quantum number crucial for TQC using anyons ($N = 4$ holds true in this case). Non-Abelian anyons are needed by TQC, and the existence of long range classical electro-weak fields predicted by TGD is an essential prerequisite of non-Abelianity.

Negative energies and zero energy states are of crucial importance of TQC in TGD. The possibility of phase conjugation for fermions would resolve the puzzle of matter-antimatter asymmetry in an elegant manner. Anti-fermions would be present but have negative energies. Quite generally, it is possible to interpret scattering as a creation of pair of positive and negative energy states, the latter representing the final state. One can characterize precisely the deviations of this Eastern world view with respect to the Western world view assuming an objective reality with a positive definite energy and understand why the Western illusion apparently works. In the case of TQC the initial *resp.* final state of braided anyon system would correspond to positive *resp.* negative energy state.

The light-like boundaries of magnetic flux tubes are ideal for TQC. The point is that 3-dimensional light-like quantum states can be interpreted as representations for the time evolution of a two-dimensional system and thus represented self-reflective states being "about something". The light-likeness (no geometric time flow) is a space-time correlate for the ceasing of subjective time flow during macro-temporal quantum coherence. The S-matrices of TQC can be coded to these light-like states such that each elementary braid operation corresponds to positive energy anyons near the boundary of the magnetic flux tube A and negative energy anyons with opposite topological charges residing near the boundary of flux tube B and connected by braided threads representing the quantum gate. Light-like boundaries also force Chern-Simons action as the only possible general coordinate invariant action since

the vanishing of the metric determinant does not allow any other candidate. Chern-Simons action indeed defines the modular functor for braid coding for a TQC program.

The comparison of the concrete model for TQC in terms of magnetic flux tubes with the structure of DNA gives tantalizing hints that DNA double strand is a topological quantum computer. Strand *resp.* conjugate strand would carry positive *resp.* negative energy anyon systems. The knotting and linking of DNA double strand would code for 2-gates realized as a unique maximally entangling Yang-Baxter matrix R for 2-state system. The pairs A-T, T-A, C-G, G-C in active state would code for the four braid operations of 3-braid group in 1-qubit Temperley Lieb representation associated with quantum group $SL(2)_q$. On basis of this picture one can identify N-O hydrogen bonds between DNA strands as structural correlates of 3-braids responsible for the nontrivial 1-gates whereas N-N hydrogen bonds would be correlates for the return gates acting as identity gates. Depending on whether the nucleotide is active or not it codes for nontrivial 1-gate or for identity gate so that DNA strand can program itself or be programmed dynamically.

1 Introduction

Quantum computation is perhaps one of the most rapidly evolving branches of theoretical physics. TGD inspired theory of consciousness has led to new insights about quantum computation and in this chapter I want to discuss these ideas in a more organized manner.

There are three mathematically equivalent approaches to quantum computation [21]: quantum Turing machines, quantum circuits, and topological quantum computation (TQC). In fact, the realization that TGD Universe seems to be ideal place to perform TQC [22, 23] served as the stimulus for writing this chapter.

Quite generally, quantum computation allows to solve problems which are NP hard, that is the time required to solve the problem increases exponentially with the number of variables using classical computer but only polynomially using quantum computer. The topological realization of the computer program using so called braids resulting when threads are weaved to 2-dimensional patterns is very robust so that de-coherence, which is the basic nuisance of quantum computation, ceases to be a problem. More precisely, the error probability is proportional to $exp(-\alpha l)$, where l is the length scale characterizing the distance between strands of the braid [23] .

1.1 Evolution of basic ideas of quantum computation

The notion of quantum computation goes back to Feynman [24] who demonstrated that some computational tasks boil down to problems of solving quantum evolution of some physical system, say electrons scattering from each other. Many of these computations are NP hard, which means that the number of computational steps required grows exponentially with the number of variables involved so that they become quickly unsolvable using ordinary computers. A quicker manner to do the computation is to make a physical experiment. A further bonus is that if you can solve one NP hard problem, you can solve many equivalent NP hard problems. What is new that quantum computation is not deterministic so that computation must be carried out several times and probability distribution for the outcomes allows to deduce the answer. Often however the situation is such that it is easy to check whether the outcome provides the sought for solution.

Years later David Deutch [25] transformed Feynman's ideas into a detailed theory of quantum computation demonstrating how to encode quantum computation in a quantum system and researchers started to develop applications. One of the key factors in the computer security is cryptography which relies on the fact that the factorization of large integers to primes is a NP hard problem. Peter Shor [26] discovered an algorithm, which allows to carry out the factorization in time, which is exponentially shorter than by using ordinary computers. A second example is problem of searching a particular from a set of N items, which requires time proportional to N classically but quantally only a time proportional to $\sqrt{N}$.

The key notion is quantum entanglement which allows to store information in the relationship between systems, qubits in the simplest situation. This means that information storage capacity increases exponentially as a function of number of qubits rather than only linearly. This explains why NP hard problems which require time increasing exponentially with the number of variables can be solved using quantum computers. It also means exponentially larger information storage capacity than possible classically.

Recall that there are three equivalent approaches to quantum computation: quantum Turing machine, quantum circuits, and topology based unitary modular functor approach. In quantum circuit approach the unitary time evolution defining the quantum computation is assumed to be decomposable to a product of more elementary operations defined by unitary operators associated with quantum gates. The number of different gates needed is surprisingly small: only 1-gates generating unitary transformations of single qubit, and a 2-gate representing a transformation which together with 1-gates is able to generate entanglement are needed to generate a dense subgroup of unitary group $U(2^n)$ in the case of n-qubit system. 2-gate could be conditional NOT (CNOT). The first 1-gate can induce a phase factor to the qubit 0 and do nothing for qubit 1. Second 1-gate could form orthogonal square roots of bits 1 and 0 as superposition of 1 and 0 with identical probabilities.

The formal definition of the quantum computation using quantum circuit is as a computation of the value of a Boolean function of n Boolean arguments, for instance the k:th bit of the largest prime factor of a given integer. The unitary operator U is constructed as a product of operators associated with the basic gates. It is said that the function coding the problem belongs to the class BQP (function is computable with a bounded error in polynomial time) if there exists a classical polynomial-time (in string length) algorithm for specifying the quantum circuit. The first qubit of the outgoing n-qubit is measured and the probability that the the value is 0 determines the value of the bit to be calculated. For instance, for $p(0) \geq 2/3$ the bit is 0 and for $p(0) \geq 1/3$ the bit is 1. The evaluation of the outcome is probabilistic and requires a repeat the computation sufficiently many times.

The basic problem of quantum computation is the extremely fragility of the physical qubit (say spin). The fragility can be avoided by mapping q-bits to logical qubits realized as highly entangled states of many qubits and quantum error-correcting codes and fault tolerant methods [27, 28, 29] rely on this.

The space W of the logical qubits is known as a code space. The sub-space W of physical states of space $Y = V \otimes V ..... \otimes V$ is called k-code if the effect of any k-local operator (affecting only k tensor factors of Y linearly but leaving the remaining factors invariant) followed by an orthogonal projection to W is multiplication by scalar. This means that k-local operator modify the states only in directions orthogonal to W.

These spaces indeed exist and it can be shown that the quantum information coded in W is not affected by the errors operating in fewer than $k/2$ of the n particles. Note that $k = 3$ is enough to guarantee stability with respect to 1-local errors. In this manner it is possible to correct the errors by repeated quantum measurements and by a suitable choice of the sub-space eliminate the errors due to the local changes of qubits by just performing a projection of the state back to the subspace (quantum measurement).

If the the error magnitude is below so called accuracy threshold, arbitrary long quantum computations are reliable. The estimates for this constant vary between 10^{-5} and 10^{-3}. This is beyond current technologies. Error correction is based on the representation of qubit as a logical qubit defined as a state in a linear sub-space of the tensor product of several qubits.

Topological quantum computation [23] provides an alternative approach to minimize the errors caused by de-coherence. Conceptually the modular functor approach [37, 23] is considerably more abstract than quantum circuit approach. Unitary modular functor is the S-matrix of a topological quantum field theory. It defines a unitary evolution realizing the quantum computation in macroscopic topological ground states degrees of freedom. The nice feature of this approach is that the notion of physical qubit becomes redundant and the code space defined by the logical qubits can be represented in terms topological and thus non-local degrees of freedom which are stable against local perturbations as required.

1.2 Quantum computation and TGD

Concerning quantum computation [21] in general, TGD TGD inspired theory of consciousness provides several new insights.

1.2.1 Quantum jump as elementary particle of consciousness and cognition

Quantum jump is interpreted as a fundamental cognitive process leading from creative confusion via analysis to an experience of understanding, and involves TGD counterpart of the unitary process followed by state function reduction and state preparation. One can say that quantum jump is the elementary particle of consciousness and that selves consists of sequences of quantum jump just like hadrons, nuclei, atoms, molecules,... consist basically of elementary particles. Self loses its consciousness when it generates bound state entanglement with environment. The conscious experience of self is in a well-defined sense a statistical average over the quantum jump during which self exists. During macro-temporal quantum coherence during macro-temporal quantum coherence a sequence of quantum jumps integrates effectively to a single moment of consciousness and effectively defines single unitary time evolution followed by state function reduction and preparation. This means a fractal hierarchy of consciousness very closely related to the corresponding hierarchy for bound states of elementary particles and structure formed from them.

1.2.2 Negentropy Maximization Principle guarantees maximal entanglement

Negentropy Maximization Principle is the basic dynamical principle constraining what happens in state reduction and self measurement steps of state preparation. Each self measurement involves a decomposition of system into two parts. The decomposition is dictated by the requirement that the reduction of entanglement entropy in self measurement is maximal. Self measurement can lead to either unentangled state or to entangled state with density matrix which is proportional to unit matrix (density matrix is the observable measured). In the latter case maximally entangled state typically involved with quantum computers results as an outcome. Hence Nature itself would favor maximally entangling 2-gates. Note however that self measurement occurs only if it increases the entanglement negentropy.

1.2.3 Number theoretical information measures and extended rational entanglement as bound state entanglement

The emerging number theoretical notion of information allows to interpret the entanglement for which entanglement probabilities are rational (or belong to an extension of rational numbers defining a finite extension of p-adic numbers) as bound state entanglement with positive information content. Macro-temporal quantum coherence corresponds to a formation of bound entanglement stable against state function reduction and preparation processes.

Spin glass degeneracy, which is the basic characteristic of the variational principle defining space-time dynamics, implies a huge number of vacuum degrees of freedom, and is the key mechanism behind macro-temporal quantum coherence. Spin glass degrees of freedom are also ideal candidates qubit degrees of freedom. As a matter fact, p-adic length scale hierarchy suggests that qubit represents only the lowest level in the hierarchy of qupits defining p-dimensional state spaces, p prime.

1.2.4 Time mirror mechanism and negative energies

The new view about time, in particular the possibility of communications with and control of geometric past, suggests the possibility of circumventing the restrictions posed by time for quantum computation. Iteration based on initiation of quantum computation again and again in geometric past would make possible practically instantaneous information processing.

Space-time sheets with negative time orientation carry negative energies. Also the possibility of phase conjugation of fermions is strongly suggestive. It is also possible that anti-fermions possess negative energies in phases corresponding to macroscopic length scales. This would explain matter-antimatter asymmetry in elegant manner. Zero energy states would be ideal for quantum computation purposes and could be even created intentionally by first generating a p-adic surface representing the state and then transforming it to a real surface.

The most predictive and elegant cosmology assumes that the net quantum numbers of the Universe vanish so that quantum jumps would occur between different kinds of vacua. Crossing symmetry makes this option almost consistent with the idea about objective reality with definite conserved total quantum numbers but requires that quantum states of 3-dimensional quantum theory represent S-matrices of 2-dimensional quantum field theory. These quantum states are thus about something. The boundaries of space-time surface are most naturally light-like 3-surfaces space-time surface and are limiting cases of space-like 3-surface and time evolution of 2-surface. Hence they would act naturally as space-time correlates for the reflective level of consciousness.

1.3 TGD and the new physics associated with TQC

TGD predicts the new physics making possible to realized braids as entangled flux tubes and also provides a detailed model explaing basic facts about anyons.

1.3.1 Topologically quantized magnetic flux tube structures as braids

Quantum classical correspondence suggests that the absolute minimization of Kähler action corresponds to a space-time representation of second law and that the 4-surfaces approach asymptotically space-time representations of systems which do not dissipate anymore. The correlate for the absence of dissipation is the vanishing of Lorentz 4-force associated with the induced Kähler field. This condition can be regarded as a generalization of Beltrami condition for magnetic fields and leads to very explicit general solutions of field equations [D1].

The outcome is a general classification of solutions based on the dimension of CP_2 projection. The most unstable phase corresponds to $D = 2$-dimensional projection and is analogous to a ferromagnetic phase. $D = 4$ projection corresponds to chaotic demagnetized phase and $D = 3$ is the extremely complex but ordered phase at the boundary between chaos and order. This phase was identified as the phase responsible for the main characteristics of living systems [I4, I5]. It is also ideal for quantum computations since magnetic field lines form extremely complex linked and knotted structures.

The flux tube structures representing topologically quantized fields, which have $D = 3$ - dimensional CP_2 projection, are knotted, linked and braided, and carry an infinite number of conserved topological charges labelled by representations of color group. They seem to be tailor-made for defining the braid structure needed by TQC. The boundaries of the magnetic flux tubes correspond to light-like 3-surfaces with respect to the induced metric (being thus metrically 2-dimensional and allowing conformal invariance) and can be interpreted either as 3-surfaces or time-evolutions of 2-dimensional systems so that S-matrix of 2-D system can be coded into the quantum state of conformally invariant 3-D system.

1.3.2 Anyons in TGD

TGD suggests a many-sheeted model for anyons used in the modelling of quantum Hall effect [42, 44, 43]. Quantum-classical correspondence requires that dissipation has space-time correlates. Hence a periodic motion should create a permanent track in space-time. This kind of track would be naturally magnetic flux tube like structure surrounding the Bohr orbit of the charged particle in the magnetic field. Anyon would be electron plus its track.

The magnetic field inside magnetic flux tubes impels the anyons to the surface of the magnetic flux tube and a highly conductive state results. The partial fusion of the flux tubes along their boundaries makes possible delocalization of valence anyons localized at the boundaries of flux tubes and implies a dramatic increase of longitudinal conductivity. When magnetic field is gradually increased the radii of flux tubes and the increase of the net flux brings in new flux tubes. The competition of these effects leads to the emergence of quantum Hall plateaus and sudden increase of the longitudinal conductivity σ_{xx}.

The simplest model explains only the filling fractions $\nu = 1/m$, m odd. The filling fractions $\nu = N/m$, m odd, require a more complex model. The transition to chaos means that periodic orbits become gradually more and more non-periodic: closed orbits fail to close after the first turn and do so only after N 2π rotations. Tracks would become N-branched surfaces. In N-branched space-time the single-valued analytic two particle wave functions $(\xi_k - \xi_l)^m$ of Laughlin [43] correspond to multiple valued wave functions $(z_k - z_l)^{m/N}$ at its M^4_+ projection and give rise to a filling fraction $\nu = N/m$. The filling fraction $\nu = N/m$, m even, requires composite fermions [48]. Anyon tracks can indeed contain up to $2N$ electrons if both directions of spin are allowed so that a rich spectroscopy is predicted: in particular anyonic super-conductivity becomes possible by 2-fermion composites. The branching gives rise to Z_N-valued topological charge.

One might think that fractional charges could be only apparent and result from the multi-branched character as charges associated with a single branch. This does not seem to be the case. Rather, the fractional charges result from the additional contribution of the vacuum Kähler charge of the anyonic flux tube to the charge of anyon. For $D = 3$ Kähler charge is topologized in the sense that the charge density is proportional to the Chern-Simons. Also anyon spin could become genuinely fractional due to the vacuum contribution of the Kähler field to the spin. Besides electronic anyons also anyons associated with various ions are predicted and certain strange experimental findings about fractional Larmor frequencies of proton in water environment [60, 55] have an elegant explanation in terms of protonic anyons with $\nu = 3/5$. In this case however the magnetic field was weaker than the Earth's magnetic field so that the belief that anyons are possible only in systems carrying very strong magnetic fields would be wrong.

In TGD framework anyons as punctures of plane would be replaced by wormhole like tubes connecting different points of the boundary of the magnetic flux tube and are predicted to always appear as pairs as they indeed do. Detailed arguments demonstrate that TGD anyons are for $N = 4$ ($\nu = 4/m$) ideal for realizing the scenario of [23] for TQC.

The TGD inspired model of non-Abelian anyons is consistent with the model of anyons based on spontaneous symmetry breaking of a gauge symmetry G to a discrete sub-group H dynamically [49]. The breaking of electro-weak gauge symmetry for classical electro-weak gauge fields occurs at the space-time sheets associated with the magnetic flux tubes defining the strands of braid. Symmetry breaking implies that elements of holonomy group span H. This group is also a discrete subgroup of color group acting as isotropy group of the many-branched surface describing anyon track inside the magnetic flux tube. Thus the elements of the holonomy group are mapped to a elements of discrete subgroup of the isometry group leading from branch to another one but leaving many-branched surface invariant.

1.3.3 Witten-Chern-Simons action and light-like 3-surfaces

The magnetic field inside magnetic flux tube expels anyons at the boundary of the flux tube. In quantum TGD framework light-like 3-surfaces of space-time surface and future light cone are in key role since they define causal determinants for Kähler action. They also provide a universal manner to satisfy boundary conditions. Hence also the boundaries of magnetic flux tube structures could be light like surfaces with respect to the induced metric of space-time sheet and would be somewhat like black hole horizons. By their metric 2-dimensionality they allow conformal invariance and due the vanishing of the metric determinant the only coordinate invariant action is Chern-Simons action associated Kähler gauge potential or with the induced electro-weak gauge potentials.

The quantum states associated with the light-like boundaries would be naturally "self-reflective" states in the sense that they correspond to S-matrix elements of the Witten-Chern-Simons topological field theory. Modular functors could results as restriction of the S-matrix to ground state degrees of freedom and Chern-Simons topological quantum field theory is a promising candidate for defining the modular functors [32, 33].

Braid group B_n is isomorphic to the first homotopy group of the configuration space $C_n(R^2)$ of n particles. $C_n(R^2)$ is $((R^2)_n - D)/S_n$, where D is the singularity represented by the configurations in which the positions of 2 or more particles. and be regarded also as the configuration associated with plane with $n+1$ punctures with $n+1$:th particle regarded as inert. The infinite order of the braid group is solely due to the 2-dimensionality. Hence the dimension $D = 4$ for space-time is unique also in the sense that it makes possible TQC.

1.4 TGD and TQC

Many-sheeted space-time concept, the possibility of negative energies, and Negentropy Maximization Principle inspire rather concrete ideas about TQC. NMP gives good hopes that the laws of Nature could take care of building fine-tuned entanglement generating 2-gates whereas 1-gates could be reduced to 2-gates for logical qubits realized using physical qubits realized as Z^4 charges and not existing as free qubits.

1.4.1 Only 2-gates are needed

The entanglement of qubits is algebraic which corresponds in TGD Universe to bound state entanglement. Negentropy Maximization Principle implies that maximal entanglement results automatically in quantum jump. This might saves from the fine-tuning of the 2-gates. In particular, the maximally entangling Yang-Baxter R-matrix is consistent with NMP.

TGD suggests a rather detailed physical realization of the model of [23] for anyonic quantum computation. The findings about strong correlation between quantum entanglement and topological entanglement are apparently contradicted by the Temperley-Lie representations for braid groups using only single qubit. The resolution of the paradox is based on the observation that in TGD framework batches containing anyon Cooper pair (AA) and single anyon (instead of two anyons as in the model of [23]) allow to represent single qubit as a logical qubit, and that mixing gate and phase gate can be represented as swap operations s_1 and s_2. Hence also 1-gates are induced by the purely topological 2-gate action, and since NMP maximizes quantum entanglement, Nature itself would take care of the fine-tuning also in this case. The quantum group representation based on $q = exp(i2\pi/5)$ is the simplest representation satisfying various constraints and is also physically very attractive. [23, 37].

1.4.2 TGD makes possible zero energy TQC

TGD allows also negative energies: besides phase conjugate photons also phase conjugate fermions and anti-fermions are possible, and matter-antimatter asymmetry might be only apparent and due to the ground state for which fermion energies are positive and anti-fermion energies negative.

This would make in principle possible zero energy topological quantum computations. The least one could hope wold be the performance of TQC in doubles of positive and negative energy computations making possible error detection by comparison. The TGD based model for anyon computation however leads to expect that negative energies play much more important role.

The idea is that the quantum states of light-like 3-surfaces represent 2-dimensional time evolutions (in particular modular functors) and that braid operations correspond to zero energy states with initial state represented by positive energy anyons and final state represented by negative energy anyons. The simplest manner to realize braid operations is by putting positive *resp.* negative energy anyons near the boundary of tube T_1 *resp.* T_2. Opposite topological charges are at the ends of the magnetic threads connecting the positive energy anyons at T_1 with the negative energy anyons at T_2. The braiding for the threads would code the quantum gates physically.

Before continuing a humble confession is in order: I am not a professional in the area of quantum information science. Despite this, my hope is that the speculations below might serve as an inspiration for real professionals in the field and help them to realize that TGD Universe provides an ideal arena for quantum information processing, and that the new view about time, space-time, and information suggests a generalization of the existing paradigm to a much more powerful one.

2 Existing view about topological quantum computation

In the sequel the evolution of ideas related to topological quantum computation, dance metaphor, and the idea about realizing the computation using a system exhibiting so called non-Abelian Quantum Hall effect, are discussed.

2.1 Evolution of ideas about TQC

The history of the TQC paradigm is as old as that of QC and involves the contribution of several Fields Medalists. At 1987 to-be Fields Medalist Vaughan Jones [34] demonstrated that the von Neumann algebras encountered in quantum theory are related to the theory of knots and allow to distinguish between very complex knots. Vaughan also demonstrated that a given knot can be characterized in terms an array of bits. The knot is oriented by assigning an arrow to each of its points and projected to a plane. The bit sequence is determined by a sequence of bits defined by the self-intersections of the knot's projection to plane. The value of the bit in a given intersection changes when the orientation of either line changes or when the line on top of another is moved under it. Since the logic operations performed by the gates of computer can be coded to matrices consisting of 0s and 1s, this means that tying a know can encode the logic operations necessary for computation.

String theorist Edward Witten [32, 33], also a Fields Medalist, connected the work of Jones to quantum physics by showing that performing measurements to a system described by a 3-dimensional topological quantum field theory defined by non-Abelian Chern-Simons action is equivalent with performing the computation that a particular braid encodes. The braids are determined by linked word lines of the particles of the topological quantum field theory. What makes braids and quantum computation so special is that the coding of the braiding pattern to a bit sequence gives rise to a code, which corresponds to a code solving NP hard problem using classical computer.

1989 computer scientist Alexei Kitaev [40] demonstrated that Witten's topological quantum field theory could form a basis for a computer. Then Fields Medalist Michael Freedman entered the scene and in collaboration with Kitaev, Michael Larson and Zhenghan Wang developed a vision of how to build a topological quantum computer [37, 23] using system exhibiting so called non-Abelian quantum Hall effect [51].

The key notion is Z_4 valued topological charge which has values 1 and 3 for anyons and 0 and 2 for their Cooper pairs. For a system of $2n$ non-Abelian anyon pairs created from vacuum there are n-1 anyon qubits analogous to spin . The notion of physical qubit is not needed at all and logical qubit is coded to the topological charge of the anyon Cooper pair. The basic idea is to utilize entanglement between Z_4 valued topological charges to achieve quantum information storage stable against de-coherence. The swap of neighboring strands of the braid is the topological correlate of a 2-gate which as such does not generate entanglement but can give rise to a transformation such as CNOT. When combined with 1-gates taking square root of qubit and relative phase, this 2-gate is able to generate $U(2^n)$.

The swap can be represented as the so called braid Yang-Baxter R-matrix characterizing also the deviation of quantum groups from ordinary groups [35]. Quite generally, all unitary Yang-Baxter R-matrices are entangling when combined with square root gate except for special values of parameters characterizing them and thus there is a rich repertoire of topologically realized quantum gates. Temperley-Lieb representation provides a 1-qubit representation for swaps in 3-braid system [35, 37]. The measurement of qubit reduces to the measurement of the topological charge of the anyon Cooper pair: in the case that it vanishes (qubit 0) the anyon Cooper pair can annihilate and this serves as the physical signature.

2.2 Topological quantum computation as quantum dance

Although topological quantum computation involves very abstract and technical mathematical thinking, it is possible to illustrate how it occurs by a very elegant metaphor. With tongue in cheek one could say that topological quantum computation occurs like a dance. Dancers form couples and in this dancing floor the partners can be also of same sex. Dancers can change their partners. If the partners are of the same sex, they define bit 1 and if they are of opposite sex they define bit 0.

To simplify things one can arrange dancers into a row or several rows such that neighboring partners along the row form a couple. The simplest situation corresponds to a single row of dancers able to make twists of 180 degrees permuting the dancers and able to change the partner to a new one any time. Dance corresponds to a pattern of tracks of dancers at the floor. This pattern can be lifted to a three-dimensional pattern introducing time as a third dimension. When one looks the tracks of a row of dancers in this 2+1-dimensional space-time, one finds that the tracks of the dancers form a complex weaved pattern known as braiding. The braid codes for the computation. The braiding consists of primitive swap operations in which two neighboring word lines twist around each other.

The values of the bits giving the result of the final state of the calculation can be detected since there is something very special which partners with opposite sex can do and do it sooner or later. Just by looking which pairs do it allows to deduce the values of the bits. The alert reader has of course guessed already now that the physical characterization for the sex is as a Z^4 valued topological charge, which is of opposite sign for the different sexes forming Cooper pairs, and that the thing that partners of opposite sex can do is to annihilate! All that is needed to look for those pairs which annihilate after the dance evening to detect the 0s in the row of bits. The coding of the sex to the sign of the topological charge implies also robustness.

It is however essential that the value of topological charge for a given particle in the final state is not completely definite (this is completely general feature of all quantum computations). One can tell only with certain probability that given couple in the final state is male-female or male-male or female-female and the probabilities in question code for the braid pattern in turn coding for quantum logic circuit. Hence one must consider an ensemble of braid calculations to deduce these probabilities.

The basic computational operation permuting the neighboring topological charges is topological so that the program represented by the braiding pattern is very stable against perturbations. The values of the topological charges are also stable. Hence the topological quantum computation is a very robust process and immune to quantum de-coherence even in the standard physics context.

2.3 Braids and gates

In order to understand better how braids define gates one must introduce some mathematical notions related to the braids.

2.3.1 Braid groups

Artin introduced the braid groups bearing his name as groups generated by the elements, which correspond to the cross section between neighboring strands of the braid. The definition of these groups is discussed in detail in [35]. For a braid having $n+1$ strands the Artin group B_{n1} has n generators s_i. The generators satisfy certain relations. Depending on whether the line coming from left is above the the line coming from right one has s_i or s_i^{-1}. The elements s_i and s_j commute for $i<j$ and $i>j+1$: $s_is_j = s_js_i$, which only says that two swaps which do not have common lines commute. For $i=j$ and $i=j+1$ commutativity is not assumed and this correspond to the situation in which the swaps act on common lines.

As already mentioned, Artin's braid group B_n is isomorphic with the homotopy group $\pi_1((R^2)^n/S_{n+1})$ of plane with $n+1$ punctures. B_n is infinite-dimensional because the conditions $s_i^2=1$ added to the defining relations in the case of permutation group S_n are not included. The infinite-dimensionality of homotopy groups reflects the very special topological role of 2-dimensional spaces.

One must consider also variants of braid groups encountered when all particles in question are not identical particles. The reason is that braid operation must be replaced by a 2π rotation of particle A around B when the particles are not identical.

1. Consider first the situation in which all particles are non-identical. The first homotopy group of $R^2)^n - D$, where D represents points configurations for which two or more points are identical is identical with the colored braid group B_n^c defined by $n+1$ punctures in plane such that $n+1$:th is passive (punctures are usually imagined to be located on line). Since particles are not identical the braid operation must be replaced by monodromy in which i:th particle makes 2π rotation around j:th particle. This group has generators

$$\gamma_{ij} = s_i...s_{j-2}s_{j-1}^2s_{j-2}...s_i^{-1} \quad ,i<j \quad , \tag{1}$$

   and can be regarded as a subgroup of the braid group.

2. When several representatives of a given particle species are present the so called partially colored braid group B_n^{pc} is believed to describe the situation. For pairs of identical particles the generators are braid generators and for non-identical particles monodromies appear as generators. It will be found later that in case of anyon bound states, the ordinary braid group with the assumption that braid operation can lead to a temporary decay and recombination of anyons to a bound state, might be a more appropriate model for what happens in braiding.

3. When all particles are identical, one has the braid group B_n, which corresponds to the fundamental group of $C_n(R^2) = ((R^2)^n - D)/S_n$. Division by S_n expresses the identicality of particles.

2.3.2 Extended Artin's group

Artin's group can be extended by introducing any group G and forming its tensor power $G^{\otimes^n} = G \otimes ... \otimes G$ by assigning to every strand of the braid group G. The extended group is formed from elements of $g_1 \otimes g_2 .... \otimes g_n$ and s_i by posing additional relations $g_i s_j = s_j g_i$ for $i < j$ and $i > j+1$. The interpretation of these relations is completely analogous to the corresponding one for the Artin's group.

If G allows representation in some space V one can look for the representations of the extended Artin's group in the space $V^{\otimes^n}$. In particular, unitary representations are possible. The space in question can also represent physical states of for instance anyonic system and the element g_i associated with the lines of the braid can represent the unitary operators characterizing the time development of the strand between up to the moment when it experiences a swap operation represented by s_i after this operation g_i becomes $s_i g_i s_i^{-1}$.

2.3.3 Braids, Yang-Baxter relations, and quantum groups

Artin's braid groups can be related directly to the so called quantum groups and Yang-Baxter relations. Yang-Baxter relations follow from the relation $s_1 s_2 s_1 = s_2 s_1 s_2$ by noticing that these operations permute the lines 123 of the braid to the order 321. By assigning to a swap operation permuting i:th and j:th line group element R_{ij} when i:th line goes over the j:th line, and noticing that $R_{ij}i$ acts in the tensor product $V_i \otimes V_j$, one can write the relation for braids in a form

$$R_{32}R_{13}R_{12} = R_{12}R_{13}R_{23} \quad .$$

Braid Yang-Baxter relations are equivalent with the so called algebraic Yang-Baxter relations encountered in quantum group theory. Algebraic R can be written as $R_a = RS$, where S is the matrix representing swap operation as a mere permutation. For a suitable choice R_a provides the fundamental representations for the elements of the quantum group $SL(n)_q$ when V is n-dimensional.

The equations represent n^6 equations for n^4 unknowns and are highly over-determined so that solving the equations is a difficult challenge. Equations have symmetries which are obvious on basis of the topological interpretation. Scaling and automorphism induced by linear transformations of V act as symmetries, and the exchange of tensor factors in $V \otimes V$ and transposition are symmetries as also shift of all indices by a constant amount (using modulo N arithmetics).

2.3.4 Unitary R-matrices

Quite a lot is known about the general solutions of the Yang-Baxter equations and for $n = 2$ the general unitary solutions of equations is known [36]. All of these solutions are entangling and define thus universal 1-gates except for certain parameter values.

The first solution is

$$R = \frac{1}{\sqrt{2}} \left(\begin{array}{cccc} 1 & \cdot & \cdot & 1 \\ \cdot & 1 & -1 & \cdot \\ \cdot & 1 & 1 & \cdot \\ -1 & \cdot & \cdot & 1 \end{array} \right) \tag{2}$$

and contains no free parameters (dots denote zeros). This R-matrix is strongly entangling. Note that the condition $R^8 = 1$ is satisfied. The defining relations for Artin's braid group allow also more general solutions obtained by multiplying R with an arbitrary phase factor. This would mean that $R^8 = 1$ constraint is not satisfied anymore. One can argue that over-all phase does not matter: on the other hand, the over all phase is visible in knot invariants defined by the trace of R.

The second and third solution come as families labelled four phases a, b, c and d:

$$R'(a,b,c,d) = \frac{1}{\sqrt{2}} \begin{pmatrix} a & \cdot & \cdot & \cdot \\ \cdot & & b & \cdot \\ \cdot & c & \cdot & \cdot \\ \cdot & \cdot & \cdot & d \end{pmatrix}$$

$$R''(a,b,c,d) = \frac{1}{\sqrt{2}} \begin{pmatrix} \cdot & \cdot & \cdot & a \\ \cdot & b & \cdot & \cdot \\ \cdot & \cdot & c & \cdot \\ d & \cdot & \cdot & \cdot \end{pmatrix} \tag{3}$$

These matrices are not as such entangling. The products $U_1 \otimes U_2 R V_1 \otimes V_2$, where U_i and V_i are 2×2 unitary matrices, are however entangling matrices and thus act as universal gates for $ad - bc \neq 0$ guaranteeing that the state $a|11\rangle + b|10\rangle + |01\rangle + |00\rangle$ is entangled.

It deserves to be noticed that the swap matrix

$$S = R'(1,1,1,1) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & & 1 & \cdot \\ \cdot & 1 & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \end{pmatrix} \tag{4}$$

permuting the qubits does not define universal gate. This is understandable since in this representation of braid group reduces it to permutation group and situation becomes completely classical.

One can write all solutions R of braid Yang-Baxter equation in the form $R = R_a$, where R_a is the solution of so called algebraic Yang-Baxter equation. The interpretation is that the swap matrix S represents the completely classical part of the swap operation since it acts as a mere permutation whereas R_a represents genuine quantum effects related to the swap operation.

In the article of Kauffman [35] its is demonstrated explicitly how to construct CNOT gate as a product MRN, where M and N are products of single particle gates. This article contains also a beautiful discussion about how the traces of the unitary matrices defined by the braids define knot invariants. For instance, the matrix R satisfies $R^8 = 1$ so that the invariants constructed using R as 2-gate cannot distinguish between knots containing n and $n + 8k$ sub-sequent swaps. Note however that the multiplication of R with a phase factor allows to get rid of the 8-periodicity.

2.3.5 Knots, links, braids, and quantum 2-gates

In [35] basic facts about knots, links, and their relation to braids are discussed. Knot diagrams are introduced, the so called Reidermeister moves and homeomorphisms of plane as isotopies of knots and links are discussed. Also the notion of braid closure producing knots or links is introduced together with the theorem of Markov stating that any knot and link corresponds to some (not unique) braid. Markov moves as braid deformations leaving corresponding knots and links invariant are discussed and it the immediate implication is that traces of the braid matrices define knot invariants. In particular, the traces of the unitary matrices defined by R-matrix define invariants having same value for the knots and links resulting in the braid closure.

In [35] the state preparation and quantum measurement allowing to deduce the absolute value of the trace of the unitary matrix associated with the braid defining the quantum computer is discussed as an example how quantum computations could occur in practice. The braid in question is product of the braid defining the invariant and trivial braid with same number n of strands. The incoming state is maximally entangled state formed $\sum_n |n\rangle \otimes |n\rangle$, where n runs over all possible bit sequences defined by the tensor product of n qubits. Quantum measurement performs a projection to this state and from the measurements it is possible to deduce the absolute value of the trace defining the knot invariant.

2.4 About quantum Hall effect and theories of quantum Hall effect

Using the dance metaphor for TQC, the system must be such that it is possible to distinguish between the different sexes of dancers. The proposal of [23] is that the system exhibiting so called non-Abelian Quantum Hall effect [44, 50] could make possible realization of the topological computation.

The most elegant models of quantum Hall effect are in terms of anyons regarded as singularities due to the symmetry breaking of gauge group G down to a finite sub-group H, which can be also non-Abelian. Concerning the description of the dynamics of topological degrees of freedom topological quantum field theories based on Chern-Simons action are the most promising approach.

2.4.1 Quantum Hall effect

Quantum Hall effect [42, 44] occurs in 2-dimensional systems, typically a slab carrying a longitudinal voltage V causing longitudinal current j. A magnetic field orthogonal to the slab generates a transversal current component j_T by Lorentz force. j_T is proportional to the voltage V along the slab and the dimensionless coefficient is known as transversal conductivity. Classically the coefficients is proportional ne/B, where n is 2-dimensional electron density and should have a continuous spectrum. The finding that came as surprise was that the change of the coefficient as a function of parameters like magnetic field strength and temperature occurred as discrete steps of same size. In integer quantum Hall effect the coefficient is quantized to $2\nu\alpha$, $\alpha = e^2/4\pi$, such that ν is integer.

Later came the finding that also smaller steps corresponding to the filling fraction $\nu = 1/3$ of the basic step were present and could be understood if the charge of electron would have been replaced with $\nu = 1/3$ of its ordinary value. Later also QH effect with wide large range of filling fractions of form $\nu = k/m$ was observed.

The model explaining the QH effect is based on pseudo particles known as anyons [49, 44]. According to the general argument of [42] anyons have fractional charge νe. Also the TGD based model for fractionization to be discussed later suggests that the anyon charge should be νe quite generally. The braid statistics of anyon is believed to be fractional so that anyons are neither bosons nor fermions. Non-fractional statistics is absolutely essential for the vacuum degeneracy used to represent logical qubits.

In the case of Abelian anyons the gauge potential corresponds to the vector potential of the divergence free velocity field or equivalently of incompressible anyon current. For non-Abelian anyons the field theory defined by Chern-Simons action is free field theory and in well-defined sense trivial although it defines knot invariants. For non-Abelian anyons situation would be different. They would carry non-Abelian gauge charges possibly related to a symmetry breaking to a discrete subgroup H of gauge group [49] each of them defining an incompressible hydrodynamical flow. Non-Abelian QH effect has not yet been convincingly demonstrated experimentally. According to [23] the anyons associated with the filling fraction $\nu = 5/2$ are a good candidate for non-Abelian anyons and in this case the charge of electron is reduced to $Q = 1/4$ rather than being $Q = \nu e$.

Non-Abelian anyons [51, 44] are always created in pairs since they carry a conserved topological charge. In the model of [23] this charge should have values in 4-element group Z_4 so that it is conserved only modulo 4 so that charges +2 and -2 are equivalent as are also charges 3 and -1. The state of n anyon pairs created from vacuum can be show to possess 2^{n-1}-dimensional vacuum degeneracy [50]: later a TGD based argument for why this is the case is constructed. When two anyons fuse the 2^{n-1}-dimensional state space decomposes to 2^{n-2}-dimensional tensor factors corresponding to anyon Cooper pairs with topological charges 2 and 0. The topological "spin" is ideal for representing logical qubits. Since free topological charges are not possible the notion of physical qubit does not make sense (note the analogy with quarks). The measurement of topological qubit reduces to a measurement of whether anyon Cooper pair has vanishing topological charge or not.

2.4.2 Quantum Hall effect as a spontaneous symmetry breaking down to a discrete subgroup of the gauge group

The system exhibiting quantum Hall effect is effectively 2-dimensional. Fractional statistics suggests that topological defects, anyons, allowing a description in terms of the representations of the homotopy group of $((R^2)^n - D)/S_n$. The gauge theory description would be in terms of spontaneous symmetry breaking of the gauge group G to a finite subgroup H by a Higgs mechanism [49, 44]. This would make all gauge degrees of freedom massive and leave only topological degrees of freedom. What is unexpected that also non-Abelian topological degrees of freedom are in principle possible. Quantum Hall effect is Abelian or non-Abelian depending on whether the group H has this property.

In the symmetry breaking $G \rightarrow H$ the non-Abelian gauge fluxes defined as non-integrable phase factors $Pexp(i\oint A_\mu dx^\mu)$ around large circles (surrounding singularities (so that field approaches a pure gauge configuration) are elements of the first homotopy group of G/H, which is H in the case that H is discrete group and G is simple. An idealized manner to model the situation [44] is to assume that the connection is pure gauge and defined by an H-valued function which is many-valued such that the values for different branches are related by a gauge transformation in H. In the general case a gauge transformation of a non-trivial gauge field by a multi-valued element of the gauge group would give rise to a similar situation.

One can characterize a given topological singularity magnetically by an element in conjugacy class C of H representing the transformation of H induced by a 2π rotation around singularity. The elements of C define states in given magnetic representation. Electrically the particles are characterized by an irreducible representations of the subgroup of $H_C \subset H$ which commutes with an arbitrarily chosen element of the conjugacy class C.

The action of $h(B)$ resulting on particle A when it makes a closed turn around B reduces in magnetic degrees of freedom to translation in conjugacy class combined with the action of element of H_C in electric degrees of freedom. Closed paths correspond to elements of the braid group $B_n(X^2)$ identifiable as the mapping class group of the punctured 2-surface X^2 and this means that symmetry breaking $G \rightarrow H$ defines a representation of the braid group. The construction of these representations is discussed in [44] and leads naturally via the group algebra of H to the so called quantum double $D(H)$ of H, which is a quasi-triangular Hopf algebra allowing non-trivial representations of braid group.

Anyons could be singularities of gauge fields, perhaps even non-Abelian gauge fields, and the latter ones could be modelled by these representations. In particular, braid operations could be represented using anyons.

2.4.3 Witten-Chern-Simons action and topological quantum field theories

The Wess-Zumino-Witten action used to model 2-dimensional critical systems consists of a 2-dimensional conformally invariant term for the chiral field having values in group G combined with 2+1-dimensional term defined as the integral of Chern-Simons 3-form over a 3-space containing 2-D space as its boundary. This term is purely topological and identifiable as winding number for the map from 3-dimensional space to G. The coefficient of this term is integer k in suitable normalization. k gives the value of central extension of the Kac-Moody algebra defined by the theory.

One can couple the chiral field $g(x)$ to gauge potential defined for some subgroup of G_1 of G. If the G_1 coincides with G, the chiral field can be gauged away by a suitable gauge transformation and the theory becomes purely topological Witten-Chern-Simons theory. Pure gauge field configuration represented either as flat gauge fields with non-trivial holonomy over homotopically non-trivial paths or as multi-valued gauge group elements however remain and the remaining degrees of freedom correspond to the topological degrees of freedom.

Witten-Chern-Simons theories are labelled by a positive integer k giving the value of central extension of the Kac-Moody algebra defined by the theory. The connection with Wess-Zumino-Witten theory come from the fact that the highest weight states associated with the representations of the Kac-Moody algebra of WZW theory are in one-one correspondence with the representations R_i possible for Wilson loops in the topological quantum field theory.

In the Abelian case case 2+1-dimensional Chern-Simons action density is essentially the inner product $A\wedge dA$ of the vector potential and magnetic field known as helicity density and the theory in question is a free field theory. In the non-Abelian case the action is defined by the 3-form

$$\frac{k}{4\pi}Tr\left(A\wedge(dA+\frac{2}{3}A\wedge A)\right)$$

and contains also interaction term so that the field theory defined by the exponential of the interaction term is non-trivial.

In topological quantum field theory the usual n-point correlation functions defined by the functional integral are replaced by the functional averages for $Diff^3$ invariant quantities defined in terms of non-integrable phase factors defined by ordered exponentials over closed loops. One can consider arbitrary number of loops which can be knotted, linked, and braided. These quantities define both knot and 3-manifold invariants (the functional integral for zero link in particular). The perturbative calculation of the quantum averages leads directly to the Gaussian linking numbers and infinite number of perturbative link and not invariants.

The experience gained from topological quantum field theories defined by Chern-Simons action has led to a very elegant and surprisingly simple category theoretical approach to the topological quantum field theory [30, 31] allowing to assign invariants to knots, links, braids, and tangles and also to 3-manifolds for which braids as morphisms are replaced with cobordisms. The so called modular Hopf algebras, in particular quantum groups $Sl(2)_q$ with q a root of unity, are in key role in this approach. Also the connection between links and 3-manifolds can be understood since closed, oriented, 3-manifolds can be constructed from each other by surgery based on links.

Witten's article [32] "Quantum Field Theory and the Jones Polynomial" is full of ingenious constructions, and for a physicist it is the easiest and certainly highly enjoyable manner to learn about knots and 3-manifolds. For these reasons a little bit more detailed sum up is perhaps in order.

1. Witten discusses first the quantization of Chern-Simons action at the weak coupling limit $k\rightarrow\infty$. First it is shown how the functional integration around flat connections defines a topological invariant for 3-manifolds in the case of a trivial Wilson loop. Next a canonical quantization is performed in the case $X^3=\Sigma^2\times R^1$: in the Coulomb gauge $A_3=0$ the

action reduces to a sum of $n = dim(G)$ Abelian Chern-Simons actions with a non-linear constraint expressing the vanishing of the gauge field. The configuration space consists thus of flat non-Abelian connections, which are characterized by their holonomy groups and allows Kähler manifold structure.

2. Perhaps the most elegant quantal element of the approach is the decomposition of the 3-manifold to two pieces glued together along 2-manifold implying the decomposition of the functional integral to a product of functional integrals over the pieces. This together with the basic properties of Hilbert of complex numbers (to which the partition functions defined by the functional integrals over the two pieces belong) allows almost a miracle like deduction of the basic results about the behavior of 3-manifold and link invariants under a connected sum, and leads to the crucial skein relations allowing to calculate the invariants by decomposing the link step by step to a union of unknotted, unlinked Wilson loops, which can be calculated exactly for $SU(N)$. The decomposition by skein relations gives rise to a partition function like representation of invariants and allows to understand the connection between knot theory and statistical physics [16]. A direct relationship with conformal field theories and Wess-Zumino-Witten model emerges via Wilson loops associated with the highest weight representations for Kac Moody algebras.

3. A similar decomposition procedure applies also to the calculation of 3-manifold invariants using link surgery to transform 3-manifolds to each other, with 3-manifold invariants being defined as Wilson loops associated with the homology generators of these (solid) tori using representations R_i appearing as highest weight representations of the loop algebra of torus. Surgery operations are represented as mapping class group operations acting in the Hilbert space defined by the invariants for representations R_i for the original 3-manifold. The outcome is explicit formulas for the invariants of trivial knots and 3-manifold invariant of S^3 for $G = SU(N)$, in terms of which more complex invariants are expressible.

4. For $SU(N)$ the invariants are expressible as functions of the phase $q = exp(i2\pi/(k+N))$ associated with quantum groups. Note that for $SU(2)$ and $k = 3$, the invariants are expressible in terms of Golden Ratio. The central charge $k = 3$ is in a special position since it gives rise to $k + 1 = 4$-vertex representing naturally 2-gate physically. Witten-Chern-Simons theories define universal unitary modular functors characterizing quantum computations [37].

2.4.4 Chern-Simons action for anyons

In the case of quantum Hall effect the Chern-Simons action has been deduced from a model of electrons as a 2-dimensional incompressible fluid [43]. Incompressibility requires that the electron current has a vanishing divergence, which makes it analogous to a magnetic field. The expressibility of the current as a curl of a vector potential b, and a detailed study of the interaction Lagrangian leads to the identification of an Abelian Chern-Simons for b as a low energy effective action. This action is Abelian, whereas the anyonic realization of quantum computation would suggest a non-Abelian Chern-Simons action.

Non-Abelian Chern-Simons action could result in the symmetry breaking of a non-Abelian gauge group G, most naturally electro-weak gauge group, to a non-Abelian discrete subgroup H [49] so that states would be labelled by representations of H and anyons would be characterized magnetically H-valued non-Abelian magnetic fluxes each of them defining its own incompressible hydro-dynamical flow. As will be found, TGD predicts a non-Abelian Chern-Simons term associated with electroweak long range classical fields.

2.5 Topological quantum computation using braids and anyons

By the general mathematical results braids are able to code all quantum logic operations [41]. In particular, braids allow to realize any quantum circuit consisting of single particle gates acting on qubits and two particle gates acting on pairs of qubits. The coding of braid requires a classical computation which can be done in polynomial time. The coding requires that each dancer is able to remember its dancing history by coding it into its own state.

The general ideas are following.

1. The ground states of anyonic system characterize the logical qubits, One assumes non-Abelian anyons with Z_4 -valued topological charge so that a system of n anyon pairs created from vacuum allows 2^{n-1}-fold anyon degeneracy [50]. The system is decomposed into blocks containing one anyonic Cooper pair with $Q_T \in \{2,0\}$ and two anyons with such topological charges that the net topological charge vanishes. One can say that the states $(0,1-1)$ and $(0,-1,+1))$ represent logical qubit 0 whereas the states $(2,-1,-1)$ and $(2,+1,+1)$ represent logical qubit 1. This would suggest 2^2-fold degeneracy but actually the degeneracy is 2-fold.

   Free physical qubits are not possible and at least four particles are indeed necessarily in order to represent logical qubit. The reason is that the conservation of Z^4 charge would not allow mixing of qubits 1 and 0, in particular the Hadamard 1-gate generating square root of qubit would break the conservation of topological charge. The square root of qubit can be generated only if 2 units of topological charge is transferred between anyon and anyon Cooper pair. Thus qubits can be represented as entangled states of anyon Cooper pair and anyon and the fourth anyon is needed to achieve vanishing total topological charge in the batch.

2. In the initial state of the system the anyonic Cooper pairs have $Q_T = 0$ and the two anyons have opposite topological charges inside each block. The initial state codes no information unlike in ordinary computation but the information is represented by the braid. Of course, also more general configurations are possible. Anyons are assumed to evolve like free particles except during swap operations and their time evolution is described by single particle Hamiltonians.

   Free particle approximation fails when the anyons are too near to each other as during braid operations. The space of logical qubits is realized as k-code defined by the 2^{n-1} ground states, which are stable against local single particle perturbations for $k = 3$ Witten-Chern-Simons action. In the more general case the stability against n-particle perturbations with $n < [k/2]$ is achieved but the gates would become $[k/2]$-particle gates (for $k = 5$ this would give 6-particle vertices).

3. Anyonic system provides a unitary modular functor as the S-matrix associated with the anyon system whose time evolution is fixed by the pre-existing braid structure. What this means that the S-matrices associated with the braids can be multiplied and thus a unitary representation for the group formed by braids results. The vacuum degeneracy of anyon system makes this representation non-trivial. By the NP complexity of braids it is possible to code any quantum logic operation by a particular braid [39]. There exists a powerful approximation theorem allowing to achieve this coding classically in polynomial time [41]. From the properties of the R-matrices inducing gate operations it is indeed clear that two gates can be realized. The Hadamard 1-gate could be realized as 2-gate in the system formed by anyon Cooper pair and anyon.

4. In [23] the time evolution is regarded as a discrete sequence of modifications of single anyon Hamiltonians induced by swaps [38]. If the modifications define a closed loop in the space of Hamiltonians the resulting unitary operators define a representation of braid group in a

dense discrete sub-group of $U(2^n)$. The swap operation is 2-local operation acting like a 2-gate and induces quantum logical operation modifying also single particle Hamiltonians. What is important that this modification maps the space of the ground states to a new one and only if the modifications correspond to a closed loop the final state is in the same code space as the initial state. What time evolution does is to affect the topological charges of anyon Cooper pairs representing qubits inside the 4-anyon batches defined by the braids.

In quantum field theory the analog but not equivalent of this description would be following. Quite generally, a given particle in the final state has suffered a unitary transformation, which is an ordered product consisting of two kinds of unitary operators. Unitary single particle operators $U_n = Pexp(i\int_{t_n}^{t_{n+1}} H_0 dt)$ are analogs of operators describing single qubit gate and play the role of anyon propagators during no-swap periods. Two-particle unitary operators $U_{swap} = Pexp(i\int H_{swap}dt)$ are analogous to four-particle interactions and describe the effect of braid operations inducing entanglement of states having opposite values of topological charge but conserving the net topological charge of the anyon pair. This entanglement is completely analogous to spin entanglement. In particular, the braid operation mixes different states of the anyon. The unitary time development operator generating entangled state of anyons and defined by the braid structure represents the operation performed by the quantum circuit and the quantum measurement in the final state selects a particular final state.

5. Formally the computation halts with a measurement of the topological charge of the left-most anyon Cooper pair when the outcome is just single bit. If decay occurs with sufficiently high probability it is concluded that the value of the computed bit is 0, otherwise 1.

3 General implications of TGD for quantum computation

TGD based view about time and space-time could have rather dramatic implications for quantum computation in general and these implications deserve to be discussed briefly.

3.1 Time need not be a problem for quantum computations in TGD Universe

Communication with and control of the geometric past is the basic mechanism of intentional action, sensory perception, and long term memory in TGD inspired theory of consciousness. The possibility to send negative energy signals to the geometric past allows also instantaneous computations with respect to subjective time defined by a sequence of quantum jumps. The physicist of year 2100 can induce the quantum jump to turn on the quantum computer at 2050 to perform a simulation of field equations defined by the absolute minimization of Kähler action and lasting 50 geometric years, and if this is not enough iterate the process by sending the outcome of computation back to the past where it defines initial values of the next round of iteration. Time would cease to be a limiting factor to computation.

3.2 New view about information

The notion of information is very problematic even in the classical physics and in quantum realm this concept becomes even more enigmatic. TGD inspired theory consciousness has inspired number theoretic ideas about quantum information which are still developing. The standard definition of entanglement entropy relies on the Shannon's formula: $S = -\sum_k p_k log(p_k)$. This entropy is always non-negative and tells that the best one can achieve is entanglement with zero entropy.

The generalization of the notion of entanglement entropy to the p-adic context however led to realization that entanglement for which entanglement probabilities are rational or in an extension of rational numbers defining a finite extension of p-adics allows a hierarchy of entanglement entropies S_p labelled by primes. These entropies are defined as $S_p = -\sum_k p_k log(|p_k|_p)$, where $|p_k|_p$ denotes the p-adic norm of probability. S_p can be negative and in this case defines a genuine information measure. For given entanglement probabilities S_p has a minimum for some value p_0 of prime p, and S_{p_0} could be taken as a measure for the information carried by the entanglement in question whereas entanglement in real and p-adic continua would be entropic. The entanglement with negative entanglement entropy is identified as bound state entanglement.

Since quantum computers by definition apply states for which entanglement coefficients belong to a finite algebraic extension of rational numbers, the resulting states, if ideal, should be bound states. Also finite-dimensional extensions of p-adic numbers by transcendentals are possible. For instance, the extension by the $p-1$ first powers of e (e^p is ordinary p-adic number in R_p). As an extension of rationals this extension would be discrete but infinite-dimensional. Macro-temporal quantum coherence can be identified as being due to bound state formation in appropriate degrees of freedom and implying that state preparation and state function reduction effectively ceases to occur in these degrees of freedom.

Macro-temporal quantum coherence effectively binds a sequence of quantum jumps to single quantum jump so that the effective duration of unitary evolution is stretched from about 10^4 Planck times to arbitrary long time span. Also quantum computations can be regarded as this kind of extended moments of consciousness.

3.3 Number theoretic vision about quantum jump as a building block of conscious experience

The generalization of number concept resulting when reals and various p-adic number fields are fused to a book like structure obtaining by gluing them along rational numbers common to all these number fields leads to an extremely general view about what happens in quantum jump identified as basic building block of conscious experience. First of all, the unitary process U generates a formal superposition of states belonging to different number fields including their extensions. Negentropy Maximization Principle [H2] constrains the dynamics of state preparation and state function reduction following U so that the final state contains only rational or extended rational entanglement with positive information content. At the level of conscious experience this process can be interpreted as a cognitive process or analysis leading to a state containing only bound state entanglement serving as a correlate for the experience of understanding. Thus quantum information science and quantum theory of consciousness seem to meet each other.

In the standard approach to quantum computing entanglement is not bound state entanglement. If bound state entanglement is really the entanglement which is possible for quantum computer, the entanglement of qubits might not serve as a universal entanglement currency. That is, the reduction of the general two-particle entanglement to entanglement between N qubits might not be possible in TGD framework.

The conclusion that only bound state entanglement is possible in quantum computation in human time scales is however based on the somewhat questionable heuristic assumption that subjective time has the same universal rate, that is the average increment Δt of the geometric time in single quantum jump does not depend on the space-time sheet, and is of order CP_2 time about 10^4 Planck times. The conclusion could be circumvented if one assumes that Δt depends on the space-time sheet involved: for instance, instead of CP_2 time Δt could be of order p-adic time scale T_p for a space-time sheet labelled by p-adic prime p and increase like $\sqrt{p}$. In this case the unitary operator defining quantum computation would be simply that defining the unitary process U.

3.4 Dissipative quantum parallelism?

The new view about quantum jump implies that state function reduction and preparation process decomposes into a hierarchy of these processes occurring in various scales: dissipation would occur in quantum parallel manner with each p-adic scale defining one level in the hierarchy. At space-time level this would correspond to almost independent quantum dynamics at parallel space-time sheets labelled by p-adic primes. In particular, dissipative processes can occur in short scales while the dynamics in longer scales is non-dissipative. This would explain why the description of hadrons as dissipative systems consisting of quarks and gluons in short scales is consistent with the description of hadrons as genuine quantum systems in long scales. Dissipative quantum parallelism would also mean that thermodynamics at shorter length scales would stabilize the dynamics at longer length scales and in this manner favor scaled up quantum coherence.

NMR systems [21] might represent an example about dissipative quantum parallelism. Room temperature NMR (nuclear magnetic resonance) systems use highly redundant replicas of qubits which have very long coherence times. Quantum gates using radio frequency pulses to modify the spin evolution have been implemented, and even effective Hamiltonians have been synthesized. Quantum computations and dynamics of other quantum systems have been simulated and quantum error protocols have been realized. These successes are unexpected since the energy scale of cyclotron states is much below the thermal energy. This has raised fundamental questions about the power of quantum information processing in highly mixed states, and it might be that dissipative quantum parallelism is needed to explain the successes.

Magnetized systems could realize quite concretely the renormalization group philosophy in the sense that the magnetic fields due to the magnetization at the atomic space-time sheets could define a return flux along larger space-time sheets as magnetic flux quanta (by topological flux quantization) defining effective block spins serving as thermally stabilized qubits for a long length scale quantum parallel dynamics. For an external magnetic field $B \sim 10$ Tesla the magnetic length is $L \sim 10$ nm and corresponds to the p-adic length scale $L(k = 151)$. The induced magnetization is $M \sim n\mu^2 B/T$, where n is the density of nuclei and $\mu = ge/2m_p$ is the magnetic moment of nucleus. For solid matter density the magnetization is by a factor ~ 10 weaker than the Earth's magnetic field and corresponds to a magnetic length $L \sim 15$ μm: the p-adic length scale is around $L(171)$. For 10^{22} spins per block spin used for NMR simulations the size of block spin should be $\sim$ 1mm solid matter density so that single block spin would contain roughly 10^6 magnetization flux quanta containing 10^{16} spins each. The magnetization flux quanta serving as logical qubits could allow to circumvent the standard physics upper bound for scaling up of about 10 logical qubits [21].

3.5 Negative energies and quantum computation

In TGD universe space-times are 4-surfaces so that negative energies are possible due to the fact that the sign of energy depends on time orientation (energy momentum tensor is replaced by a collection of conserved momentum currents). This has several implications. Negative energy photons having phase conjugate photons as physical correlates of photons play a key role in TGD inspired theory of consciousness and living matter and there are also indications that magnetic flux tubes structures with negative energies are important.

Negative energies makes possible communications to the geometric past, and time mirror mechanism involving generation of negative energy photons is the key mechanism of intentional action and plays central role in the model for the functioning of bio-systems. In principle this could allow to circumvent the problems due to the time required by computation by initiating computation in the geometric past and iterating this process. The most elegant and predictive cosmology is that for which the net conserved quantities of the universe vanish due the natural boundary condition that nothing flows into the future light cone through its boundaries representing the moment of big bang.

Also topological quantum field theories describe systems for which conserved quantities associated with the isometries of space-time, such as energy and momentum, vanish. Hence the natural question is whether negative energies making possible zero energy states might also make possible also zero energy quantum computations.

3.5.1 Crossing symmetry and Eastern and Western views about what happens in scattering

The hypothesis that all physical states have vanishing net quantum numbers (Eastern view) forces to interpret the scattering events of particle physics as quantum jumps between different vacua. This interpretation is in a satisfactory consistency with the assumption about existence of objective reality characterized by a positive energy (Western view) if crossing symmetry holds so that configuration space spinor fields can be regarded as S-matrix elements between initial state defined by positive energy particles and negative energy state defined by negative energy particles. As a matter fact, the proposal for the S-matrix of TGD at elementary particle level relies on this idea: the amplitudes for the transition from vacuum to states having vanishing net quantum numbers with positive and negative energy states interpreted as incoming and outgoing states are assumed to be interpretable as S-matrix elements.

More generally, one could require that scattering between any pair of states with zero net energies and representing S-matrix allows interpretation as a scattering between positive energy states. This requirement is satisfied if their exists an entire self-reflective hierarchy of S-matrices in the sense that the S-matrix between states representing S-matrices S_1 and S_2 would be the tensor product $S_1 \otimes S_2$. At the observational level the experience the usual sequence of observations $|m_1\rangle \rightarrow |m_2\rangle ..... \rightarrow |m_n\rangle ...$ based on belief about objective reality with non-vanishing conserved net quantum numbers would correspond to a sequence $(|m_1 \rightarrow m_2\rangle \rightarrow |m_2 \rightarrow m_3\rangle ...$ between "self-reflective" zero energy states. These sequences are expected to be of special importance since the contribution of the unit matrix to S-matrix $S = 1 + iT$ gives dominating contribution unless interactions are strong. This sequence would result in the approximation that $S_2 = 1 + iT_2$ in $S = S_1 \otimes S_2$ is diagonal. The fact that the scattering for macroscopic systems tends to be in forward direction would help to create the materialistic illusion about unique objective reality.

It should be possible to test whether the Eastern or Western view is correct by looking what happens strong interacting systems where the western view should fail. The Eastern view is consistent with the basic vision about quantum jumps between quantum histories having as a counterpart the change of the geometric past at space-time level.

3.5.2 Negative energy anti-fermions and matter-antimatter asymmetry

The assumption that space-time is 4-surface means that the sign of energy depends on time orientation so that negative energies are possible. Phase conjugate photons [52] are excellent candidates for negative energy photons propagating into geometric past.

Also the phase conjugate fermions make in principle sense and one can indeed perform Dirac quantization in four manners such that a) both fermions and anti-fermions have positive/negative energies, b) fermions (anti-fermions) have positive energies and anti-fermions (fermions) have negative energies. The corresponding ground state correspond to Dirac seas obtained by applying the product of a) all fermionic and anti-fermionic annihilation (creation) operators to vacuum, b) all fermionic creation (annihilation) operators and anti-fermionic annihilation (creation) operators to vacuum. The ground states of a) have infinite vacuum energy which is either negative or positive whereas the ground states of b) have vanishing vacuum energy. The case b) with positive fermionic and negative anti-fermionic energies could correspond to long length scales in which are matter-antisymmetric due to the effective absence of anti-fermions ("effective" meaning that no-one has tried to detect the negative energy anti-fermions). The case a) with positive energies could naturally correspond to the phase studied in elementary particle physics.

If gravitational and inertial masses have same magnitude and same sign, consistency with empirical facts requires that positive and negative energy matter must have been separated in cosmological length scales. Gravitational repulsion might be the mechanism causing this. Applying naively Newton's equations to a system of two bodies with energies $E_1 > 0$ and $-E_2 < 0$ and assuming only gravitational force, one finds that the sign of force for the motion in relative coordinates is determined by the sign of the reduced mass $-E_1E_2/(E_1 - E_2)$, which is negative for $E_1 > |E_2|$: positive masses would act repulsively on smaller negative masses. For $E_1 = -E_2$ the motion in the relative coordinate becomes free motion and both systems experience same acceleration which for E_1 corresponds to a repulsive force. The reader has probably already asked whether the observed acceleration of the cosmological expansion interpreted in terms of cosmological constant due to vacuum energy could actually correspond to a repulsive force between positive and negative energy matter.

It is possible to create pairs of positive energy fermions and negative energy fermions from vacuum. For instance, annihilation of photons and phase conjugate photons could create electron and negative energy positron pairs with a vanishing net energy. Magnetic flux tubes having positive and negative energies carrying fermions and negative energy positrons pairs of photons and their phase conjugates via fermion anti-fermion annihilation. The obvious idea is to perform zero energy topological quantum computations by using anyons of positive energy and anti-anyons of negative energy plus their Cooper pairs. This idea will be discussed later in more detail.

4 TGD based new physics related to topological quantum computation

The absolute minimization of Kähler action is the basic dynamical principle of space-time dynamics. For a long time it remained an open question whether the known solutions of field equations are building blocks of the absolute minima of Kähler action or represent only the simplest extremals one can imagine and perhaps devoid of any real significance. Quantum-classical correspondence meant a great progress in the understanding the solution spectrum of field equations. Among other things, this principle requires that the dissipative quantum dynamics leading to non-dissipating asymptotic self-organization patterns should have the vanishing of the Lorentz 4-force as space-time correlate. The absence of dissipation in the sense of vanishing of Lorentz 4-force is a natural correlate for the absence of dissipation in quantum computations. Furthermore, absolute minimization, if it is really a fundamental principle, should represent the second law of thermodynamics at space-time level. Of course, one cannot exclude the possibility that second law of thermodynamics at space-time level could replace absolute minimization as the basic principle.

The vanishing of Lorentz 4-force generalizes the so called Beltrami conditions [17, 19] stating the vanishing of Lorentz force for purely magnetic field configurations and these conditions reduce in many cases to topological conditions. The study of classical field equations predicts three phases corresponding to non-vacuum solutions of field equations possessing vanishing Lorentz force. The dimension D of CP_2 projection of the space-time sheet serves as classifier of the phases.

1. $D = 2$ phase is analogous to ferro-magnetic phase possible in low temperatures and relatively simple, $D = 4$ phase is in turn analogous to a chaotic de-magnetized high temperature phase.

2. $D = 3$ phase represents spin glass phase, kind of boundary region between order and chaos possible in a finite temperature range and is an ideal candidate for the field body serving as a template for living systems. $D = 3$ phase allows infinite number of conserved topological

charges having interpretation as invariants describing the linking of the magnetic field lines. This phase is also the phase in which topological quantum computations are possible.

4.1 Topologically quantized generalized Beltrami fields and braiding

From the construction of the solutions of field equations in terms topologically quantized fields it is obvious that TGD Universe is tailor made for TQC.

4.1.1 $D = 3$ phase allows infinite number of topological charges characterizing the linking of magnetic field lines

When space-time sheet possesses a $D = 3$-dimensional CP_2 projection, one can assign to it a non-vanishing and conserved topological charge characterizing the linking of the magnetic field lines defined by Chern-Simons action density $A \wedge dA/4\pi$ for induced Kähler form. This charge can be seen as classical topological invariant of the linked structure formed by magnetic field lines. For $D = 2$ the topological charge densities vanish identically, for $D = 3$ they are in general non-vanishing and conserved, whereas for $D = 4$ they are not conserved. The transition to $D = 4$ phase can thus be used to erase quantum computer programs realized as braids. The 3-dimensional CP_2 projection provides an economical manner to represent the braided world line pattern of dancers and would be the space where the 3-dimensional quantum field theory would be defined.

The topological charge can also vanish for $D = 3$ space-time sheets. In Darboux coordinates for which Kähler gauge potential reads as $A = P_k dQ^k$, the surfaces of this kind result if one has $Q^2 = f(Q^1)$ implying $A = fdQ^1$, $f = P_1 + P_2\partial_{Q_1}Q^2$, which implies the condition $A \wedge dA = 0$. For these space-time sheets one can introduce Q^1 as a global coordinate along field lines of A and define the phase factor $exp(i \int A_\mu dx^\mu)$ as a wave function defined for the entire space-time sheet. This function could be interpreted as a phase of an order order parameter of super-conductor like state and there is a high temptation to assume that quantum coherence in this sense is lost for more general $D = 3$ solutions. Note however that in boundaries can still remain super-conducting and it seems that this occurs in the case of anyons.

Chern-Simons action is known as helicity in electrodynamics [20]. Helicity indeed describes the linking of magnetic flux lines as is easy to see by interpreting magnetic field as incompressible fluid flow having A as vector potential: $B = \nabla \times A$. One can write A using the inverse of $\nabla\times$ as $A = (1/\nabla\times)B$. The inverse is non-local operator expressible as

$$\frac{1}{\nabla\times}B(r) = \int dV' \frac{(r - r')}{|r - r'|^3} \times B(r') \ ,$$

as a little calculation shows. This allows to write $\int A \cdot B$ as

$$\int dV A \cdot B = \int dV dV' B(r) \cdot \left(\frac{(r - r')}{|r - r'|^3} \times B(r') \right) \ ,$$

which is completely analogous to the Gauss formula for linking number when linked curves are replaced by a distribution of linked curves and an average is taken.

For $D = 3$ field equations imply that Kähler current is proportional to the helicity current by a factor which depends on CP_2 coordinates, which implies that the current is automatically divergence free and defines a conserved charge for $D = 3$-dimensional CP_2 projection for which the instanton density vanishes identically. Kähler charge is not equal to the helicity defined by the inner product of magnetic field and vector potential but to a more general topological charge.

The number of conserved topological charges is infinite since the product of any function of CP_2 coordinates with the helicity current has vanishing divergence and defines a topological charge. A very natural function basis is provided by the scalar spherical harmonics of $SU(3)$ defining Hamiltonians of CP_2 canonical transformations and possessing well defined color quantum numbers. These functions define and infinite number of conserved charges which are also classical knot invariants in the sense that they are not affected at all when the 3-surface interpreted as a map from CP_2 projection to M^4_+ is deformed in M^4_+ degrees of freedom. Also canonical transformations induced by Hamiltonians in irreducible representations of color group affect these invariants via Poisson bracket action when the $U(1)$ gauge transformation induced by the canonical transformation corresponds to a single valued scalar function. These link invariants are additive in union whereas the quantum invariants defined by topological quantum field theories are multiplicative.

Also non-Abelian topological charges are well-defined. One can generalize the topological current associated with the Kähler form to a corresponding current associated with the induced electro-weak gauge fields whereas for classical color gauge fields the Chern-Simons form vanishes identically. Also in this case one can multiply the current by CP_2 color harmonics to obtain an infinite number of invariants in $D = 3$ case. The only difference is that $A \wedge dA$ is replaced by $Tr(A \wedge (dA + 2A \wedge A/3))$.

There is a strong temptation to assume that these conserved charges characterize colored quantum states of the conformally invariant quantum theory as a functional of the light-like 3-surface defining boundary of space-time sheet or elementary particle horizon surrounding wormhole contacts. They would be TGD analogs of the states of the topological quantum field theory defined by Chern-Simons action as highest weight states associated with corresponding Wess-Zumino-Witten theory. These charges could be interpreted as topological counterparts of the isometry charges of configuration space of 3-surfaces defined by the algebra of canonical transformations of CP_2.

The interpretation of these charges as contributions of light-like boundaries to configuration space Hamiltonians would be natural. The dynamics of the induced second quantized spinor fields relates to that of Kähler action by a super-symmetry, so that it should define super-symmetric counterparts of these knot invariants. The anti-commutators of these super charges would contribute to configuration space metric a part which would define a Kähler magnetic knot invariant. These Hamiltonians and their super-charge counterparts would be responsible for the topological sector of quantum TGD.

The color partial wave degeneracy of topological charges inspires the idea that also anyons could move in color partial waves identifiable in terms of "rigid body rotation" of the magnetic flux tube of anyon in CP_2 degrees of freedom. Their presence could explain non-Abelianity of Chern-Simons action and bring in new kind bits increasing the computational capacity of the topological quantum computer. The idea about the importance of macroscopic color is not new in TGD context. The fact that non-vanishing Kähler field is always accompanied by a classical color field (proportional to it) has motivated the proposal that colored excitations in macroscopic length scales are important in living matter and that colors as visual qualia correspond to increments of color quantum numbers in quantum phase transitions giving rise to visual sensations.

4.1.2 Knot theory, 3-manifold topology, and $D = 3$ solutions of field equations

Topological quantum field theory (TQFT) [30] demonstrates a deep connection between links and 3-topology, and one might hope that this connection could be re-interpreted in terms of imbeddings of 3-manifolds to $H = M^4_+ \times CP_2$ as surfaces having 3-dimensional CP_2 projection, call it X^3 in the sequel. $D = 3$ suggests itself because in this case Chern-Simons action density for the induced Kähler field is generically non-vanishing and defines an infinite number of classical charges identifiable as Kähler magnetic canonical covariants invariant under $Diff(M^4_+)$. The field topology of Kähler magnetic field should be in a key role in the understanding of these invariants.

1. Could 3-D CP_2 projection of 3-surface provide a representation of 3-topology?

Witten-Chern-Simons theory for a given 3-manifold defines invariants which characterize both the topology of 3-manifold and the link. Why this is the case can be understood from the construction of 3-manifolds by drilling a tubular neighborhood of a link in S^3 and by gluing the tori back to get a new 3-manifolds. The links with some moves defining link equivalences are known to be in one-one correspondence with closed 3-manifolds and the axiomatic formulation of TQFT [30] as a modular functor clarifies this correspondence. The question is whether the CP_2 projection of the 3-surface could under some assumptions be represented by a link so that one could understand the connection between the links and topology of 3-manifolds.

In order to get some idea about what might happen consider the CP_2 projection X^3 of 3-surface. Assume that X^3 is obtained from S^3 represented as a 3-surface in CP_2 by removing from S^3 a tubular link consisting of linked and knotted solid tori $D^2 \times S^1$. Since the 3-surface is closed, it must have folds at the boundaries being thus representable as a two-valued map $S^3 \rightarrow M^4_+$ near the folds. Assume that this is the case everywhere. The two halves of the 3-surface corresponding to the two branches of the map would be glued together along the boundary of the tubular link by identification maps which are in the general case characterized by the mapping class group of 2-torus. The gluing maps are defined inside the overlapping coordinate batches containing the boundary $S^1 \times S^1$ and are maps between the pairs (Ψ_i, Φ_i), $i = 1, 2$ of the angular coordinates parameterizing the tori.

Define longitude as a representative for the $a + nb$ of the homology group of the 2-torus. The integer n defines so called framing and means that the longitude twists n times around torus. As a matter fact, TQFT requires bi-framing: at the level of Chern-Simons perturbation theory bi-framing is necessary in order to define self linking numbers. Define meridian as the generator of the homology group of the complement of solid torus in S^3. It is enough to glue the carved torus back in such a manner that meridian is mapped to longitude and longitude to minus meridian. This map corresponds to the $SL(2, C)$ element

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} .$$

Also other identification maps defined by $SL(2, Z)$ matrices are possible but one can do using only this. Note that the two component $SL(2, Z)$ spinors defined as superpositions of the generators (a, b) of the homology group of torus are candidates for the topological correlates of spinors. In the gluing process the tori become knotted and linked when seen in the coordinates of the complement of the solid tori.

This construction would represent the link surgery of 3-manifolds in terms of CP_2 projections of 3-surfaces of H. Unfortunately this representation does not seem to be the only one. One can construct closed three-manifolds also by the so called Heegaard splitting. Remove from S^3 D_g, a solid sphere with g handles having boundary S_g, and glue the resulting surface with its oppositely oriented copy along boundaries. The gluing maps are classified by the mapping class group of S_g. Any closed orientable 3-manifold can be obtained by this kind of procedure for some value of g. Also this construction could be interpreted in terms of a fold at the boundary of the CP_2 projection for a 2-valued graph $S^3 \rightarrow M^4_+$. Whether link surgery representation and Heegaard splitting could be transformed to each other by say pinching D_g to separate tori is not clear to me.

When the graph $CP_2 \rightarrow M^4_+$ is at most 2-valued, the intricacies due to the imbedding of the 3-manifold are at minimum, and the link associated with the projection should give information about 3-topology and perhaps even characterize it. Also the classical topological charges associated with Kähler Chern-Simons action could give this kind of information.

2. Knotting and linking for 3-surfaces

The intricacies related to imbedding become important in small co-dimensions and it is of considerable interest to find what can happen in the case of 3-surfaces. For 1-dimensional links and knots the projection to a plane, the shadow of the knot, characterizes the link/knot and allows to deduce link and knot invariants purely combinatorially by gradually removing the intersection points and writing a contribution to the link invariant determined by the orientations of intersecting strands and by which of them is above the other. Thus also the generalization of knot and link diagrams is of interest.

Linking of m- and n-dimensional sub-manifolds of D-dimensional manifold H_D occurs when the condition $m+n=D-1$ holds true. The n-dimensional sub-manifold intersects $m+1$-dimensional surfaces having m-dimensional manifold as its boundary at discrete points, and it is usually not possible to remove these points by deforming the surfaces without intersections in some intermediate stage. The generalization of the link diagram results as a projection $D-1$-dimensional disk D^{D-1} of H_D.

3-surfaces link in dimension $D=7$ so that the linking of 3-surfaces occurs quite generally in time=constant section of the imbedding space. A link diagram would result as a projection to $E^2\times CP_2$, E^2 a 2-dimensional plane: putting CP_2 coordinates constant gives ordinary link diagram in E^2. For magnetic flux tubes the reduction to 2-dimensional linking by idealizing flux tubes with 1-dimensional strings makes sense.

Knotting occurs in codimension 2 that is for an n-manifold imbedded in $D=n+2$-dimensional manifold. Knotting can be understood as follows. Knotted surface spans locally $n+1$-dimensional 2-sided n+1-disk D^{n+1} (disk for ordinary knot). The portion of surface going through D^{n+1} can be idealized with a 1-dimensional thread going through it and by $n+2=D$ knotting is locally linking of this 1-dimensional thread with n-dimensional manifold. N-dimensional knots define n+1-dimensional knots by so called spinning. Take an n-knot with the topology of sphere S^n such that the knotted part is above $n+1$-plane of $n+2$-dimensional space R^{n+2} $(z\geq 0)$, cut off the part below plane $(z<0)$, introduce an additional dimension (t) and make a 2π rotation for the resulting knot in $z-t$ plane. The resulting manifold is a knotted S^{n+1}. The counterpart of the knot diagram would be a projection to $n+1$-dimensional sub-manifold, most naturally disk D^{n+1}, of the imbedding space.

3-surfaces could become knotted under some conditions. Vacuum extremals correspond to 4-surfaces $X^4\subset M^4_+\times Y^2$ whereas the four-surfaces $X^4\subset M^4_+\times S^2$, S^2 homologically non-trivial geodesic sphere, define their own "sub-theory". In both cases 3-surfaces in time=constant section of imbedding space can get knotted in the sense that un-knotting requires giving up the defining condition temporarily. The counterpart of the knot diagram is the projection to $E^2\times X^2$, $X^2=Y^2$ or S^2, where E^2 is plane of M^4_+. For constant values of CP_2 coordinates ordinary knot diagram would result. Reduction to ordinary knot diagrams would naturally occur for $D=2$ magnetic flux tubes. The knotting occurs also for 4-surfaces themselves in $M^4_+\times X^2$: knot diagram is now defined as projection to $E^3\times X^2$.

3. Could the magnetic field topology of 3-manifold be able to mimic other 3-topologies?

In $D=3$ case the topological charges associated with Kähler Chern-Simons term characterize the linking of the field lines of the Kähler gauge potential A. What $dA\wedge A\neq 0$ means that field lines are linked and it is not possible to define a coordinate varying along the field lines of A. This is impossible even locally since the $dA\wedge A\neq 0$ condition is equivalent with non existence of a scalar functions k and Φ such that $\nabla\Phi=kA$ guaranteing that Φ would be the sought for global coordinate.

One can idealize the situation a little bit and think of a field configuration for which magnetic flux is concentrated at one-dimensional closed lines. The vector potential would in this case be simply $A=\nabla(k\Psi+l\Phi)$, where Ψ is an angle coordinate around the singular line and Φ a coordinate along the singular circle. In this idealized situation the failure to have a global coordinate would be due to the singularities of otherwise global coordinates along one-dimensional linked and knotted circles. The reason is that the field lines of A and B rotate helically around the singular circle and the points (x,y,z) with constant values of x,y are on a helix which becomes singular at z-axis. Since the replacement of a field configuration with a non-singular field configuration but having same field line topology does not affect the global field line topology, one might hope of characterizing the field topology by its singularities along linked and knotted circles also in the general case.

Just similar linked and knotted circles are used to construct 3-manifolds in the link surgery which would suggest that the singularities of the field line topology of X^3 code the non-trivial 3-topology resulting when the singularities are removed by link surgery. Physically the longitude defining the framing $a + nb$ would correspond to the field line of A making an $n2\pi$ twist along the singular circle. Meridian would correspond to a circle in the plane of B. The bi-framing necessitated by TQFT would have a physical interpretation in terms of the helical field lines of A and B rotating around the singular circle. At the level of fields the gluing operation would mean a gauge transformation such that the meridians would become the field lines of the gauge transformed A and being non-helical could be continued to the the interior of the glued torus without singularities. Simple non-helical magnetic torus would be in question.

This means that the magnetic field patterns of a given 3-manifold could mimic the topologies of other 3-manifolds. The topological mimicry of this kind would be a very robust manner to represent information and might be directly relevant to TQC. For instance, the computation of topological invariants of 3-manifold Y^3 could be coded by the field pattern of X^3 representing the link surgery producing the 3-manifold from S^3, and the physical realization of TQC program could directly utilize the singularities of this field pattern. Topological magnetized flux tubes glued to the back-ground 3-surface along the singular field lines of A could provide the braiding.

This mimicry could also induce transitions to the new topology and relate directly to 3-manifold surgery performed by a physical system. This transition would quite concretely mean gluing of simple $D = 2$ magnetic flux tubes along their boundaries to the larger $D = 3$ space-time sheet from which similar flux tube has been cut away.

4. A connection with anyons?

There is also a possible connection with anyons. Anyons are thought to correspond to singularities of gauge fields resulting in a symmetry breaking of gauge group to a finite subgroup H and are associated with homotopically non-trivial loops of $C_n = ((R^2)^n - D)/S_n$ represented as elements of H. Could the singularities of gauge fields relate to the singularities of the link surgery so that the singularities would be more or less identifiable as anyons? Could N-branched anyons be identified in terms of framings $a + Nb$ associated with the gluing map? $D = 3$ solutions allow the so called contact structure [D1], which means a decomposition of the coordinates of CP_2 projection to a longitudinal coordinate s and a complex coordinate w. Could this decomposition generalize the notion of effective 2-dimensionality crucial for the notion of anyon?

5. What about Witten's quantal link invariants?

Witten's quantal link invariants define natural multiplicative factors of configuration space spinor fields identifiable as representations of two 2-dimensional topological evolution. In Witten's approach these invariants are defined as functional averages of non-integrable phase factors associated with a given link in a given 3-manifold. TGD does not allow any natural functional integral over gauge field configurations for a fixed 3-surface unless one is willing to introduce fictive non-Abelian gauge fields. Although this is not a problem as such, the representation of the invariants in terms of inherent properties of the 3-surface or corresponding 4-surfaces would be highly desirable.

Functional integral representation is not the only possibility. Quantum classical correspondence combined with topological field quantization implied by the absolute minimization of Kähler action generalizing Bohr rules to the field context gives hopes that the 3-surfaces themselves might be able to represent 3-manifold invariants classically. In $D = 3$ case the quantized exponents of Kähler-Chern-Simons action and $SU(2)_L$ Chern-Simons action could define 3-manifold invariants.

These invariants would satisfy the obvious multiplicativity conditions and could correspond to the phase factors due to the framing dependence of Witten's invariants identifying the loops of surgery link as Wilson loops. These phase factors are powers of $U = exp(i2\pi c/24)$, where c is the central charge of the Virasoro representation defined by Kac Moody representation. One has $c = k \times dim(g)/(k + c_g/2)$, which gives $U = exp(i2\pi k/8(k + 2))$ for $SU(2)$. The dependence on k differs from what one might naively expect. For this reason, and also because the classical Wilson loops do not depend explicitly on k, the value of k appearing in Chern-Simons action should be fixed by the internal consistency and be a constant of Nature according to TGD. The guess is that k possesses the minimal value $k = 3$ allowing a universal modular functor for $SU(2)$ with $q = exp(i2\pi/5)$.

The loops associated with the topological singularities of the Kähler gauge potential (typically the center lines of helical field configurations) would in turn define natural Wilson loops, and since the holonomies around these loops are also topologically quantized, they could define invariants of 3-manifolds obtained by performing surgery around these lines. The behavior of the induced gauge fields should be universal near the singularities in the sense that the holonomies associated with the CP_2 projections of the singularities to CP_2 would be universal. This expectation is encouraged by the notion of quantum criticality in general and in particular, by the interpretation of $D = 3$ phase as a critical system analogous to spin glass.

The exponent of Chern-Simons action can explain only the phase factors due to the framing, which are usually regarded as an unavoidable nuisance. This might be however all that is needed. For the manifolds of type $X^2 \times S^1$ all link invariants are either equal to unity or vanish. Surgery would allow to build 3-manifold invariants from those of $S^2 \times S^1$. For instance, surgery gives the invariant $Z(S^3)$ in terms of $Z(S^2 \times S^1, R_i)$ and mapping class group action coded into the linking of the field lines.

Holonomies can be also seen as multi-valued $SU(2)_L$ gauge transformations and can be mapped to a multi-valued transformations in the $SU(2)$ subgroup of $SU(3)$ acting on 3-surface as a geometric transformations and making it multi-branched. This makes sense if the holonomies define a finite group so that the gauge transformation is finitely many-valued. This description might apply to the 3-manifold resulting in a surgery defined by the Wilson loops identifiable as branched covering of the initial manifold.

The construction makes also sense for the holonomies defined by the classical $SU(3)$ gauge fields defined by the projections of the isometry currents. Furthermore, the fact that any CP_2 Hamiltonian defines a conserved topological charge in $D = 3$ phase should have a deep significance. At the level of the configuration space geometry the finite-dimensional group defining Kac Moody algebra is replaced with the group of canonical transformations of CP_2. Perhaps one could extend the notion of Wilson loop for the algebra of canonical transformations of CP_2 so that the representations R_i of the gauge group would be replaced by matrix representations of the canonical algebra. That the trace of the identity matrix is infinite in this case need not be a problem since one can simply redefine the trace to have value one.

4.1.3 Braids as topologically quantized magnetic fields

$D = 3$ space-time sheets would define complex braiding structures with flux tubes possessing infinite number of topological charges characterizing the linking of field lines. The world lines of the quantum computing dancers could thus correspond to the flux tubes that can get knotted, linked, and braided. This idea conforms with the earlier idea that the various knotted and linked structures formed by linear bio-molecules define some kind of computer programs.

1. Boundaries of magnetic flux tubes as light-like 3-surfaces

Field equations for Kähler action are satisfied identically at boundaries if the boundaries of magnetic flux tubes (and space-time sheet in general) are light-like in the induced metric. In M^4_+

metric the flux tubes could look static structures. Light-likeness allows an interpretation of the boundary state either as a 3-dimensional quantum state or as a time-evolution of a 2-dimensional quantum state. This conforms with the idea that quantum computation is cognitive, self reflective process so that quantum state is about something rather than something. There would be no need to force particles to flow through the braid structure to build up time-like braid whereas for time-like boundaries of magnetic flux tubes a time-like braid results only if the topologically charged particles flow through the flux tubes with the same average velocity so that the length along flux tubes is mapped to time.

Using the terminology of consciousness theory, one could say that during quantum dance the dancers are in trance being entangled to a single macro-temporally coherent state which represents single collective consciousness, and wake up to individual dancers when the dance ends. Quantum classical correspondence suggests that the generation of bound state entanglement between dancers requires tangled join along boundaries bonds connecting the space-time sheets of anyons (braid of flux tubes again!): dancers share mental images whereas direct contact between magnetic flux tubes defining the braid is not necessary. The bound state entanglement between sub-systems of unentangled systems is made possible by the many-sheeted space-time. This kind of entanglement could be interpreted as entanglement not visible in scales of larger flux tubes so that the notion is natural in the philosophy based on the idea of length scale resolution.

2. How braids are generated?

The encoding of the program to a braid could be a mechanical process: a bundle of magnetic flux tubes with one end fixed would be gradually weaved to a braid by stretching and performing the needed elementary twists. The time to perform the braiding mechanically requires classical computer program and the time needed to carry out the braiding depends polynomially on the number of strands.

The process could also occur by a quantum jump generating the braided flux tubes in single flash and perhaps even intentionally in living systems (flux tubes with negative topological charge could have negative energy so that it would require no energy to generate the structure from vacuum). The interaction with environment could be used to select the desired braids. Also ensembles of braids might be imagined. Living matter might have discovered this mechanism and used int intentionally.

3. Topological quantization, many-sheetedness, and localization

Localization of modular functors is one of the key problems in topological quantum computation (see the article of Freedman [38]. For anyonic computation this would mean in the ideal case a decomposition of the system into batches containing 4 anyons each so that these anyon groups interact only during swap operations.

The role of topological quantization would be to select of a portion of the magnetic field defining the braid as a macroscopic structure. Topological field quantization realizes elegantly the requirement that single particle time evolutions between swaps involve no interaction with other anyons.

Also many-sheetedness is important. The (AA) pair and two anyons would correspond braids inside braids and as it turns out this gives more flexibility in construction of quantum computation since the 1-gates associated with logical qubits of 4-batch can belong to different representation of braid group than that associated with braiding of the batches.

4.2 Quantum Hall effect and fractional charges in TGD

In fractional QH effect anyons possess fractional electromagnetic charges. Also fractional spin is possible. TGD explains fractional charges as being due to multi-branched character of space-time sheets. Also the Z_n-valued topological charge associated with anyons has natural explanation.

4.2.1 Basic TGD inspired ideas about quantum Hall effect

Quantum Hall effect is observed in low temperature systems when the intensity of a strong magnetic field perpendicular to the current carrying slab is varied adiabatically. Classically quantum Hall effect can be understood as a generation of a transversal electric field, which exactly cancels the magnetic Lorentz force. This gives $E = -j \times B/ne$. The resulting current can be also understood as due to a drift velocity proportional to $E \times B$ generated in electric and magnetic fields orthogonal to each other and allowing to cancel Lorentz force. This picture leads to the classical expression for transversal Hall conductivity as $\sigma_{xy} = ne/B$. σ_{xy} should vary continuously as a function of the magnetic field and 2-dimensional electron density n.

In quantum Hall effect σ_{xy} is piece-wise constant and quantized with relative precision of about 10^{-10}. The second remarkable feature is that the longitudinal conductivity σ_{xx} is very high at plateaus: variations by 13 orders of magnitude are observed. The system is also very sensitive to small perturbations.

Consider now what these qualitative observations might mean in TGD context.

1. Sensitivity to small perturbations means criticality. TGD Universe is quantum critical and quantum criticality reduces to the spin glass degeneracy due to the enormous vacuum degeneracy of the theory. The $D = 2$ and $D = 3$ non-vacuum phases predicted by the generalized Beltrami ansatz are this in-stability might play important role in the effect.

2. The magnetic fields are genuinely classical fields in TGD framework, and for $D = 2$ proportional to induced Kähler magnetic field. The canonical symmetries of CP_2 act like $U(1)$ gauge transformations on the induced gauge field but are not gauge symmetries since canonical transformations change the shape of 3-surface and affect both classical gravitational fields and electro-weak and color gauge fields. Hence different gauges for classical Kähler field represent magnetic fields for which topological field quanta can have widely differing and physically non-equivalent shapes. For instance, tube like quanta act effectively as insulators whereas magnetic walls parallel to the slab act as conducting wires.

   Wall like flux tubes parallel to the slab perhaps formed by a partial fusion of magnetic flux tubes along their boundaries would give rise to high longitudinal conductivity. For disjoint flux tubes the motion would be around the flux tubes and the electrons would get stuck inside these tubes. By quantum criticality and by $D < 4$ property the magnetic flux tube structures are unstable against perturbations, in particular the variation of the magnetic field strength itself. The transitions from a plateau to a new one would correspond to the decay of the magnetic walls back to disjoint flux tubes followed by a generation of walls again so that conductivity is very high outside transition regions. The variation of any parameter, such as temperature, is expected to be able to cause similar effects implying dramatic changes in Hall conductivity.

   The percolation model for the quantum Hall effect represents slab as a landscape with mountains and valleys and the varied external parameter, say B or free electron density, as the sea level. For the critical values of sea level narrow regions carrying so called edge states allow liquid to fill large regions appear and implies increase of conductivity. Obviously percolation model differs from the model based on criticality for which the landscape itself is highly fragile and a small perturbation can develop new valleys and mountains.

3. The effective 2-dimensionality implies that the solutions of Schrödinger equation of electron in external magnetic field are products of any analytic function with a Gaussian representing the ground state of a harmonic oscillator. Analyticity means that the kinetic energy is completely degenerate for these solutions. The Lauhglin ansatz for the state functions of electron in the external magnetic field is many-electron generalization of these solutions: the

wave functions consists of products of terms of form $(z_i - z_j)^m$, m odd integer from Fermi statistics.

The N-particle variant of Laughlin's ansatz allows to deduce that the system is incompressible. The key observation is that the probability density for the many-particle state has an interpretation as a Boltzmann factor for a fictive two-dimensional plasma in electric field created by constant charge density [42, 43]. The probability density is extremely sensitive to the changes of the positions of electrons giving rise to the constant electron density. The screening of charge in this fictive plasma implies the filling fraction $\nu = 1/m$, m odd integer and requires charge fractionization $e \rightarrow e/m$. The explanation of the filling fractions $\nu = N/m$ would require multi-valued wave functions $(z_i - z_j)^{N/m}$. In single-sheeted space-time this leads to problems. TGD suggests that these wave functions are single valued but defined on N-branched surface.

The degeneracy with respect to kinetic energy brings in mind the spin glass degeneracy induced by the vacuum degeneracy of the Kähler action. The Dirac equation for the induced spinors is not ordinary Dirac equation but super-symmetrically related to the field equations associated with Kähler action. Also it allows vacuum degeneracy. One cannot exclude the possibility that also this aspect is involved at deeper level.

4. The fractionization of charge in quantum Hall effect challenges the idea that charged particles of the incompressible liquid are electrons and this leads to the notion of anyon. Quantum-classical correspondence inspires the idea that although dissipation is absent, it has left its signature as a track associated with electron. This track is magnetic flux tube surrounding the classical orbit of electron and electron is confined inside it. This reduces the dissipative effects and explains the increase of conductivity. The rule that there is single electron state per magnetic flux quantum follows if Bohr quantization is applied to the radii of the orbits. The fractional charge of anyon would result from a contribution of classical Kähler charge of anyon flux tube to the charge of the anyon. This charge is topologized in $D = 3$ phase.

4.2.2 Anyons as multi-branched flux tubes representing charged particle plus its track

Electrons (in fact, any charged particles) moving inside magnetic flux tubes move along circular paths classically. The solutions of the field equations with vanishing Lorentz 4-force correspond to asymptotic patterns for which dissipation has already done its job and is absent. Dissipation has however definite effects on the final state of the system, and one can argue that the periodic motion of the charged particle has created what might called its "track". The track would be realized as a circular or helical flux tube rotating around field lines of the magnetic field. The corresponding cyclotron states would be localized inside tracks. Simplest tracks are circular ones and correspond to absence of motion in the direction of the magnetic field. Anyons could be identified as systems formed as particles plus the tracks containing them.

1. Many-branched tracks and approach to chaos

When the system approaches chaos one expects the the periodic circular tracks become non-periodic. One however expects that this process occurs in steps so that the tracks are periodic in the sense that they close after N 2π rotations with the value of N increasing gradually. The requirement that Kähler energy stays finite suggests also this. A basic example of this kind of track is obtained when the phase angles Ψ and Φ of complex CP_2 coordinates (ξ^1, ξ^2) have finitely multi-valued dependence on the coordinate ϕ of cylindrical coordinates: $(\Psi, \Phi) = (m_1/N, m_2/N)\phi)$. The space-sheet would be many-branched and it would take N turns of 2π to get back to the point were one started. The phase factors behave as a phase of a spinning particle having effective fractional spin $1/N$. I have proposed this kind of mechanism as an explanation of so called hydrino atoms claimed to have the spectrum of hydrogen atom but with energies scaled up by N^2 [56, G2]. The first guess that N corresponds to m in $\nu = 1/m$ is wrong. Rather, N corresponds to N in $\nu = N/m$ which means many-valued Laughlin wave functions in single branched space-time.

Similar argument applies also in CP_2 degrees of freedom. Only the N-multiples of 2π rotations by CP_2 isometries corresponding to color hyper charge and color iso-spin would affect trivially the point of multi-branched surface. Since the contribution of Kähler charge to electromagnetic charge corresponds also to anomalous hyper-charge of spinor field in question, an additional geometric contribution to the anomalous hypercharge would mean anomalous electromagnetic charge.

It must be emphasized the fractionization of the isometry charges is only effective and results from the interpretation of isometries as space-time transformations rather than transformation rotating entire space-time sheet in imbedding space. Also classical charges are effectively fractionized in the sense that single branch gives in a symmetric situation a fraction of $1/n$ of the entire charge. Later it will be found that also a genuine fractionization occurs and is due to the classical topologized Kähler charge of the anyon track.

2. Modelling anyons in terms of gauge group and isometry group

Anyons can be modelled in terms of the gauge symmetry breaking $SU(2)_L \rightarrow H$, where H is discrete sub-group. The breaking of gauge symmetry results by the action of multi-valued gauge transformation $g(x)$ such that different branches of the multi-valued map are related by the action of H.

1. The standard description of anyons is based on spontaneous symmetry breaking of a gauge symmetry G to a discrete sub-group H dynamically [49]. The gauge field has suffered multi-valued gauge transformation such that the elements of H permute the different branches of $g(x)$. The puncture is characterized by the element of the H associated with the loop surrounding puncture. In the idealized situation that gauge field vanishes, the parallel translation of a particle around puncture affects the particle state, itself a representation of G, by the element of the homotopy $\pi_1(G/H) = H$ identifiable as non-Abelian magnetic charge. Thus holonomy group corresponds to homotopy group of G/H which in turn equals to H. This in turn implies that the infinite-dimensional braid group whose elements define holonomies in turn is represented in H.

2. In TGD framework the multi-valuedness of $g(x)$ corresponds to a many-branched character of 4-surface. This in turn induces a branching of both magnetic flux tube and anyon tracks describable in terms of $H \subset SU(2)_L$ acting as an isotropy group for the boundaries of the magnetic flux tubes. H can correspond only to a non-Abelian subgroup $SU(2)_L$ of the electro-weak gauge group for the induced (classical) electro-weak gauge fields since the Chern-Simons action associated with the classical color gauge fields vanishes identically. The electro-weak holonomy group would reduce to a discrete group H around loops defined by anyonic flux tubes surrounding magnetic field lines inside the magnetic flux tubes containing anyons. The reduction to H need to occur only at the boundaries of the space-time sheet where conducting anyons would reside: boundaries indeed correspond to asymptotia in well-defined sense. Electro-weak symmetry group can be regarded as a sub-group of color group of isometries in a well-defined sense so that H can be regarded also as a subgroup of color group acting as isotropies of the multi-branched surface at least in the in regions where gauge field vanishes.

3. For branched surfaces the points obtained by moving around the puncture correspond in a good approximation to some elements of $h \in H$ leading to a new branch but the 2-surface as a whole however remains invariant. The braid group of the punctured 2-surface would be also now represented as transformations of H. The simplest situation is obtained when H

is a cyclic group Z_N of the $U(1)$ group of CP_2 geodesic in such a manner that 2π rotation around symmetry axis corresponds to the generating element $exp(i2\pi/N)$ of Z_N.

Dihedral group D_n having order $2n$ and acting as symmetries of n-polygon of the plane is especially interesting candidate for H. For $n = 2$ the group is Abelian group $Z_2 \times Z_2$ whereas for $n > 2$ D_n is a non-Abelian sub-group of the permutation group S_n. The cyclic group Z_4 crucial for TQC is a sub-group of D_4 acting as symmetries of square. D_4 has a 2-dimensional faithful representation. The numbers of elements for the conjugacy classes are 1,1,2,2,2. The sub-group commuting with a fixed element of a conjugacy class is D_4 for the 1-element conjugacy classes and cyclic group Z_4 for 2-element conjugacy classes. Hence 2-valued magnetic flux would be accompanied by Z_4 valued "electric charge" identifiable as a cyclic group permuting the branches.

3. Can one understand the increase in conductivity and filling fractions at plateaus?

Quantum Hall effect involves the increase of longitudinal conductivity by a factor of order 10^{13}[42]. The reduction of dissipation could be understood as being caused by the fact that anyonic electrons are closed inside the magnetic flux tubes representing their tracks so that their interactions with matter and thus also dissipation are reduced.

Laughlin's theory [43, 42] gives almost universal description of many aspects of quantum Hall effect and the question arises whether Laughlin's wave functions are defined on possibly multi-branched space-time sheet X^4 or at projection of X^4 to M^4_+. Since most theoreticians that I know still live in single sheeted space-time, one can start with the most conservative assumption that they are defined at the projection to M^4_+. The wave functions of one-electron state giving rise filling fraction $\nu = 1/m$ are constructed of $(z_i - z_j)^m$, where m is odd by Fermi statistics.

Also rational filling fractions of form $\nu = 1/m = N/n$ have been observed. These could relate to the presence of states whose projections to M^4 are multi-valued and which thus do not have any "classical" counterpart. For N-branched surface the single-valued wave functions $(\xi_i - \xi_j)^n$, n odd by Fermi statistics, correspond to apparently multi-valued wave functions $(z_i - z_j)^{n/N}$ at M^4 projection with fractional relative angular momenta $m = n/N$. The filling fraction would be $\nu = N/n$, n odd. All filling fractions reported in [42] have n odd with n varying in the range $1-7$. N has the values $1, 2, 3, 4, 5, 7, 9$. Also values $N = 12, 13$ for which $n = 5$ are reported [23].

The filling fractions $\nu = N/n = 5/2, 3/8, 3/10$ reported in [48] would require even values of n conflicting with Fermi statistics. Obviously Lauhglin's model fails in this case and the question is whether one these fractions could correspond to bosonic anyons, perhaps Cooper pairs of electrons inside track flux tubes. The Z_N valued charge associated with N-branched surfaces indeed allows the maximum $2N$ electrons per anyon. Bosonic anyons are indeed the building block of the TQC model of [23]. The anyon Cooper pairs could be this kind of states and their BE condensation would make possible genuine super-conductivity rather than only exceptionally high value of conductivity.

One can imagine also more complex multi-electron wave functions than those of Laughlin. The so called conformal blocks representing correlation functions of conformal quantum field theories are natural candidates for the wave functions [50] and they appear naturally as state functions of in topological quantum field theories. For instance, wave functions which are products of factors $(z^k - z^l)^2$ with the Pfaffian $Pf(A_{kl})$ of the matrix $A_{kl} = 1/(z_k - z_l)$ guaranteing anti-symmetrization have been used to explain even values of m [50].

4. N-branched space-time surfaces make possible Z_N valued topological charge

According to [50] that $2n$ non-Abelian anyon pairs with charge $1/4$ created from vacuum gives rise to a 2^{n-1}-fold degenerate ground state. It is also argued that filling fraction $5/2$ could correspond to this charge [23]. TGD suggests somewhat different interpretation. 4-fold branching implies automatically the Z_4-valued topological charge crucial for anyonic quantum computation. For 4-branched space-time surface the contribution of a single branch to electron's charge is indeed

1/4 units but this has nothing to do with the actual charge fractionization. The value of ν is of form $\nu = /m$ and electromagnetic charge equals to $\nu = 4e/m$ in this kind of situation.

If anyons (electron plus flux tube representing its track) have Z_4 charges 1 and 3, their Cooper pairs have charges 0 and 2. The double-fold degeneracy for anyon's topological charge means that it possesses topological spin conserved modulo 4. In presence of $2n$ anyon pairs one would expect 2^n-fold degeneracy. The requirement that the net topological charge vanishes modulo 4 however fixes the topological charge of n:th pair so that 2^{n-1} fold degeneracy results.

A possible interpretation for Z_N-valued topological charge is as fractional angular momenta k/N associated with the phases $exp(ik2\pi/N)$, $k = 0, 1, ..., n-1$ of particles in multi-branched surfaces. The projections of these wave functions to single-branched space-time would be many-valued. If electro-weak gauge group breaks down to a discrete subgroup H for magnetic flux tubes carrying anyonic "tracks", this symmetry breakdown could induce their multi-branched property in the sense rotation by 2π would correspond to H isometry leading to a different branch.

4.2.3 Topologization of Kähler charge as an explanation for charge fractionization

The argument based on what happens when one adds one anyon to the anyon system by utilizing Faraday's law [42] leads to the conclusion that anyon charge is fractional and given by νe. The anyonic flux tube along boundary of the flux tube corresponds to the left hand side in the Faraday's equation

$$\oint E \cdot dl = -\frac{d\Phi}{dt} \ .$$

By expressing E in turns of current using transversal conductivity and integrating with respect to time, one obtains

$$Q = \nu e$$

for the charge associated with a single anyon. Hence the addition of the anyon means an addition of a fractional charge νe to the system. This argument should survive as such the 1-branched situation so that at least in this case the fractional charges should be real.

In N-branched case the closed loop $\oint E{\cdot}dl$ around magnetic flux tube corresponds to N-branched anyon and surrounds the magnetic flux tube N times. This would suggest so that net magnetic flux should be N times the one associated with single but unclosed 2π rotation. Hence the formula would seem to hold true as such also now for the total charge of the anyon and the conclusion is that charge fractionization is real and cannot be an effective effect due to fractionization of charge at single branch of anyon flux tube.

One of the basic differences between TGD and Maxwell's theory is the possibility of vacuum charges and this provides an explanation of the effect is in terms of vacuum Kähler charge. Kähler charge contributes $e/2$ to the charge of electron. Anyon flux tube can generate vacuum Kähler charge changing the net charge of the anyon. If the anyon charge equals to νe the conclusions are following.

1. The vacuum Kähler charge of the anyon track is $q = (\nu - 1)e$.

2. The dimension of the CP_2 projection of the anyon flux tube must be $D = 3$ since only in this case the topologization of anyon charge becomes possible so that the charge density is proportional to the Chern-Simons term $A \wedge dA/4\pi$. Anyon flux tubes cannot be super-conducting in the sense that non-integrable phase factor $exp(\int A{\cdot}dl)$ would define global order parameter. The boundaries of anyonic flux tubes could however remain potentially super-conducting and anyon Cooper pairs would be expelled there by Meissner effect. This gives super-conductivity in length scale of single flux tube. Conductivity and super-conductivity in

long length scales requires that magnetic flux tubes are glued together along their boundaries partially.

3. By Bohr quantization anyon tracks can have $r_n = \sqrt{n} \times r_B$, $n \leq m$, where r_m corresponds to the radius of the magnetic flux tube carrying m flux quanta. Only the tracks with radius r_m contribute to boundary conductivity and super-conductivity giving $\nu = 1/m$ for singly branched surfaces.

   The states with $\nu = N/m$ cannot correspond to non-super-conducting anyonic tracks with radii r_n, $n < m$, n odd, since these cannot contribute to boundary conductivity. The many-branched character however allows an N-fold degeneracy corresponding to the fractional angular momentum states $exp(ik\phi/N)$, $k = 0, ..., N-1$ of electron inside anyon flux tubes of radius r_m. k is obviously a an excellent candidate for the Z_N-valued topological charge crucial for anyonic quantum computation. Z_4 is uniquely selected by the braid matrix R.

   Only part of the anyonic Fermi sea need to be filled so that filling fractions $\nu = k/m$, $k = 1, ..., N$ are possible. Charges νe are possible if each electron inside anyon track contributes $1/m$ units to the fractional vacuum Kähler charge. This is achieved if the radius of the anyonic flux tube grows as $\sqrt{k/m}$ when electrons are added. The anyon tracks containing several electrons give rise to composite fermions with fermion number up to $2N$ if both directions of electron spin are allowed.

4. Charge fractionization requires vacuum Kähler charge has rational values $Q_K = (\nu - 1)e$. The quantization indeed occurs for the helicity defined by Chern-Simons term $A \wedge dA/4\pi$. For compact 3-spaces without boundary the helicity can be interpreted as an integer valued invariant characterizing the linking of two disjoint closed curves defined by the magnetic field lines. This topological charge can be also related to the asymptotic Hopf invariant proposed by Arnold [18], which in non-compact case has a continuum of values. Vacuum Kähler current is obtained from the topological current $A \wedge dA/4\pi$ by multiplying it with a function of CP_2 coordinates completely fixed by the field equations. There are thus reasons to expect that vacuum Kähler charge and also the topological charges obtained by multiplying Chern Simons current by $SU(3)$ Hamiltonians are quantized for compact 3-surfaces but that the presence of boundaries replaces integers by rationals.

4.2.4 What happens in quantum Hall system when the strength of the external magnetic field is increased?

The proposed mechanism of anyonic conducicitivity allows to understand what occurs in quantum Hall system when the intensity of the magnetic field is gradually increased.

1. Percolation picture encourages to think that magnetic flux tubes fuse partially along their boundaries in a transition to anyon conductivity so that the anyonic states localized at the boundaries of flux tubes become delocalized much like electrons in metals. Laughlin's states provide an idealized description for these states. Also anyons, whose tracks have Bohr radii r_m smaller than the radius r_B of the magnetic flux tube could be present but they would not participate in this localization. Clearly, the anyons at the boundaries of magnetic flux tubes are highly analogous to valence electrons in atomic physics.

2. As the intensity of the magnetic field B increases, the areas a of the flux tubes decreases as $a \propto 1/B$: this means that the existing contacts between neighboring flux tubes tend to be destroyed so that anyon conductivity is reduced. On the other hand, new magnetic flux tubes must emerge by the constancy of the average magnetic flux implying $dn/da \propto B$ for the average density of flux tubes. This increases the probability that the newly generated flux tubes can partially fuse with the existing flux tubes.

3. If the flux tubes are not completely free to move and change their shape by area preserving transformations, one can imagine that for certain value ranges of B the generation of new magnetic flux tubes is not favored since there is simply no room for the newcomers. The Fermi statistics of the anyonic electrons at the boundaries of flux tubes might relate to this non-hospitable behavior. At certain critical values of the magnetic field the sizes of flux tubes become however so small that the situation changes and the new flux tubes penetrate the system and via the partial fusion with the existing flux tubes increase dramatically the conductivity.

4.2.5 Also protonic anyons are possible

According to the TGD based model, any charged particle can form anyons and the strength of the magnetic field does not seem to be crucial for the occurrence of the effect and it could occur even in the Earth's magnetic field. The change of the cyclotron and Larmor frequencies of the charged particle in an external magnetic field to a value corresponding to the fractional charge provides a clear experimental signature for both the presence of anyons and for their the fractional charge.

Interestingly, water displays a strange scaling of proton's cyclotron frequency in an external magnetic field [60, 55]. In an alternating magnetic field of .1551 Gauss (Eearth's field has a nominal value of .58 Gauss) a strong absorption at frequency $f = 156$ Hz was observed. The frequency was halved when D_2O was used and varied linearly with the field strength. The resonance frequency however deviated from proton's Larmor frequency, which suggests that a protonic anyon is in question. The Larmor frequency would be in this case $f_L = r \times \nu eB/2m_p$, where $r = \mu_p/\mu_B = 2.2792743$ is the ratio of proton's actual magnetic moment to its value for a point like proton. The experimental data gives $\nu = .6003 = 3/5$ with the accuracy of 5×10^{-4} so that 3-branched protonic anyons with $m = 5$ would be responsible for the effect.

If this interpretation is correct, entire p-adic hierarchy of anyonic NMR spectroscopies associated with various atomic nuclei would become possible. Bosonic anyon atoms and Cooper pairs of fermionic anyon atom could also form macroscopic quantum phases making possible superconductivity very sensitive to the value of the average magnetic field and bio-systems and brain could utilize this feature.

4.3 Does the quantization of Planck constant transform integer quantum Hall effect to fractional quantum Hall effect?

The model for topological quantum computation inspired the idea that Planck constant might be dynamical and quantized. The work of Nottale [53] gave a strong boost to concrete development of the idea and it took year and half to end up with a proposal about how basic quantum TGD could allow quantization Planck constant associated with M^4 and CP_2 degrees of freedom such that the scaling factor of the metric in M^4 degrees of freedom corresponds to the scaling of $\hbar$ in CP_2 degrees of freedom and vice versa [A9]. The dynamical character of the scaling factors of M^4 and CP_2 metrics makes sense if space-time and imbedding space, and in fact the entire quantum TGD, emerge from a local version of an infinite-dimensional Clifford algebra existing only in dimension $D = 8$ [A8].

The predicted scaling factors of Planck constant correspond to the integers n defining the quantum phases $q = exp(i\pi/n)$ characterizing Jones inclusions. A more precise characterization of Jones inclusion is in terms of group $G_b \subset SU(2) \subset SU(3)$ in CP_2 degrees of freedom and $G_a \subset SL(2, C)$ in M^4 degrees of freedom. In quantum group phase space-time surfaces have exact symmetry such that to a given point of M^4 corresponds an entire G_b orbit of CP_2 points and vice versa. Thus space-time sheet becomes $N(G_a)$ fold covering of CP_2 and $N(G_b)$-fold covering of M^4. This allows an elegant topological interpretation for the fractionization of quantum numbers. The integer n corresponds to the order of maximal cyclic subgroup of G.

In the scaling $\hbar_0 \rightarrow n\hbar_0$ of M^4 Planck constant fine structure constant would scale as

$$\alpha = \frac{e^2}{4\pi\hbar c} \rightarrow \frac{\alpha}{n} \ ,$$

and the formula for Hall conductance would transform to

$$\sigma_H \rightarrow \frac{\nu}{n}\alpha \ .$$

Fractional quantum Hall effect would be integer quantum Hall effect but with scaled down α. The apparent fractional filling fraction $\nu = m/n$ would directly code the quantum phase $q = exp(i\pi/n)$ in the case that m obtains all possible values. A complete classification for possible phase transitions yielding fractional quantum Hall effect in terms of finite subgroups $G \subset SU(2) \subset SU(3)$ given by ADE diagrams would emerge (A_n, D_{2n}, E_6 and E_8 are possible). What would be also nice that CP_2 would make itself directly manifest at the level of condensed matter physics.

4.4 Why 2+1-dimensional conformally invariant Witten-Chern-Simons theory should work for anyons?

Wess-Zumino-Witten theories are 2-dimensional conformally invariant quantum field theories with dynamical variables in some group G. The action contains the usual 2-dimensional kinetic term for group variables allowing conformal group action as a dynamical symmetry plus winding number defined associated with the mapping of 3-surface to G which is $Diff^4$ invariant. The coefficient of this term is quantized to integer.

If one couples this theory to a gauge potential, the original chiral field can be transformed away and only a Chern-Simons term defined for the 3-manifold having the 2-dimensional space as boundary remains. Also the coefficient k of Chern-Simons term is quantized to integer. Chern-Simons-Witten action has close connection with Wess-Zumino-Witten theory. In particular, the states of the topological quantum field theory are in one-one correspondence with highest weights of the WZW action.

The appearance of 2+1-dimensional $Diff^3$ invariant action can be understood from the fundamentals of TGD.

1. Light-like 3-surfaces of both future light-cone M^4_+ and of space-time surface X^4 itself are in a key role in the construction of quantum TGD since they define causal determinants for Kähler action.

2. At the space-time level both the boundaries of X^4 and elementary particle horizons surrounding the orbits of wormhole contacts define light-like 3-surfaces. The field equations are satisfied identically at light-like boundaries. Of course, the projections of the the light-like surfaces of X^4 to Minkowski space need not look light-like at all, and even boundaries of magnetic flux tubes could be light-like.

   Light-like 3-surfaces are metrically 2-dimensional and allow a generalized conformal invariance crucial for the construction of quantum TGD. At the level of imbedding space conformal super-canonical invariance results. At the space-time level the outcome is conformal invariance highly analogous to the Kac Moody symmetry of super string models [B2, B3, E2]. In fact, there are good reasons to believe that the three-dimensional Chern-Simons action appears even in the construction of configuration space metric and give an additional contribution to the configuration space metric when the light-like boundaries of 3-surface have 3-dimensional CP_2 projection.

3. By the effective two-dimensionality the Wess-Zumino-Witten action containing Chern-Simons term is an excellent candidate for the quantum description of S-matrix associated with the light-like 3-surfaces since by the vanishing of the metric determinant one cannot define any general coordinate invariant 3-dimensional action other than Chern-Simons action. The boundaries of the braid formed by the magnetic flux tubes having light-like boundaries, perhaps having join along boundaries bonds between swapped flux tubes would define the 2+1-dimensional space-time associated with a braid, would define the arena of Witten-Chern-Simons theory describing anyons. This S-matrix can be interpreted also as characterizing either a 3-dimensional quantum state since light-like boundaries are limiting cases of space-like 3-surfaces.

4. Kähler action defines an Abelian Chern-Simons term and the induced electroweak gauge fields define a non-Abelian variant of this term. The Chern-Simons action associated with the classical color degrees of freedom vanishes as is easy to find. The classical color fields are identified as projections of Killing vector fields of color group: $A^c_\alpha = j^A_k \partial_\alpha s^k \tau_A = J_k{}^r \partial_r H^A \partial_\alpha s^k$. The classical color gauge field is proportional to the induced Kähler form: $F^c_{\alpha\beta} = H^A J_{\alpha\beta}\tau_A$. A little calculation shows that the instanton density vanishes by the identity $H_A H^A = 1$ (this identity is forced by the necessary color-singletness of the YM action density and is easy to check in the simpler case of S^2.

5. Since qubit realizes the fundamental representation of the quantum group $SU(2)_q$, $SU(2)$ is in a unique role concerning the construction of modular functors and quantum computation using Chern-Simons action. The quantum group corresponding to $q = exp(i2\pi/r)$, $r = 5$ is realized for the level $k = 3$ Chern-Simons action and satisfies the constraint $r = k + c_g$, where $c_g = 2$ is the so called dual Coxeter number of $SU(2)$ [37, 39, 23].

   The exponent non-Abelian $SU(2)_L \times U(1)$ Chern-Simons action combined with the corresponding action for Kähler form so that effective reduction to $SU(2)_L$ occurs, could appear as a multiplicative factor of the configuration space spinor fields defined in the configuration space of 3-surfaces. Since 3-dimensional quantum state would represent a 2-dimensional time evolution the role of these phase factor would be very analogous to the role of ordinary Chern-Simons action.

5 Topological quantum computation in TGD Universe

The general philosophy behind TQC inspires the dream that the existence of basic gates, in particular the maximally entangling 2-gate R, is guaranteed by the laws of Nature so that no fine tuning would be needed to build the gates. Negentropy Maximization Principle, originally developed in context of TGD inspired theory of consciousness, is a natural candidate for this kind of Law of Nature.

5.1 Concrete realization of quantum gates

The bold dream is that besides 2-gates also 1-gates are realized by the basic laws of Nature. The topological realization of the 3-braid representation in terms of Temperley-Lie algebra allows the reduction of 1-gates to 2-gates.

5.1.1 NMP and TQC

Quantum jump involves a cascade of self measurements in which the system under consideration can be though of as decomposing to two parts which are either un-entangled or possess rational or extended rational entanglement in the final state. The sub-system is selected by the requirement that entanglement negentropy gain is maximal in the measurement of the density matrix characterizing the entanglement of the sub-system with its complement.

In the case case that the density matrix before the self measurement decomposes into a direct sum of matrices of dimensions N_i , such that $N_i > 1$ holds true for some values of i, say i_0, the final state is a rationally entangled and thus a bound state. i_0 is fixed by the requirement that the number theoretic entropy for the final state maximally negative and equals to $klog(p)$, where p^k is the largest power of prime dividing N_{i_0}. This means that maximally entangled state results and the density matrix is proportional to a unit matrix as it is also for the entanglement produced by R. In case of R the density matrix is 1/2 times 2-dimensional unit matrix so that bound state entanglement negentropy is 1 bit.

The question is what occurs if the density matrix contains a part for which entanglement probabilities are extended rational but not identical. In this case the entanglement negentropy is positive and one could argue that no self-measurement occurs for this state and it remains entangled. If so then the measurement of the density matrix would occur only when it increases entanglement negentropy. This looks the only sensible option since otherwise only bound state entanglement with identical entanglement probabilities would be possible. This question is relevant also because Temperley-Lieb representation using $(AA)-A-A$ system involves entanglement with entanglement probabilities which are not identical.

In the case that the 2-gate itself is not directly entangling as in case of R' and R'', NMP should select just the quantum history, that single particle gates at it guarantee maximum entanglement negentropy. Thus NMP would come in rescue and give hopes that various gates are realized by Nature.

Non-Abelian anyon systems are modelled in terms of punctures of plane and Chern-Simons action for the incompressible vector potential of hydrodynamical flow. It is interesting to find how these ideas relate to the TGD description.

5.1.2 Non-Abelian anyons reside at boundaries of magnetic flux tubes in TGD

In [23] anyons are modelled in terms of punctures of plane defined by the slab carrying Hall current. In TGD the punctures correspond naturally to magnetic flux tubes defining the braid. It is now however obvious under what conditions the braid containing the TGD counterpart of (AA)-A-A system can be described as a punctured disk if the flux tubes describing the tracks of valence anyons are very near to the boundaries of the magnetic flux tubes. Rather, the punctured disk is replaced with the closed boundary of the magnetic flux tube or of the structure formed by the partial fusion of several magnetic flux tubes. This microscopic description and is consistent with Laughlin's model only if it is understood as a long length scale description.

Non-Abelian charges require singularities and punctures but a two-surface which is boundary does not allow punctures. The punctures assigned with an anyon pair would become narrow wormhole threads traversing through the interior of the magnetic flux tube and connecting the punctures like wormholes connect two points of an apple. It is also possible that the threads connect the surfaces of two nearby magnetic flux tubes. The wormhole like character conforms with the fact that non-Abelian anyons appear always in pairs.

The case in which which the ends of the wormhole thread belong to different neighboring magnetic flux tubes, call them T_1 and T_2, is especially interesting as far as the model for TQC is considered. The state of $(AA) - A - A$ system before (after) the 3-braid operation would be identifiable as anyons near the surface of T_1 (T_2). If only sufficiently local operations are allowed, the braid group would be same as for anyons inside disk. This means consistency with the anyon model of [23] for TQC requiring that the dimension for the space of ground states is 2^{n-1} in a system consisting of n anyon pairs.

The possibility of negative energies allows inspires the idea that the anyons at T_2 have negative energies so that the anyon system would have a vanishing net energy. This would conform with the idea that the scattering from initial to final state is equivalent with the creation of zero energy state for which initial (final) state particles have positive (negative) energies, and with the fact that the boundaries of magnetic flux tubes are light-like systems for which 3-D quantum state is representation for a 2-D time evolution.

Since the correlation between anyons at the ends of the wormhole thread is purely topological, the most plausible option is that they behave as free anyons dynamically. Assuming 4-branched anyon surfaces, the charges of anyons would be of form $Q = \nu_A e$, $\nu_A = 4/m$, m odd.

Consider now the representation of 3-braid group. That the mapping class group for the 3-braid system should have a 2-dimensional representation is obvious from the fact that the group has same generators as the mapping class group for torus which is represented by as $SL(2, Z)$ matrices acting on the homology of torus having two generators a, b corresponding to the two non-contractible circles around torus. 3-braid group would be necessarily represented in Temperley-Lieb representation.

The character of the anyon bound state is important for braid representations.

1. If anyons form loosely bound states (AA), the electrons are at different tracks and the charge is additive in the process so that one has $Q_{AA} = 2Q_A = 8/m$, m odd, which is at odds with statistics. It might be that the naive rule of assigning fractional charge to the state does not hold true for loosely bound bosonic anyons. In this case $(AA) - A$ system with charge states $((1, -1), 1)$ and $((1, 1), -1)$ would be enough for realizing 1-gates in TQC. The braid operation s_2 of Temperley-Lieb representation represented $(A_1A_2) - A_3 \rightarrow (A_1A_3) - A_2$ would correspond to an exchange of the dance partner by a temporary decay of (A_1A_2) followed by a recombination to a quantum superposition of (A_1A_2) and (A_1A_3) and could be regarded as an ordinary braid operation rather than monodromy. The relative phase 1-gate would correspond to s_1 represented as braid operation for A_1 and A_2 inside (A_1A_2).

2. If anyons form tightly bound states (AA) in the sense that single anyonic flux tube carries two electrons, charge need not be additive so that bound states could have charges $Q = 4/2m_1$ so that the vacuum Kähler charge $Q_K = 4(1/m_1 - 2/m)$ would be created in the process. This would stabilize (AA) state and would mean that the braid operation $(A_1A_2) - A_3 \rightarrow (A_1A_3) - A_2$ cannot occur via a temporary decay to free anyons and it might be necessary to replace 3-braid group by a partially colored 3-braid group for $(AA) - A - A$ system which is sub-group of 3-braid group and has generators s_1^2 (two swaps for $(AA) - A$) and s_2 (swap for $A - A$) instead of s_1 and s_2. Also in this case a microscopic mechanism changing the value of (AA) Z^4 charge is needed and the situation might reduce to the case a) after all.

   The Temperley Lieb representation for this group is obtained by simply taking square of the generator inducing entanglement (s_2 rather than s_1 in the notation used!). The topological charge assignments for $(AA) - A - A$ system are $((1, -1), 1, -1)$ and $((1, 1), -1, -1)$. s_1^2 would correspond to the group element generating $(AA) - A$ entanglement and s_2 acting on $A - A$ pair would correspond to phase generating group element.

5.1.3 Braid representations and 4-branched anyon surfaces

Some comments about braid representations in relation to Z_N- valued topological charges are in order.

1. Yang-Baxter braid representation using the maximally entangling braid matrix R is especially attractive option. For anyonic computation with Z_4-valued topological charge R is the unique 2-gate conserving the net topological charge (note that the mixing of the $|1, 1\rangle$ and $|-1, -1\rangle$

is allowed). On the other, R allows only the conservation of Z_4 value topological charge. This suggests that the the entanglement between logical qubits represented by $(AA)-A-A$ batches is is generated by R. The physical implication is that only $\nu=4/n$ 4-branched anyons could be used for TQC.

2. In TGD framework the entangling braid representation inside batches responsible for 1-gates need not be the same since batches correspond to magnetic flux tubes. In standard physics context it would be harder to defend this kind of assumption. As will be found 3-braid Temperley-Lieb representation is very natural for 1-gates. The implication is that the n-braid system with braids represented as 4-batches would have 2^n-dimensional space of logical qubits in fact identical with the space of realizable qubits.

3. Also n-braid Temperley-Lieb representations are possible and the explicit expressions of the braiding matrices for 6-braid case suggest that Z_4 topological charge is conserved also now [37]. In this case the dimension of the space of logical qubits is for highly favored value of quantum group parameter $q=exp(i\pi/5)$ given by the Fibonacci number $F(n)$ for n-braid case and behaves as Φ^{4n} asymptotically so that this option would be more effective. From $\Phi^4=1+3\Phi\simeq 8.03$ one can say that single 4-batch carries 3 bits of information instead of one. This is as it must be since topological charge is not conserved inside batches separately for this option.

4. $(AA)-A$ representation based on Z_4-valued topological charge is unique in that the space of logical qubits would be the space of topologically realizable qubits. Quantum superposition of logical qubits could could be represented $(AA)-A$ entangled state of form $a|2,-1\rangle+b|0,1\rangle$ generated by braid action. Relative phase could be generated by braid operation acting on the entangled state of anyons of (AA) Cooper pair. Since the superposition of logical cubits corresponds to an entangled state $a|2,-1\rangle+b|0,1\rangle$ for which coefficients are extended rational numbers, the number theoretic realization of the bound state property could pose severe conditions on possible relative phases.

5.2 Temperley-Lieb representations

The articles of Kaufmann [35] and Freedman [37, 38] provide enjoyable introduction to braid groups and to Tempeley-Lie representations. In the sequel Temperley-Lieb representations are discussed from TGD view point.

5.2.1 Temperley-Lieb representation for 3-braid group

In [35] it is explained how the so called Temperley-Lie algebra defined by 2×2-matrices I, U_1, U_2 satisfying the relations $U_1^2=dU_1$, $U_2^2=dU_2$, $U_1U_2U_1=U_2$, $U_2U_1U_2=U_1$ allows a unitary representation of Artin's braid group by unitary 2×2 matrices. The explicit representations of the matrices U_1 and U_2 (note that U_i/d acts as a projector) given by

$$\begin{aligned} U_1 &= \begin{pmatrix} d & 0 \\ 0 & 0 \end{pmatrix} , \\ U_2 &= \begin{pmatrix} \frac{1}{d} & \sqrt{1-\frac{1}{d^2}} \\ \sqrt{1-\frac{1}{d^2}} & d-\frac{1}{d} \end{pmatrix} . \end{aligned} \tag{5}$$

Note that the eigenvalues of U_i are d and 0. The representation of the elements s_1 and s_2 of the 3-braid group is given by

$$\begin{aligned}
\Phi(s_1) &= AI + A^{-1}U_1 = \begin{pmatrix} -U^{-3} & 0 \\ 0 & U \end{pmatrix} \ , \\
\Phi(s_2) &= AI + A^{-1}U_2 = \begin{pmatrix} -\frac{U^3}{d} & \frac{U^{-1}}{\sqrt{1-(1/d)^2}} \\ \frac{U^{-1}}{\sqrt{1-(1/d)^2}} & \frac{U^{-5}}{d} \end{pmatrix} \ , \\
U &= exp(i\phi) \ . \qquad (6)
\end{aligned}$$

Here the condition $d = -A^2 - A^{-2}$ is satisfied. For $A = exp(i\phi)$, with $|\phi| \leq \pi/6$ or $|\pi - \phi| \leq \pi/6$, the representation is unitary. The constraint comes from the requirement $d > 1$. From the basic representation it follows that the eigenvalues of $\Phi(s_i)$ are $-exp(-3i\phi)$ and $exp(i\phi)$.

Tihs 3-braid representation is a special case of a more general Temperley-Lieb-Jones representation discussed in [37] using notations $A = \sqrt{-1}exp(-i2\pi/4r)$, $s = A^2$, and $q = A^4$. In this case all eigen-values of all representation matrices are -1 and $q = exp(-i2\pi/r)$. This representation results by multiplying Temperley-Lieb representation above with an over-all phase factor $exp(4i\phi)$ and by the replacement $A = exp(i\phi) \rightarrow \sqrt{-1}A$.

5.2.2 Constraints on the parameters of Temperley-Lieb representation

The basic mathematical requirement is that besides entangling 2-gate there is minimum set of 1-gates generating infinite sub-group of $U(2)$. Further conditions come from the requirement that a braid representation is in question. In the proposal of [23, 37] the 1-gates are realized using Temperley-Lieb 3-braid representation. It is found that there are strong constraints to the representation and that relative phase gate generating the phase $exp(i\phi) = exp(i2\pi/5)$ is the simplest solution to the constraints.

The motivation comes from the findings made already by Witten in his pioneering work related to the topological quantum field theories and one can find a good representation about what is involve din [30].

Topological quantum field theories can produce unitary modular functors when the $A = q^{1/4} = exp(i\phi)$ characterizing the quantum group multiplication is a root of unity so that the quantum enveloping algebra $U(Sl(2))_q$ defined as the quantum version of the enveloping algebra $U(Sl(2))$ is not homomorphic with $U(Sl(2))$ and theory does not trivialize. Besides this, q must satisfy some consistency conditions. First of all, $A^{4n} = 1$ must be satisfied for some value of n so that A is either a primitive l:th, $2l$:th of unity for l odd, or $4l$:th primitive root of unity.

This condition relates directly to the fact that the quantum integers $[n]_q = (A^{2n} - A^{-2n})/(A^2 - A^{-2})$ vanish for $n \geq l$ so that the representations for a highest weight n larger than l are not irreducible. This implies that the theory simplifies dramatically since these representations can be truncated away but can cause also additional difficulties in the definition of link invariants. Indeed, as Witten found in his original construction, the topological field theories are unitary for $U(Sl(2))_q$ only for $A = exp(ik\pi/2l)$, k not dividing $2l$, and $A = exp(i\pi/l)$, l odd (no multiples are allowed) [30]. $n = 2l = 10$, which is the physically favored choice, corresponds to the relative phase $4\phi = 2\pi/5$.

5.2.3 Golden Mean and quantum computation

Temperley-Lieb representation based on $q = exp(i2\pi/5)$ is highly preferred physically.

1. One might hope that the Yang-Baxter representation based on maximally entangling braid matrix R might work. $R^8 = 1$ constraint is however not consistent with Temperley-Lieb representations. The reason is that $\Phi^8(s_1) = 1$ gives $\phi = \pi/4 > \pi/6$ so that unitarity constraint is not satisfied. $\phi = exp(i2\pi/16)$ corresponding $r = 4$ and to the matrix

$\Phi(s_2) = \hat{R} = exp(i2\pi/16) \times R$ allows to satisfy the unitarity constraint. This would look like a very natural looking selection since $\Phi(s_2)$ would act as a Hadamard gate and NMP would imply identical entanglement probabilities if a bound state results in a quantum jump. Unfortunately, s_1 and s_2 do not generate a dense subgroup of $U(2)$ in this case as shown in [37].

2. $\phi = \pi/10$ corresponding to $r = 5$ and Golden Mean satisfies all constraints coming from quantum computation and knot theory. That is it spans a dense subgroup of $U(2)$, and allows the realization of modular functor defined by Witten-Chern-Simons $SU(2)$ action for $k = 3$, which is physically highly attractive since the condition

$$r = k + c_g(SU(2))$$

   connecting r, k and the dual Coxeter number $c_g(SU(N)) = n$ in WCS theories is satisfied for $SU(2)$ in this case for $r = 5$ and $k = 3$.

   $SU(2)$ would have interpretation as the left-handed electro-weak gauge group $SU(2)_L$ associated with classical electro-weak gauge fields. The symmetry breaking of $SU(2)_L$ down to a discrete subgroup of $SU(2)_L$ yielding anyons would relate naturally to this. The conservation of the topologized Kähler charge would correlate with the fact that there is no symmetry breaking in the classical $U(1)$ sector. $k = 3$ Chern-Simons theory is also known to share the same universality class as simple 4-body Hamiltonian [23] (larger values of k would correspond to $k + 1$-body Hamiltonians).

3. Number theoretical vision about intentional systems suggests that the preferred relative phases are algebraic numbers or more generally numbers which belong to a finite-dimensional extension of p-adic numbers. The idea about p-adic cognitive evolution as a gradual generation of increasingly complex algebraic extensions of rationals allows to see the extension containing Golden Mean $\Phi = (1 + \sqrt{5})/2$ as one of the simplest extensions. The relative phase $exp(i4\phi) = exp(i2\pi/5)$ is expressible in an extension containing $\sqrt{\Phi}$ and Φ: one has $cos(4\phi) = (\Phi - 1)/2$ and $sin(4\phi) = \sqrt{5\Phi}/2$.

The general number theoretical ideas about cognition support the view that Golden Mean is in a very special role in the number theoretical world order. This would be due to the fact that $log(\Phi)/\pi$ is a rational number. This hypothesis would explain scaling hierarchies based on powers of Golden Mean. One could argue that the geometry of the braid should reflect directly the value of the $A = exp(i2\phi)$. The angle increment per single DNA nucleotide is $\phi/2 = 2\pi/10$ for DNA double strand (note that q would be $exp(i\pi/10)$, which raises the question whether DNA might be a topological quantum computer.

5.2.4 Bratteli diagram for $n = 5$ case, Fibonacci numbers, and microtubuli

Finite-dimensional von Neumann algebras can be conveniently characterized in terms of Bratteli diagrams [16]. For instance, the diagram a) of the figure 5.2.4 at the end of the chapter represents the inclusion $N \subset M$, where $N = M_2(C) \otimes C$, $M = M_6(C) \otimes M_3(C) \otimes C$. The diagram expresses the imbeddings of elements $A \otimes x$ of $M_2(C) \otimes C$ to $M_6(C)$ as a tensor product $A_1 \otimes A_2 \otimes x$

$$\begin{aligned} A_1 &= \begin{pmatrix} A & . & . \\ . & A & . \\ . & . & A \end{pmatrix} , \\ A_2 &= \begin{pmatrix} A & . \\ . & x \end{pmatrix} . \end{aligned}$$

(7)

Bratteli diagrams of infinite-dimensional von Neumann algebras are obtained as limiting cases of finite-dimensional ones.

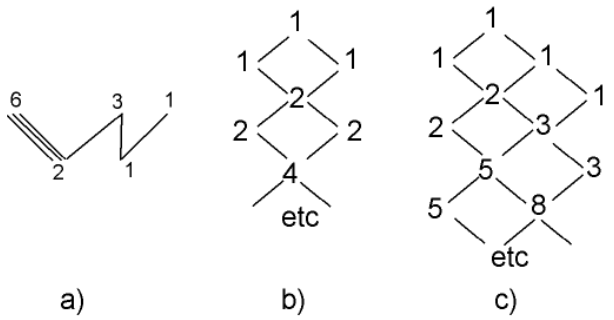

Figure 1: a) Illustration of Bratteli diagram. b) and c) give Bratteli diagrams for $n = 4$ and $n = 5$ Temperley Lieb algebras

2. Temperley Lieb algebras approximate II_1 factors

The hierarchy of inclusions of with $|M_{i+1} : M_i| = r$ defines a hierarchy of Temperley-Lieb algebras characterizable using Bratteli diagrams. The diagrams b) and c) of the figure 5.2.4 at the end of the chapter characterize the Bratteli diagrams for $n = 4$ and $n = 5$. For $n = 4$ the dimensions of algebras come in powers of 2 in accordance with the fact $r = 2$ is the dimension of the effective tensor factor of II_1.

For $n = 5$ and $B_m = \{1, e_1, ..., e_m\}$ the dimensions of the two tensor factors of the Temperley Lieb-representation are two subsequent Fibonacci numbers F_{m-1}, F_m ($F_{m+1} = F_m + F_{m-1}$, $F_1 = 1, F_2 = 1$) so that the dimension of the tensor product is $dim(B_m) = F_m F_{m-1}$. One has $dim(B_{m+1})/dim(B_m) = F_m/F_{m-2} \rightarrow \Phi^2 = 1 + \Phi$, the dimension of the effective tensor factor for the corresponding hierarchy of II_1 factors. Hence the two dimensional hierarchies "approximate" each other. In fact, this result holds completely generally.

The fact that r is approximated by an integer in braid representations is highly interesting from the point of view of TQC. For 3-braid representation the dimension of Temperley-Lieb representation is 2 for all values of n so that 3-braid representation defines single (topo)logical qubit as $(AA) - A - A$ realization indeed assumes. One could optimistically say that TGD based physics automatically realizes topological qubit in terms of 3-braid representation and the challenge is to understand the details of this realization.

2. Why Golden Mean should be favored?

The following argument suggests a physical reason for why just Golden Mean should be favored in the magnetic flux tube systems.

1. Arnold [18] has shown that if Lorentz 3-force satisfies the condition $F_B = q(\nabla \times B) \times B = q\nabla\Phi$, then the field lines of the magnetic field lie on $\Phi = constant$ tori. On the other hand, the vanishing of the Lorentz 4-forces for solutions of field equations representing asymptotic self-organized states, which are the "survivors" selected by dissipation, equates magnetic force with the negative of the electric force expressible as qE, $E = -\nabla\Phi + \partial_t A$, which is

gradient if the vector potential does not depend on time. Since the vector potential depends on three CP_2 coordinates only for $D = 3$, this seems to be the case.

2. The celebrated Kolmogorov-Arnold-Moser (KAM) theorem is about the stability of systems, whose orbits are on invariant tori characterized by the frequencies associated with the n independent harmonic oscillator like degrees of freedom. The theorem states that the tori for which the frequency ratios are rational are highly unstable against perturbations: this is due to resonance effects. The more "irrational" the frequencies are, the higher the stability of the orbits is, and the most stable situation corresponds to frequencies whose ratio is Golden Mean. In quantum context the frequencies for wave motion on torus would correspond to multiples $\omega_i = n2\pi/L_i$, L_i the circumference of torus. This poor man's argument would suggest that the ratio of the circumferences of the most stable magnetic tori should be given by Golden Mean in the most stable situation: perhaps one might talk about Golden Tori!

3. Golden Mean and microtubuli

What makes this observation so interesting is that Fibonacci numbers appear repeatedly in the geometry of living matter. For instance, micro-tubuli, which are speculated to be systems performing quantum computation, represent in their structure the hierarchy Fibonacci numbers $5, 8, 13$, which brings in mind the tensor product representation $5 \otimes 8$ of B_5 (5 braid strands!) and leads to ask whether this Temperley-Lieb representation could be somehow realized using microtubular geometry.

According to the arguments of [23] the state of n anyons corresponds to 2^{n-1} topological degrees of freedom and code space corresponds to F_n-dimensional sub-space of this space. The two conformations of tubulin dimer define the standard candidate for qubit, and one could assume that the conformation correlates strongly with the underlying topological qubit. A sequence of 5 *resp.* 8 tubulin dimers would give 2^4 *resp.* 2^7-dimensional space with $F_5 = 5$- *resp.* $F_7 = 13$-dimensional code sub-space so that numbers come out nicely. The changes of tubulin dimer conformations would be induced by the braid groups B_4 and B_7. B_4 would be most naturally realized in terms of a unit of 5-dimers by regarding the 4 first tubulins as braided punctures and 5th tubulin as the passive puncture. B_7 would be realized in a similar manner using a unit of 8 tubulin dimers.

Flux tubes would connect the subsequent dimers along the helical 5-strand *resp.* 8-strand defined by the microtubule. Nearest neighbor swap for the flux tubes would induce the change of the tubulin conformation and induce also entanglement between neighboring conformations. A full 2π helical twist along microtubule would correspond to 13 basic steps and would define a natural TQC program module. In accordance with the interpretation of II_1 factor hierarchy, (magnetic or electric) flux tubes could be assumed to correspond to $r = 2$ II_1 factor and thus carry 2-dimensional representations of $n = 5$ or $n = 4$ 3-braid group. These qubits could be realized as topological qubits using $(AA) - A$ system.

5.2.5 Topological entanglement as space-time correlate of quantum entanglement

Quantum-classical correspondence encourages to think that bound state formation is represented at the space-time level as a formation of join along boundaries bonds connecting the boundaries of 3-space sheets. In particular, the formation of entangled bound states would correspond to a topological entanglement for the join along boundaries bonds forming braids. The light-likeness of the boundaries of the bonds gives a further support for this identification. During macro-temporal quantum coherence a sequence of quantum jumps binds effectively to single quantum jump and subjective time effectively ceases to run. The light-likeness for the boundaries of bonds means that geometric time stops and is thus natural space-time correlate for the subjective experience during macro-temporal quantum coherence.

Also the work with TQC lends support for a a deep connection between quantum entanglement and topological entanglement in the sense that the knot invariants constructed using entangling 2-gate R can detect linking. Temperley-Lieb representation for 3-braids however suggests that topological entanglement allows also single qubit representations for with quantum entanglement plays no role. One can however wonder whether the entanglement might enter into the picture in some natural manner in the quantum computation of Temperley-Lieb representation. The idea is simple: perhaps the physics of $(AA) - A - A$ system forces single qubit representation through the simple fact that the state space reduces in 4-batch to single qubit by topological constraints.

For TQC the logical qubits correspond to entangled states of anyon Cooper pair (AA) and second anyon A so that the quantum superposition of qubits corresponds to an entangled state in general. Several arguments suggest that logical qubits would provide Temperley-Lieb representation in a natural manner.

1. The number of braids inside 4-anyon batch (or 3-anyon batch in case that (AA) can decay temporarily during braid operation) 3 so that by the universality this system allows to compute the unitary Temperley-Lieb braid representation. The space of logical qubits equals to the entire state space since the number of qubits represented by topological ground state degeneracy is 1 instead of the expected three since $2n$ anyon system gives rise to 2^{n-1}-fold vacuum degeneracy. The degeneracy is same even when two of the anyons fuse to anyon Cooper pair. Thus it would seem that the 3-braid system in question automatically produces 1-qubit representation of 3-braid group.

2. The braiding matrices $\Phi(s_1)$ and $\Phi(s_2)$ are different and only $\Phi(s_2)$ mixes qubit values. This can be interpreted as the presence of two inherently different braid operations such that only the second braiding operation can generate entanglement of states serving as building blocks of logical qubits. The description of anyons as 2-dimensional wormholes led to precisely this picture. The braid group reduces to braid group for one half of anyons since anyon and its partner at the end of wormhole are head and feet of single dancer, and the anyon pair (AA) forming bound state can change partner during swap operation with anyon A and this generates quantum entanglement. The swap for anyons inside (AA) can generate only relative phase.

3. The vanishing of the topological charge in a pairwise manner is the symmetry which reduces the dimension of the representation space to 2^{n-1} as already found. For $n = 4$ only single topological qubit results. The conservation and vanishing of the net topological charge inside each batch gives a constraint, which is satisfied by the maximally entangling R-matrix R so that it could take care of braiding between different 4-batches and one would have different braid representation for 4-batches and braids consisting of them. Topological quantization justifies this picture physically. Only phase generating *physical* 1-gates are allowed since Hadamard gate would break the conservation of topological charge whereas for *logical* 1-gates entanglement generating 2-gates can generate mixing without the breaking of the conservation of topological charges.

5.2.6 Summary

It deserves to summarize the key elements of the proposed model for which the localization (in the precise sense defined in [38]) made possible by topological field quantization and Z_4 valued topological charge are absolutely essential prerequisites.

1. $2n$-anyon system has 2^{n-1}-fold ground state degeneracy, which for $n = 2$ leaves only single logical qubit. In standard physics framework $(AA)-A-A$ is minimal option because the total homology charge of the system must vanish. In TGD $(AA)-A$ system is enough to represent

3-braid system if the braid operation between AA and A can be realized as an exchange of the dancing partner. This option makes sense because the anyons with opposite topological charges at the ends of wormhole threads can be negative energy anyons representing the final state of the braid operation. A pair of magnetic flux tubes is needed to realize single anyon-system containing braid.

2. Maximally entangling R-matrix realizes braid interactions between $(AA)-A$ systems realized as 3-braids inside larger braids and the space of logical qubits is equivalent with the space of realizable qubits. The topological charges are conserved separately for each $(AA)-A$ system. Also the more general realization based on n-braid representations of Temperley-Lieb algebra is formally possible but the different topological realization of braiding operations does not support this possibility.

3. Temperley-Lieb 3-braid representation for $(AA)-A-A$ system allows to realize also 1-gates as braid operations so that topology would allow to avoid the fine-tuning associated with 1-gates. Temperley-Lieb representation for $\phi = exp(i\pi/10)$ satisfies all basic constraints and provides representation of the modular functor expressible using $k = 3$ Witten-Chern-Simons action. Physically 1-gates are realizable using Φ_1 acting as phase gate for anyon pair inside (AA) and $\Phi(s_2)$ entangling (AA) and A by partner exchange. The existence of single qubit braid representations apparently conflicting with the identification of topological entanglement as a correlate of quantum entanglement has an explanation in terms of quantum computation under topological symmetries.

5.3 Zero energy topological quantum computations

As already described, TGD suggests a radical re-interpretation for matter antimatter asymmetry in long length scales. The asymmetry would be due to the fact that ground state for fermion system corresponds to infinite sea of negative energy fermions and positive energy anti-fermions so that fermions would have positive energies and anti-fermions negative energies.

The obvious implication is the possibility to interpret scattering between positive energy states as a creation of a zero energy state with outgoing particles represented as negative energy particles. The fact that the quantum states of 3-dimensional light-like boundaries of 3-surfaces represent evolutions of 2-dimensional quantum systems suggests a realization of topological quantum computations using physical boundary states consisting of positive energy anyons representing the initial state of anyon system and negative energy anyons representing the outcome of the braid operation.

The simplest scenario simply introduces negative energy charge conjugate of the $(AA) - A$ system so that no deviations from the proposed scenario are needed. Both calculation and its conjugate are performed. This picture is the only possible one if one assumes that given space-time sheet contains either positive or negative energy particles but not both and very natural if one assumes ordinary fermionic vacuum. The quantum computing system would could be generated without any energy costs and even intentionally by first generating the p-adic space-time sheets responsible for the magnetic flux tubes and anyons and then transformed to their real counterparts in quantum jump. This double degeneracy is analogous to that associated with DNA double strand and could be used for error correction purposes: if the calculation has been run correctly both anyon Cooper pairs and their charge conjugates should decay with the same probability.

Negative energies could have much deeper role in TQC. This option emerges naturally in the wormhole handle realization of TQC. The TGD realization of 1-gates in 3-braid Temperley-Lieb representation uses anyons of opposite topological charges at the opposite ends of threads connecting magnetic flux tube boundaries. Single 3-braid unit would correspond to positive energy electronic anyons at the first flux tube boundary and negative energy positronic anyons at the second flux tube boundary. The sequences of 1-gates represented as 3-braid operations would be coded by a sequence of 3-braids representing generators of 3-braid group along a pair of magnetic flux tubes. Of course, also n-braid operations could be coded in the similar manner in series. Hence TQC could be realized using only two magnetic flux tubes with n-braids connecting their boundaries in series.

Condensed matter physicist would probably argue that all this could be achieved by using electrons in strand and holes in the conjugate strand instead of negative energy positrons: this would require only established physics. One can however ask whether negative energy positrons could appear routinely in condensed matter physics. For instance, holes might in some circumstances be generated by a creation of an almost zero energy pair such that positron annihilates with a fermion below the Fermi surface. The signature for this would be a photon pair consisting of ordinary and phase conjugate photons.

The proposed interpretation of the S-matrix in the Universe having vanishing net quantum numbers encourages to think that the S-matrices of 2+1-dimensional field theories based on Witten-Chern-Simons action defined in the space of zero (net) energy states could define physical states for quantum TGD. Thus the 2+1-dimensional S-matrix could define quantum states of 4-dimensional theory having interpretation as states representing "self-reflective" level representing in itself the S-matrix of a lower-dimensional theory. The identification of the quantum state as S-matrix indeed makes sense for light-like surfaces which can be regarded as limiting cases of space-like 3-surfaces defining physical state and time-like surfaces defining a time evolution of the state of 2-dimensional system.

Time evolution would define also an evolution in topological degrees of freedom characterizing ground states. Quantum states associated with light-like (with respect to the induced metric of space-time sheet) 3-dimensional boundaries of say magnetic flux tubes would define quantum computations as modular functors. This conforms with quantum-classical correspondence since braids, the classical states, indeed define quantum computations.

The important implication would be that a configuration which looks static would code for the dynamic braiding. One could understand the quantum computation in this framework as signals propagating through the strands and being affected by the gate. Even at the limit when the signal propagates with light velocity along boundary of braid the situation looks static from outside. Time evolution as a state could be characterized as sequence of many-anyon states such that basic braid operations are realized as zero energy states with initial state realized using positive energy anyons and final state realized using negative energy energy anyons differing by the appropriate gate operation from the positive energy state.

In the case of n-braid system the state representing the S-matrix $S = S^1S^2....S^n$ associated with a concatenation of n elementary braid operations would look like

$$\begin{aligned} |S\rangle &= P_{k_1}S^1_{k_1k_2}P_{k_2}S^2_{k_2k_3}P_{k_3}S^3_{k_3k_4}\cdots \ , \\ P_k &= |k,<\rangle|k,>\rangle \ . \end{aligned} \tag{8}$$

Here S^k are S-matrices associated with gates representing simple braiding operations s_k for $n+1$ threads connecting the magnetic flux tubes. P_k represents a trivial transition $|k\rangle \rightarrow |k \rightarrow k\rangle$ as zero energy state $|k,>0\rangle|k,<\rangle$. The states P_k represent matrix elements of the identification map from positive energy Hilbert space to its negative energy dual.

What would happen can be visualized in two alternative manners.

1. For this option the braid maps occur always from flux tube 1 to flux tube 2. A braiding transition from 1 to 2 is represented by S^{k_1}; a trivial transition from 2 to 1 is represented by P_k; a braiding transition from 1 to 2 is represented by S^{k_2}, etc... In this case flux tube 1 contains positive energy anyons and flux tube 2 the negative energy anyons.

2. An alternative representation is the one in which P_k represents transition along the strand so that S^k *resp.* S^{k+1} corresponds to braiding transition from strand 1 to 2 *resp.* 2 to 1. In this case both flux tubes contain both positive and negative energy anyons.

6 Appendix: A generalization of the notion of imbedding space

In the following the recent view about structure of imbedding space forced by the quantization of Planck constant is described. This view has developed much before the original version of this chapter was written.

The original idea was that the proposed modification of the imbedding space could explain naturally phenomena like quantum Hall effect involving fractionization of quantum numbers like spin and charge. This does not however seem to be the case. $G_a \times G_b$ implies just the opposite if these quantum numbers are assigned with the symmetries of the imbedding space. For instance, quantization unit for orbital angular momentum becomes n_a where Z_{n_a} is the maximal cyclic subgroup of G_a.

One can however imagine of obtaining fractionization at the level of imbedding space for space-time sheets, which are analogous to multi-sheeted Riemann surfaces (say Riemann surfaces associated with $z^{1/n}$ since the rotation by 2π understood as a homotopy of M^4 lifted to the space-time sheet is a non-closed curve. Continuity requirement indeed allows fractionization of the orbital quantum numbers and color in this kind of situation.

6.1 Both covering spaces and factor spaces are possible

The observation above stimulates the question whether it might be possible in some sense to replace H or its factors by their multiple coverings.

1. This is certainly not possible for M^4, CP_2, or H since their fundamental groups are trivial. On the other hand, the fixing of quantization axes implies a selection of the sub-space $H_4 = M^2 \times S^2 \subset M^4 \times CP_2$, where S^2 is a geodesic sphere of CP_2. $\hat{M}^4 = M^4 \backslash M^2$ and $\hat{CP}_2 = CP_2 \backslash S^2$ have fundamental group Z since the codimension of the excluded sub-manifold is equal to two and homotopically the situation is like that for a punctured plane. The exclusion of these sub-manifolds defined by the choice of quantization axes could naturally give rise to the desired situation.

2. Zero energy ontology forces to modify this picture somewhat. In zero energy ontology causal diamonds (CDs) defined as the intersections of future and past directed light-cones are loci for zero energy states containing positive and negative energy parts of state at the two light-cone boundaries. The location of CD in M^4 is arbitrary but p-adic length scale hypothesis suggests that the temporal distances between tips of CD come as powers of 2 using CP_2 size as unit. Thus M^4 is replaces by CD and $\hat{M}^4$ is replaced with $\hat{CD}$ defined in obvious manner.

3. H_4 represents a straight cosmic string inside CD. Quantum field theory phase corresponds to Jones inclusions with Jones index $\mathcal{M} : \mathcal{N} < 4$. Stringy phase would by previous arguments correspond to $\mathcal{M} : \mathcal{N} = 4$. Also these Jones inclusions are labeled by finite subgroups of $SO(3)$ and thus by Z_n identified as a maximal Abelian subgroup.

   One can argue that cosmic strings are not allowed in QFT phase. This would encourage the replacement $\hat{CD} \times \hat{CP}_2$ implying that surfaces in $CD \times S^2$ and $(M^2 \cap CD) \times CP_2$ are not allowed. In particular, cosmic strings and CP_2 type extremals with M^4 projection in M^2 and thus light-like geodesic without zitterwebegung essential for massivation are forbidden. This brings in mind instability of Higgs=0 phase.

4. The covering spaces in question would correspond to the Cartesian products $\hat{CD}_{n_a} \times \hat{CP}_{2n_b}$ of the covering spaces of $\hat{CD}$ and $\hat{CP}_2$ by Z_{n_a} and Z_{n_b} with fundamental group is $Z_{n_a} \times Z_{n_b}$. One can also consider extension by replacing $M^2 \cap CD$ and S^2 with its orbit under G_a (say tedrahedral, octahedral, or icosahedral group). The resulting space will be denoted by $\hat{CD} \hat{\times} G_a$ *resp.* $\hat{CP}_2 \hat{\times} G_b$.

5. One expects the discrete subgroups of $SU(2)$ emerge naturally in this framework if one allows the action of these groups on the singular sub-manifolds $M^2 \cap CD$ or S^2. This would replace the singular manifold with a set of its rotated copies in the case that the subgroups have genuinely 3-dimensional action (the subgroups which corresponds to exceptional groups in the ADE correspondence). For instance, in the case of $M^2 \cap CD$ the quantization axes for angular momentum would be replaced by the set of quantization axes going through the vertices of tedrahedron, octahedron, or icosahedron. This would bring non-commutative homotopy groups into the picture in a natural manner.

6. Also the orbifolds $\hat{CD}/G_a \times \hat{CP}_2/G_b$ can be allowed as also the spaces $\hat{CD}/G_a \times (\hat{CP}_2 \hat{\times} G_b)$ and $(\hat{CD} \hat{\times} G_a) \times \hat{CP}_2/G_b$. Hence the previous framework would generalize considerably by the allowance of both coset spaces and covering spaces.

There are several non-trivial questions related to the details of the gluing procedure and phase transition as motion of partonic 2-surface from one sector of the imbedding space to another one.

1. How the gluing of copies of imbedding space at $(M^2 \cap CD) \times CP_2$ takes place? It would seem that the covariant metric of M^4 factor proportional to $\hbar^2$ must be discontinuous at the singular manifold since only in this manner the idea about different scaling factor of M^4 metric can make sense. This is consistent with the identical vanishing of Chern-Simons action in $M^2 \times S^2$.

2. One might worry whether the phase transition changing Planck constant means an instantaneous change of the size of partonic 2-surface in CD degrees of freedom. This is not the case. Light-likeness in $(M^2 \cap CD) \times S^2$ makes sense only for surfaces $X^1 \times D^2 \subset (M^2 \cap CD) \times S^2$, where X^1 is light-like geodesic. The requirement that the partonic 2-surface X^2 moving from one sector of H to another one is light-like at $(M^2 \cap CD) \times S^2$ irrespective of the value of Planck constant requires that X^2 has single point of $(M^2 \cap CD)$ as M^2 projection. Hence no sudden change of the size X^2 occurs.

3. A natural question is whether the phase transition changing the value of Planck constant can occur purely classically or whether it is analogous to quantum tunneling. Classical non-vacuum extremals of Chern-Simons action have two-dimensional CP_2 projection to homologically non-trivial geodesic sphere S^2_I. The deformation of the entire S^2_I to homologically trivial geodesic sphere S^2_{II} is not possible so that only combinations of partonic 2-surfaces with vanishing total homology charge (Kähler magnetic charge) can in principle move from sector to another one, and this process involves fusion of these 2-surfaces such that CP_2 projection becomes single homologically trivial 2-surface. A piece of a non-trivial geodesic sphere S^2_I of CP_2 can be deformed to that of S^2_{II} using 2-dimensional homotopy flattening the piece of S^2 to curve. If this homotopy cannot be chosen to be light-like, the phase transitions changing Planck constant take place only via quantum tunnelling. Obviously the notions of light-like homotopies (cobordisms) and classical light-like homotopies (cobordisms) are very relevant for the understanding of phase transitions changing Planck constant.

6.2 Do factor spaces and coverings correspond to the two kinds of Jones inclusions?

What could be the interpretation of these two kinds of spaces?

1. Jones inclusions appear in two varieties corresponding to $\mathcal{M} : \mathcal{N} < 4$ and $\mathcal{M} : \mathcal{N} = 4$ and one can assign a hierarchy of subgroups of $SU(2)$ with both of them. In particular, their maximal Abelian subgroups Z_n label these inclusions. The interpretation of Z_n as invariance group is natural for $\mathcal{M} : \mathcal{N} < 4$ and it naturally corresponds to the coset spaces. For $\mathcal{M} : \mathcal{N} = 4$ the interpretation of Z_n has remained open. Obviously the interpretation of Z_n as the homology group defining covering would be natural.

2. $\mathcal{M} : \mathcal{N} = 4$ should correspond to the allowance of cosmic strings and other analogous objects. Does the introduction of the covering spaces bring in cosmic strings in some controlled manner? Formally the subgroup of $SU(2)$ defining the inclusion is $SU(2)$ would mean that states are $SU(2)$ singlets which is something non-physical. For covering spaces one would however obtain the degrees of freedom associated with the discrete fiber and the degrees of freedom in question would not disappear completely and would be characterized by the discrete subgroup of $SU(2)$.

   For anyons the non-trivial homotopy of plane brings in non-trivial connection with a flat curvature and the non-trivial dynamics of topological QFTs. Also now one might expect similar non-trivial contribution to appear in the spinor connection of $\hat{CD}\hat{\times}G_a$ and $\hat{CP}_2\hat{\times}G_b$. In conformal field theory models non-trivial monodromy would correspond to the presence of punctures in plane.

3. For factor spaces the unit for quantum numbers like orbital angular momentum is multiplied by n_a *resp.* n_b and for coverings it is divided by this number. These two kind of spaces are in a well defined sense obtained by multiplying and dividing the factors of $\hat{H}$ by G_a *resp.* G_b and multiplication and division are expected to relate to Jones inclusions with $\mathcal{M} : \mathcal{N} < 4$ and $\mathcal{M} : \mathcal{N} = 4$, which both are labeled by a subset of discrete subgroups of SU(2).

4. The discrete subgroups of $SU(2)$ with fixed quantization axes possess a well defined multiplication with product defined as the group generated by forming all possible products of group elements as elements of $SU(2)$. This product is commutative and all elements are idempotent and thus analogous to projectors. Trivial group G_1, two-element group G_2 consisting of reflection and identity, the cyclic groups Z_p, p prime, and tedrahedral, octahedral, and icosahedral groups are the generators of this algebra.

By commutativity one can regard this algebra as an 11-dimensional module having natural numbers as coefficients ("rig"). The trivial group G_1, two-element group G_2¡ generated by reflection, and tedrahedral, octahedral, and icosahedral groups define 5 generating elements for this algebra. The products of groups other than trivial group define 10 units for this algebra so that there are 11 units altogether. The groups Z_p generate a structure analogous to natural numbers acting as analog of coefficients of this structure. Clearly, one has effectively 11-dimensional commutative algebra in 1-1 correspondence with the 11-dimensional "half-lattice" N^{11} (N denotes natural numbers). Leaving away reflections, one obtains N^7. The projector representation suggests a connection with Jones inclusions. An interesting question concerns the possible Jones inclusions assignable to the subgroups containing infinitely manner elements. Reader has of course already asked whether dimensions 11, 7 and their difference 4 might relate somehow to the mathematical structures of M-theory with 7 compactified dimensions. One could introduce generalized configuration space spinor fields in the configuration space labelled by sectors of H with given quantization axes. By introducing Fourier transform in N^{11} one would formally obtain an infinite-component field in 11-D space.

The question how do the Planck constants associated with factors and coverings relate is far from trivial and I have considered several options.

1. If one assumes that $\hbar^2(X)$, $X = M^4$, CP_2 corresponds to the scaling of the covariant metric tensor g_{ij} and performs an over-all scaling of metric allowed by Weyl invariance of Kähler action by dividing metric with $\hbar^2(CP_2)$, one obtains $r^2 \equiv \hbar^2/\hbar_0^2\hbar^2(M^4)/\hbar^2(CP_2)$. This puts M^4 and CP_2 in a very symmetric role and allows much more flexibility in the identification of symmetries associated with large Planck constant phases.

2. Algebraist would argue that Planck constant must define a homomorphism respecting multiplication and division (when possible) by G_i. This requires $r(X) = \hbar(X)\hbar_0 = n$ for covering and $r(X) = 1/n$ for factor space or vice versa. This gives two options.

3. Option I: $r(X) = n$ for covering and $r(X) = 1/n$ for factor space gives $r \equiv \hbar/\hbar_0 = r(M^4)/r(CP_2)$. This gives $r = n_a/n_b$ for $\hat{H}/G_a \times G_b$ option and $r = n_b/n_a$ for $\hat{H}ti\hat{m}es(G_a \times G_b)$ option with obvious formulas for hybrid cases.

4. Option II: $r(X) = 1/n$ for covering and $r(X) = n$ for factor space gives $r = r(CP_2)/r(M^4)$. This gives $r = n_b/n_a$ for $\hat{H}/G_a \times G_b$ option and $r = n_a/n_b$ for $\hat{H}ti\hat{m}es(G_a \times G_b)$ option with obvious formulas for the hybrid cases.

5. At quantum level the fractionization would come from the modification of fermionic anti-commutation (bosonic commutation) relations involving $\hbar$ at the right hand side so that particle number becomes a multiple of $1/n$ or n. If one postulates that the total number states is invariant in the transition, the increase in the number of sheets is compensated by the increase of the fundamental phase space volume proportional to $\hbar$. This would give $r(X) \to r(X)/n$ for factor space and $r(X) \to nr(X)$ for the covering space to compensate the n-fold reduction/increase of states. This would favor Option II.

6. The second manner to distinguish between these two options is to apply the theory to concrete physical situations. Since G_a and G_b act as symmetries in CD and CP_2 degrees of freedom, one might of being able to distinguish between the two options if it is possible to distinguish between the action of G as symmetry of quantum states associated with covering and factor space. Also the quantization of the orbital spin quantum number at single particle level as multiples of n can be distinguished from that in multiples of $1/n$.

6.3 A simple model of fractional quantum Hall effect

The generalization of the imbedding space suggests that it could possible to understand fractional quantum Hall effect [45] at the level of basic quantum TGD. This section represents the first rough model of QHE constructed for a couple of years ago is discussed. Needless to emphasize, the model represents only the basic idea and involves ad hoc assumption about charge fractionization.

Recall that the formula for the quantized Hall conductance is given by

$$\begin{aligned} \sigma &= \nu \times \frac{e^2}{h} \ , \\ \nu &= \frac{n}{m} \ . \end{aligned} \tag{9}$$

Series of fractions in $\nu = 1/3, 2/5, 3/7, 4/9, 5/11, 6/13, 7/15..., 2/3, 3/5, 4/7, 5/9, 6/11, 7/13..., 5/3, 8/5, 11/7, 14/9...4/$ $1/5, 2/9, 3/13..., 2/7, 3/11..., 1/7....$ with odd denominator have been observed as are also $\nu = 1/2$ and $\nu = 5/2$ states with even denominator [45].

The model of Laughlin [43] cannot explain all aspects of FQHE. The best existing model proposed originally by Jain is based on composite fermions resulting as bound states of electron and even number of magnetic flux quanta [46]. Electrons remain integer charged but due to the effective magnetic field electrons appear to have fractional charges. Composite fermion picture predicts all the observed fractions and also their relative intensities and the order in which they appear as the quality of sample improves.

The generalization of the notion of imbedding space suggests the possibility to interpret these states in terms of fractionized charge, spin, and electron number. There are four combinations of covering and factors spaces of CP_2 and three of them can lead to the increase of Planck constant. Besides this there are two options for the formula of Planck constant so that which the very meager theoretical background one can make only guesses. On the following just for fun consideration option I is considered although the conservation of number of states in the phase transition changing $\hbar$ favors option II.

1. The easiest manner to understand the observed fractions is by assuming that both M^4 and CP_2 correspond to covering spaces so that both spin and electric charge and fermion number are fractionized. This means that e in electronic charge density is replaced with fractional charge. Quantized magnetic flux is proportional to e and the question is whether also here fractional charge appears. Assume that this does not occur.

2. With this assumption the expression for the Planck constant becomes for Option II as $r = \hbar/\hbar_0 = n_a/n_b$ and charge and spin units are equal to $1/n_b$ and $1/n_a$ respectively. This gives $\nu = nn_a/n_b$. The values $m = 2, 3, 5, 7, ..$ are observed. Planck constant can have arbitrarily large values. There are general arguments stating that also spin is fractionized in FQHE.

3. The appearance of $\nu = 5/2$ has been observed [47]. The fractionized charge is $e/4$ in this case. Since $n_i > 3$ holds true if coverings are correlates for Jones inclusions, this requires to $n_b = 4$ and $n_a = 10$. n_b predicting a correct fractionization of charge. The alternative option would be $n_b = 2$ that also Z_2 would appear as the fundamental group of the covering space. Filling fraction 1/2 corresponds in the composite fermion model and also experimentally to the limit of zero magnetic field [46]. $n_b = 2$ is however inconsistent with the observed fractionization of electric charge and with the vision inspired by Jones inclusions.

4. A possible problematic aspect of the TGD based model is the experimental absence of even values of n_b except $n_b = 2$ (Laughlin's model predicts only odd values of n). A possible explanation is that by some symmetry condition possibly related to fermionic statistics (as in Laughlin model) n_a/n_b must reduce to a rational with an odd denominator for $n_b > 2$. In other words, one has $n_a \propto 2^r$, where 2^r the largest power of 2 divisor of n_b.

5. Large values of n_a emerge as B increases. This can be understood from flux quantization. One has $e \int BdS = n\hbar(M^4) = nn_a\hbar_0$. By using actual fractional charge $e_F = e/n_b$ in the flux factor would give $e_F \int BdS = n(n_a/n_b)\hbar_0 = n\hbar$. The interpretation is that each of the n_a sheets contributes one unit to the flux for e. Note that the value of magnetic field in given sheet is not affected so that the build-up of multiple covering seems to keep magnetic field strength below critical value.

6. The understanding of the thermal stability is not trivial. The original FQHE was observed in 80 mK temperature corresponding roughly to a thermal energy of $T \sim 10^{-5}$ eV. For graphene the effect is observed at room temperature. Cyclotron energy for electron is (from $f_e = 6 \times 10^5$ Hz at $B = .2$ Gauss) of order thermal energy at room temperature in a magnetic field varying in the range 1-10 Tesla. This raises the question why the original FQHE requires so low temperature. The magnetic energy of a flux tube of length L is by flux quantization

roughly $e^2B^2S \sim E_c(e)m_eL$ ($\hbar_0 = c = 1$) and exceeds cyclotron roughly by a factor L/L_e, L_e electron Compton length so that thermal stability of magnetic flux quanta is not the explanation. A possible explanation is that since FQHE involves several values of Planck constant, it is quantum critical phenomenon and is characterized by a critical temperature. The differences of the energies associated with the phase with ordinary Planck constant and phases with different Planck constant would characterize the transition temperature.

As already noticed, it is possible to imagine several other options and the identification of charge unit is rather ad hoc. Therefore this model can be taken only as a warm-up exercise.

References

Online books about TGD

[1] M. Pitkänen (2006), *Topological Geometrodynamics: Overview.* http://www.helsinki.fi/~matpitka/tgdview/tgdview.html.

[2] M. Pitkänen (2006), *Quantum Physics as Infinite-Dimensional Geometry.* http://www.helsinki.fi/~matpitka/tgdgeom/tgdgeom.html.

[3] M. Pitkänen (2006), *Physics in Many-Sheeted Space-Time.* http://www.helsinki.fi/~matpitka/tgdclass/tgdclass.html.

[4] M. Pitkänen (2006), *Quantum TGD.* http://www.helsinki.fi/~matpitka/tgdquant/tgdquant.html.

[5] M. Pitkänen (2006), *TGD as a Generalized Number Theory.* http://www.helsinki.fi/~matpitka/tgdnumber/tgdnumber.html.

[6] M. Pitkänen (2006), *p-Adic length Scale Hypothesis and Dark Matter Hierarchy.* http://www.helsinki.fi/~matpitka/paddark/paddark.html.

[7] M. Pitkänen (2006), *TGD and Fringe Physics.* http://www.helsinki.fi/~matpitka/freenergy/freenergy.html.

Online books about TGD inspired theory of consciousness and quantum biology

[8] M. Pitkänen (2006), *Bio-Systems as Self-Organizing Quantum Systems.* http://www.helsinki.fi/~matpitka/bioselforg/bioselforg.html.

[9] M. Pitkänen (2006), *Quantum Hardware of Living Matter.* http://www.helsinki.fi/~matpitka/bioware/bioware.html.

[10] M. Pitkänen (2006), *TGD Inspired Theory of Consciousness.* http://www.helsinki.fi/~matpitka/tgdconsc/tgdconsc.html.

[11] M. Pitkänen (2006), *Mathematical Aspects of Consciousness Theory.* http://www.helsinki.fi/~matpitka/genememe/genememe.html.

[12] M. Pitkänen (2006), *TGD and EEG.* http://www.helsinki.fi/~matpitka/tgdeeg/tgdeeg/tgdeeg.html.

[13] M. Pitkänen (2006), *Bio-Systems as Conscious Holograms*.
http://www.helsinki.fi/~matpitka/hologram/hologram.html.

[14] M. Pitkänen (2006), *Magnetospheric Consciousness*.
http://www.helsinki.fi/~matpitka/magnconsc/magnconsc.html.

[15] M. Pitkänen (2006), *Mathematical Aspects of Consciousness Theory*.
http://www.helsinki.fi/~matpitka/magnconsc/mathconsc.html.

References to the chapters of books

[A8] The chapter *Was von Neumann Right After All* of [4].
http://www.helsinki.fi/~matpitka/tgdview/tgdview.html#vNeumann.

[A9] The chapter *Does TGD Predict the Spectrum of Planck Constants?* of [1].
http://www.helsinki.fi/~matpitka/tgdview/tgdview.html#Planck.

[B1] The chapter *Identification of the Configuration Space Kähler Function* of [2].
http://www.helsinki.fi/~matpitka/tgdgeom/tgdgeom.html#kahler.

[B2] The chapter *Construction of Configuration Space Kähler Geometry from Symmetry Principles: Part I* of [2].
http://www.helsinki.fi/~matpitka/tgdgeom/tgdgeom.html#compl1.

[B3] The chapter *Construction of Configuration Space Kähler Geometry from Symmetry Principles: Part II* of [2].
http://www.helsinki.fi/~matpitka/tgdgeom/tgdgeom.html#compl2.

[B4] The chapter *Configuration Space Spinor Structure* of [2].
http://www.helsinki.fi/~matpitka/tgdgeom/tgdgeom.html#cspin.

[C2] The chapter *Construction of Quantum Theory: Symmetries* of [4].
http://www.helsinki.fi/~matpitka/tgdquant/tgdquant.html#quthe.

[C3] The chapter *Construction of Quantum Theory: S-matrix* of [4].
http://www.helsinki.fi/~matpitka/tgdquant/tgdquant.html#towards.

[C4] The chapter *Hyper-Finite Factors and Construction of S-matrix* of [4].
http://www.helsinki.fi/~matpitka/tgdquant/tgdquant.html#HFSmatrix.

[D1] The chapter *Basic Extremals of Kähler Action* of [3].
http://www.helsinki.fi/~matpitka/tgdclass/tgdclass.html#class.

[D6] The chapter *TGD and Astrophysics* of [3].
http://www.helsinki.fi/~matpitka/tgdclass/tgdclass.html#astro.

[E2] The chapter *TGD as a Generalized Number Theory: Quaternions, Octonions, and their Hyper Counterparts* of [5].
http://www.helsinki.fi/~matpitka/tgdnumber/tgdnumber.html#visionb.

[E6] The chapter *Fusion of p-Adic and Real Variants of Quantum TGD to a More General Theory* of [5].
http://www.helsinki.fi/~matpitka/tgdnumber/tgdnumber.html#mblocks.

[E8] The chapter *Riemann Hypothesis and Physics* of [5].
http://www.helsinki.fi/∼matpitka/tgdnumber/tgdnumber.html#riema.

[E11] The chapter *Langlands Program and TGD* of [5].
http://www.helsinki.fi/∼matpitka/tgdnumber/tgdeeg/tgdnumber.html#Langlandia.

[F9] The chapter *Nuclear String Physics* of [6].
http://www.helsinki.fi/∼matpitka/paddark/paddark.html#nuclstring.

[G2] The chapter *The Notion of Free Energy and Many-Sheeted Space-Time Concept* of [7].
http://www.helsinki.fi/∼matpitka/freenergy/freenergy.html#freenergy.

[H2] The chapter *Negentropy Maximization Principle* of [10].
http://www.helsinki.fi/∼matpitka/tgdconsc/tgdconsc.html#nmpc.

[I4] The chapter *Quantum Control and Coordination in Bio-systems: Part I* of [8].
http://www.helsinki.fi/∼matpitka/bioselforg/bioselforg.html#qcococI.

[I5] The chapter *Quantum Control and Coordination in Bio-Systems: Part II* of [8].
http://www.helsinki.fi/∼matpitka/bioselforg/bioselforg.html#qcococII.

[K4] The chapter *Bio-Systems as Conscious Holograms* of [13].
http://www.helsinki.fi/∼matpitka/hologram/hologram.html#hologram.

[L1] The chapter *Genes and Memes* of [11].
http://www.helsinki.fi/∼matpitka/genememe/genememe.html#genememec.

[L2] The chapter *Many-Sheeted DNA* of [11].
http://www.helsinki.fi/∼matpitka/genememe/genememe.html#genecodec.

[M3] The chapter *Dark Matter Hierarchy and Hierarchy of EEGs* of [12].
http://www.helsinki.fi/∼matpitka/tgdeeg/tgdeeg/tgdeeg.html#eegdark.

[N4] The chapter *Pre-Biotic Evolution in Many-Sheeted Space-Time* of [14].
http://www.helsinki.fi/∼matpitka/magnconsc/magnconsc.html#prebio.

Mathematics related references

[16] C. N. Yang, M. L. Ge (1989), *Braid Group, Knot Theory, and Statistical Mechanics*, World Scientific.

[17] A. Lakthakia (1994), *Beltrami Fields in Chiral Media*, Series in Contemporary Chemical Physics - Vol. 2, World Scientific, Singapore.

[18] V. I. Arnold (1986), Sel. Math. Sov. 5, 327.

[19] D. Reed (1995), in *Advanced Electromagnetism: Theories, Foundations, Applications*, edited by T. Barrett (Chap. 7), World Scientific, Singapore.

[20] G. E. Marsh (1995), *Helicity and Electromagnetic Field Topology* in *Advanced Electromagnetism*, Eds. T. W. Barrett and D. M. Grimes, Word Scientific.

Quantum computation

[21] *Quantum Information Science*, Report of NSF Workshop, October 28-29, 1999. Arlington Virginia. National Science Foundation.

[22] Paul Parsons (2004) , *Dancing the Quantum Dream*, New Scientist 24. January. www.newscientist.com/hottopics.

[23] M. Freedman, A. Kitaev, M. Larson, Z. Wang (200?), www.arxiv.of/quant-ph/0101025.

[24] R. Feynman (1982), *Simulating physics with computers*, Int. J. Theor. Phys. 21, 467-488. *Ibid* (1986), *Quantum mechanical computers*, Found. Phys., 16, 507-603.

[25] D. Deutch (1985), *Quantum theory, the Church-Turing principle, and the universal quantum computer*, Proc. Roy. Soc. London, A400, 97-117.

[26] P. Shor (1994), *Algorithms for quantum computation, discrete logarithms, and factoring*, Proc. 35th Annual Symposium on Foundations of Computer Science, IEEE Computer Society Press, Los Alamitos, CA, 124-134.

[27] J. Preskill (1997), *Fault tolerant quantum computation*, quant-ph/9712048.

[28] P. Shor (1995), *Scheme for reducing de-coherence in quantum computer memory*, Phys. Rev. A 52, 2493.

[29] D. Gottesman (1998), *Theory of fault tolerant quantum computation*, Phys. Lett. A57, 127-137.

[30] S. Sawin (1995), *Links, Quantum Groups, and TQFT's*, q-alg/9506002.

[31] C. Kassel (1995), *Quantum Groups*, Springer Verlag.

[32] E. Witten 1989), *Quantum field theory and the Jones polynomial*, Comm. Math. Phys. 121 , 351-399.

[33] K. Walker (1991), *On Witten's 3-manifold invariants*, preprint, http://xmission.com/ kwalker/math/.

[34] V. F. R. Jones (1987), *Hecke algebra representations of braid groups and link polynomial*, Ann. Math., 126, 335-388.

[35] L. H. Kauffmann and S. J. Lomonaco Jr. (2004), *Braiding operations are universal quantum gates*, arxiv.org/quant-ph/0401090.

[36] H. Dye (2003), *Unitary solutions to the Yang-Baxter equations in dimension four*, Quantum Information Processing, Vol 2 Ns 1,2, April, 117,150 (quant-ph/0211050).

[37] M. Freedman, H. Larsen, and Z. Wang (2002), *A modular functor which is universal for quantum computation*, Found. Comput. Math. 1, no 2, 183-204. Comm. Math. Phys. 227, no 3, 605-622. quant-ph/0001108.

[38] M. H. Freedman (2001), *Quantum Computation and the localization of Modular Functors*, Found. Comput. Math. 1, no 2, 183-204.

[39] M. H. Freedman (1998), *P/NP, and the quantum field computer*, Proc. Natl. Acad. Sci. USA 95, no. 1, 98-101.

[40] A. Kitaev (1997), Annals of Physics, vol 303, p.2. See also *Fault tolerant quantum computation by anyons*, quant-ph/9707021.

[41] A. Kitaev (1997), *Quantum computations: algorithms and error correction*, Russian Math. Survey, 52:61, 1191-1249.

Quantum Hall effect

[42] S. M. Girvin (1999), *Quantum Hall Effect, Novel Excitations and Broken Symmetries*, cond-mat/9907002.

[43] R. B. Laughlin (1990), Phys. Rev. Lett. 50, 1395.

[44] F. Wilzek (1990), *Fractional Statistics and Anyon Super-Conductivity*, World Scientific.

[45] *Fractional quantum Hall Effect*, http://en.wikipedia.org/wiki/Fractional_quantum_Hall_effect. *Fractional Quantum Hall Effect*, http://www.warwick.ac.uk/~phsbm/fqhe.htm.

[46] J.K. Jain(1989), Phys. Rev. Lett. 63, 199.

[47] J. B. Miller *et al*(2007), *Fractional Quantum Hall effect in a quantum point contact at filling fraction 5/2*, arXiv:cond-mat/0703161v2.

[48] A. Wojs, K.-S. Yi and J. J. Quinn (2003), *Fractional Quantum Hall States of Composite Fermions*, cond-mat/0312290.

[49] M. de Wild Propitius and F. A. Bais (1996), *Discrete Gauge Theories*, hep-th/9511201.

[50] C. Nayak and F. Wilczek (1996), *2n-quasihole states realize* 2^{n-1}*-dimensional spinor braiding statistics in paired quantum Hall states*, Nucl. Phys. B479, 529-533.

[51] G. Moore and N. Read (1991), *Non-Abelians in the fractional quantum Hall effect*, Nucl. Phys. B360, 362-396.

References related to physics anomalies

[52] V. V. Shkunov and B. Ya. Zeldowich (1985), *Optical Phase Conjugation*, Scientific American, December issue.

[53] D. Da Roacha and L. Nottale (2003), *Gravitational Structure Formation in Scale Relativity*, astro-ph/0310036.

[54] M. Chaplin (2005), *Water Structure and Behavior*,
http://www.lsbu.ac.uk/water/index.html.
For 41 anomalies see http://www.lsbu.ac.uk/water/anmlies.html.
For the icosahedral clustering see http://www.lsbu.ac.uk/water/clusters.html.
J. K. Borchardt(2003), *The chemical formula H2O - a misnomer*, The Alchemist 8 Aug (2003).
R. A. Cowley (2004), *Neutron-scattering experiments and quantum entanglement*, Physica B 350 (2004) 243-245.
R. Moreh, R. C. Block, Y. Danon, and M. Neumann (2005), *Search for anomalous scattering of keV neutrons from H2O-D2O mixtures*, Phys. Rev. Lett. 94, 185301.

[55] L. P. Semikhana and Yu. A. Lyubinov (1998), *Effects of Weak Magnetic Fields on Dielectric Loss in Ordinary Water and Heavy Water*, Moscow University Physics Bulletin, 43, 60-64.

[56] R. Mills *et al* (2003), *Spectroscopic and NMR identification of novel hybrid ions in fractional quantum energy states formed by an exothermic reaction of atomic hydrogen with certain catalysts.*
http://www.blacklightpower.com/techpapers.html .

Life science related references

[57] P. Gariaev *et al* (2000), *The DNA-wave-biocomputer*, CASYS'2000, Fourth International Conference on Computing Anticipatory Systems, Liege, 2000. Abstract Book, Ed. M. Dubois.

[58] A. Coghlan (2004), *Our genome 'reads' junk as well as genes*, New Scientist, 21 February.

[59] Articles about TGD inspired theory of consciousness in previous issues of Journal of Non-Locality and Remote Mental Interactions.
http://www.emergentmind.org .

[60] W. A. Tiller, W. E. Dibble Jr., and M. J. Kohane (2001), *Conscious Acts of Creation: the Emergence of a New Physics*, p. 63.
Pavior Publishing. http://www.pavior.com/.

[61] E. Lozneanu and M. Sanduloviciu (2003) *Minimal-cell system created in laboratory by self-organization*, Chaos, Solitons & Fractals, Volume 18, Issue 2, October, p. 335.
See also *Plasma blobs hint at new form of life*, New Scientist vol. 179 issue 2413 - 20 September 2003, page 16.