Fusion of p-Adic and Real Variants of Quantum TGD to a More General Theory

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Abstract

The mathematical aspects of p-adicization of quantum TGD are discussed. In a well-defined sense Nature itself performs the p-adicization and p-adic physics can be regarded as physics of cognitive regions of space-time which in turn provide representations of real space-time regions. Cognitive representations presumably involve the p-adicization of the geometry at the level of the space-time and imbedding space by a mapping of a real space time region to a p-adic one. One can differentiate between two kinds of maps: the identification induced by the common rationals of real and p-adic space time region and the representations of the external real world to internal p-adic world induced by a canonical identification type maps.

Only the identification by common rationals respects general coordinate invariance, and it leads to a generalization of the number concept. Different number fields form a book like structure with number fields and their extensions representing the pages of the book glued together along common rationals representing the rim of the book. This generalization forces also the generalization of the manifold concept: both imbedding space and configuration space are obtained as union of copies corresponding to various number fields glued together along common points, in particular rational ones. Space-time surfaces decompose naturally to real and p-adic space-time sheets. In this framework the fusion of real and various p-adic physics reduces more or less to to an algebraic continuation of rational number based physics to various number fields and their extensions.

The program makes sense only if also extensions containing transcendentals are allowed: the p-dimensional extension containing powers of e is perhaps the most important transcendental extension involved. Entire cognitive hierarchy of extension emerges and the dimension of extension can be regarded as a measure for the cognitive resolution and the higher the dimension the shorter the length scale of resolution. Cognitive resolution provides also number theoretical counterpart for the notion of length scale cutoff unavoidable in quantum field theories: now the length scale cutoffs are part of the physics of cognition rather than reflecting the practical limitations of theory building.

There is a lot of p-adicizing to do.

a) The p-adic variant of classical TGD must be constructed. Field equations make indeed sense also in the p-adic context. The strongest assumption is that real space time sheets have the same functional form as real space-time sheet so that there is non-uniqueness only due to the hierarchy of dimensions of extensions.

b) Probability theory must be generalized. Canonical identification playing central role in p-adic mass calculations using p-adic thermodynamics maps genuinely p-adic probabilities to their real counterparts. p-Adic entropy can be defined and one can distinguish between three kinds of entropies: real entropy, p-adic entropy mapped to its real counterpart by canonical identification, and number theoretical entropies applying when probabilities are in finite-dimensional extension of rationals. Number theoretic entropies can be negative and provide genuine information measures, and it turns that bound states should correspond in TGD framework to entanglement coefficients which belong to a finite-dimensional extension of rationals and have negative number theoretic entanglement entropy. These information measures generalize by quantum-classical correspondence to space-time level.

c) p-Adic quantum mechanics must be constructed. p-Adic unitarity differs in some respects from its real counterpart: in particular, p-adic cohomology allows unitary S-matrices $S = 1 + T$ such that T is hermitian and nilpotent matrix. p-Adic quantum measurement theory based on Negentropy Maximization Principle (NMP) leads to the notion of monitoring, which might have relevance for the physics of cognition.

d) Generalized quantum mechanics results as fusion of quantum mechanics in various number fields using algebraic continuation from the field of rational as a basic guiding principle. It seems possible to generalize the notion of unitary process in such a manner that unitary matrix leads from rational Hilbert space H_Q to a formal superposition of states in all Hilbert spaces H_F, where F runs over number fields. If this is accepted, state function reduction is a pure number theoretical necessity and involves a reduction to a particular number field followed by state function reduction and state preparation leading ultimately to a state containing only

entanglement which is rational or finitely-extended rational and because of its negative number theoretic entanglement entropy identifiable as bound state entanglement stable against NMP.

e) Generalization of the configuration space and related concepts is also necessary and again gluing along common rationals and algebraic continuation is the basic guide line also now. Configuration space is a union of symmetric spaces and this allows an algebraic construction of the configuration space Kähler metric and spinor structure, whose definition reduces to the super canonical algebra defined by the function basis at the light cone boundary. Hence the algebraic continuation is relatively straightforward. Even configuration space functional integral could allow algebraic continuation. The reason is that symmetric space structure together with Duistermaat Hecke theorem suggests strongly that configuration space integration with the constraints posed by infinite-dimensional symmetries on physical states is effectively equivalent to Gaussian functional integration in free field theory around the unique maximum of Kähler function using contravariant configuration space metric as a propagator. Algebraic continuation is possible for a subset of rational valued zero modes if Kähler action and Kähler function are rational functions of configuration space coordinates for rational values of zero modes.

1 Introduction

The notion of p-adicization has for a long time been a somewhat obscure attempt to provide a theoretical justification for the successes of the p-adic mass calculations. The reduction of quantum TGD to a generalized number theory and the developments in TGD inspired theory of consciousness have however led to a better understanding what the p-adicization possibly means.

1.1 What p-adic physics means?

Contrary to the original expectations finite-p p-adic physics means the physics of the p-adic cognitive representations about real physics rather than 'real physics'. This forces to update the prejudices about what p-adicization means. The original hypothesis was that p-adicization is a strict one-to-one map from real to p-adic physics and this led to technical problems with symmetries.

The new vision about quantum TGD the notion of the p-adic space-time emerges dynamically and p-adic space-time regions are absolutely 'real' and certainly not 'p-adicized' in any sense. Furthermore, the new view also encourages the hypothesis that p-adic regions provide cognitive models for the real matter like regions becoming more and more refined in the evolutionary self-organization process by quantum jumps. p-Adic region can serve as a cognitive model for particle itself or for the external world. The model is defined by some cognitive map of real region to its p-adic counterpart. This cognitive map need not be unique. At the level of TGD inspired theory of consciousness the p-adicization becomes modelling of how cognition works.

In this conceptual framework the successes of the p-adic mass calculations can be understood only if p-adic mass calculations provide a model a 'cognitive model' of an elementary particle. The successes of the p-adic mass calculations, and also the fact that they rely on the fundamental symmetries of quantum TGD, encourages the idea that one could try to mimic Nature. Thus p-adic physics could be seen as an abstract mimicry for what Nature already does by constructing explicitly p-adic cognitive representations. This new view about p-adic physics allows much more flexibility since p-adicization can be interpreted as a cognitive map mapping real world physics to p-adic physics. In this view p-adicization need not and cannot be a unique procedure.

1.2 Number theoretic vision briefly

The number theoretic vision [E1, E2, E3] about the classical dynamics of space-time surfaces is now relatively detailed although it involves unproven conjectures inspired by physical intuition.

1. Hyper-quaternions and octonions

The original idea was that space-time surfaces could be regarded as four-surfaces in 8-D imbedding space with the property that the tangent spaces of these spaces can be locally regarded as 4- *resp.* 8-dimensional number fields of quaternions and octonions.

The difficulties caused by the Euclidian metric signature of the number theoretical norm have however forced to give up the original idea as such, and to introduce complexified octonions and quaternions resulting by extending quaternionic and octonionic algebra by adding imaginary units multiplied with $\sqrt{-1}$. This spoils the number field property but the notion of prime is not lost. The sub-space of hyper-quaternions *resp.* -octonions is obtained from the algebra of ordinary quaternions and octonions by multiplying the imaginary part with $\sqrt{-1}$. The transition is the number theoretical counterpart of the transition from Riemannian to pseudo-Riemannin geometry performed already in Special Relativity.

The problem is that $H = M^4 \times CP_2$ cannot be endowed with a hyper-octonionic manifold structure. Indeed, space-time surfaces are assumed to be hyper-quaternionic or co-hyper-quaternionic 4-surfaces of 8-dimensional Minkowski space M^8 identifiable as the hyper-octonionic space HO. Since the hyper-quaternionic sub-spaces of HO with fixed complex structure are labelled by CP_2, each (co)-hyper-quaternionic four-surface of HO defines a 4-surface of $M^4 \times CP_2$. One can say that the number-theoretic analog of spontaneous compactification occurs.

2. Space-time-surface as a hyper-quaternionic sub-manifold of hyper-octonionic imbedding space?

Space-time identified as a hyper-quaternionic sub-manifold of the hyper-octonionic space in the sense that the tangent space of the space-time surface defines a hyper-quaternionic sub-algebra of the hyper-octonionic tangent space of H at each space-time point, looks an attractive idea. Second possibility is that the tangent space-algebra of the space-time surface is either associative or co-associative at each point. One can also consider possibility that the dynamics of the space-time surface is determined from the requirement that space-time surface is algebraically closed in the sense that tangent space at each point has this property. Also the possibility that the property in question is associated with the normal space at each point of X^4 can be considered. Some delicacies are caused by the question whether the induced algebra at X^4 is just the hyper-octonionic product or whether the algebra product is projected to the space-time surface. If normal part of the product is projected out the space-time algebra closes automatically.

The first guess would be that space-time surfaces are hyper-quaternionic sub-manifolds of hyper-octonionic space $HO = M^8$ with the property that complex structure is fixed and same at all points of space-time surface. This corresponds to a global selection of a preferred octonionic imaginary unit. The automorphisms leaving this selection invariant form group $SU(3)$ identifiable as color group. The selections of hyper-quaternionic sub-space under this condition are parameterized by CP_2. This means that each 4-surface in HO defines a 4-surface in $M^4 \times CP_2$ and one can speak about number-theoretic analog of spontaneous compactification having of course nothing to do with dynamics. It would be possible to make physics in two radically different geometric pictures: HO picture and $H = M^4 \times CP_2$ picture.

For a theoretical physicists of my generation it is easy to guess that the next step is to realize that it is possible to fix the preferred octonionic imaginary at each point of HO separately so that local $S^6 = G_2/SU(3)$, or equivalently the local group G_2 subject to $SU(3)$ gauge invariance, characterizes the possible choices of hyper-quaternionic structure with a preferred imaginary unit. $G_2 \subset SO(7)$ is the automorphism group of octonions, and appears also in M-theory. This local choice has interpretation as a fixing of the plane of non-physical polarizations and rise to degeneracy which is a good candidate for the ground state degeneracy caused by the vacuum extremals.

$OH - -M^4 \times CP_2$ duality allows to construct a foliation of HO by hyper-quaternionic space-time surfaces in terms of maps $HO \rightarrow SU(3)$ satisfying certain integrability conditions guaranteing that the distribution of hyper-quaternionic planes integrates to a foliation by 4-surfaces. In fact, the freedom to fix the preferred imaginary unit locally extends the maps to $HO \rightarrow G_2$ reducing to maps $HO \rightarrow SU(3) \times S^6$ in the local trivialization of G_2. This foliation defines a four-parameter family of 4-surfaces in $M^4 \times CP_2$ for each local choice of the preferred imaginary unit. The dual of this foliation defines a 4-parameter famility co-hyper-quaternionic space-time surfaces.

Hyper-octonion analytic functions $HO \rightarrow HO$ with real Taylor coefficients provide a physically motivated ansatz satisfying the integrability conditions. The basic reason is that hyper-octonion analyticity is not plagued by the complications due to non-commutativity and non-associativity. Indeed, this notion results also if the product is Abelianized by assuming that different octonionic imaginary units multiply to zero. A good candidate for the HO dynamics is free massless Dirac action with Weyl condition for an octonion valued spinor field using octonionic representation of gamma matrices and coupled to the G_2 gauge potential defined by the tensor 7×7 tensor product of the imaginary parts of spinor fields.

The basic conjecture is that the absolute minima of Kähler action correspond to the hyper-quaternion analytic surfaces. This conjecture has several variants. It could be that only asymptotic behavior corresponds to hyper-quaternion analytic function but that that hyper-quaternionicity is general property of absolute minima. It could also be that maxima of Kähler function correspond to this kind of 4-surfaces. The encouraging hint is the fact that Hamilton-Jacobi coordinates appear naturally also in the construction of general solutions of field equations.

3. The notion of Kähler calibration

Calibration is a closed p-form, whose value for a given p-plane is not larger than its volume in the induced metric. What is important that if it is maximum for tangent planes of p-sub-manifold, minimal surface with smallest volume in its homology equivalence class results.

The idea of Kähler calibration is based on a simple observation. The octonionic spinor field defines a map $M^8 \rightarrow H = M^4 \times CP_2$ allowing to induce metric and Kähler form of H to M^8. Also Kähler action is well defined for the local hyper-quaternion plane.

The idea is that the non-closed 4-form associated the wedge product of unit tangent vectors of hyper-quaternionic plane in M^8 and saturating to volume for it becomes closed by multiplication with Kähler action density L_K. If L_K is minimal for hyper-quaternion plane, hyper-quaternionic manifolds define extremals of Kähler action for which the magnitudes of positive and negative contributions to the action are separately minimized.

This variational principle is not equivalent with the absolute minimization of Kähler action. Rather, Universe would do its best to save energy, being as near as possible to vacuum. Also vacuum extremals would become physically relevant (they carry non-vanishing density gravitational energy). The non-determinism of the vacuum extremals would have an interpretation in terms of the ability of Universe to engineer itself. The attractiveness of the number theoretical variational principle from the point of calculability of TGD would be that the initial values for the time derivatives of the imbedding space coordinates at X^3 at light-like 7-D causal determinant could be computed by requiring that the energy of the solution is minimized. This could mean a computerizable construction of Kähler function.

4. The representation of infinite hyper-octonionic primes as 4-surfaces

The discovery of infinite primes suggested strongly the possibility to reduce physics to number theory. The construction of infinite primes can be regarded as a repeated second quantization of a super-symmetric arithmetic quantum field theory. This hierarchy of second quantizations means an enormous generalization of physics to what might be regarded a physical counterpart for a hierarchy of abstractions about abstractions about.... The ordinary second quantized quantum physics corresponds only to the lowest level infinite primes. This hierarchy can be identified with the corresponding hierarchy of space-time sheets of the many-sheeted space-time.

One can even try to understand the quantum numbers of physical particles in terms of infinite primes. In particular, the hyper-quaternionic primes correspond four-momenta and mass squared is prime valued for them. The properties of 8-D hyper-octonionic primes motivate the attempt to identify the quantum numbers associated with CP_2 degrees of freedom in terms of these primes. Infinite primes can be mapped to polynomial primes and this observation allows to identify completely generally the spectrum of infinite primes.

This in turn led to the idea that it might be possible represent infinite primes (integers) geometrically as surfaces defined by the polynomials associated with infinite primes (integers). Obviously, infinite primes would serve as a bridge between Fock-space descriptions and geometric descriptions of physics: quantum and classical. Geometric objects could be seen as concrete representations of infinite numbers providing amplification of infinitesimals to macroscopic deformations of space-time surface. We see the infinitesimals as concrete geometric shapes!

Since the notion of prime makes sense for the complexified octonions, it makes sense also for the hyper-octonions. It is possible to assign to infinite prime of this kind a hyper-octonion analytic polynomial $P : OH \rightarrow OH$ and hence also a foliation of OH and $H = M^4 \times CP_2$ by hyper-quaternionic 4-surfaces and notion of Kähler calibration. Therefore space-time surface could be seen as a geometric counterpart of a Fock state. The assignment is not unique but determined only up to an element of the local octonionic automorphism group G_2 acting in HO and fixing the local choices of the preferred imaginary unit of the hyper-octonionic tangent plane. In fact, a map $HO \rightarrow S^6$ characterizes the choice since $SO(6)$ acts effectively as a local gauge group.

The construction generalizes to all levels of the hierarchy of infinite primes and produces also representations for integers and rationals associated with hyper-octonionic numbers as space-time surfaces. A close relationship with algebraic geometry results and the polynomials define a natural hierarchical structure in the space of 3-surfaces. By the effective 2-dimensionality naturally associated with infinite primes represented by real polynomials 4-surfaces are determined by data given at partonic 2-surfaces defined by the intersections of 3-D and 7-D light-like causal determinants. In particular, the notions of genus and degree serve as classifiers of the algebraic geometry of the 4-surfaces. The great dream is to prove that this construction yields the solutions to the absolute minimization of Kähler action.

1.3 p-Adic space-time sheets as solutions of real field equations continued algebraically to p-adic number field

The ideas about how p-adic topology emerges from quantum TGD have varied. The first belief was that p-adic topology is only an effective topology of real space-time sheets. This belief turned out to be not quite correct. p-Adic topology emerges also as a genuine topology of the space-time and p-adic regions could be identified as correlations for cognition and intentionality. This requires a generalization of the notion of number by gluing reals and various p-adic number fields together along common rationals. This in turn implies generalization of the notion of imbedding space. p-Adic transcendentals can be regarded as infinite numbers in the real sense and thus most points of the p-adic space-time sheets would be at infinite distance and real and p-adic space-time sheets would intersect in discrete set consisting of rational points. This view in which cognition and intentionality would be literally cosmic phenomena is in a sharp contrast with the often held belief that p-adic topology emerges below Planck length scale.

1.4 The notion of pinary cutoff

The notion of pinary cutoff is central for p-adic TGD and it should have some natural definition and interpretation in the new approach. The presence of p-adic pseudo constants implies that there is large number of cognitive representations with varying degrees of faithfulness. Pinary cutoff must serve as a measure for how faithful the p-adic cognitive representation is. Since the cognitive maps are not unique, one cannot even require any universal criterion for the faithfullness of the cognitive map. One can indeed imagine two basic criteria corresponding to self-representations and representations for external world.

1. The subset of rationals common to the real and p-adic space-time surface could define the resolution. In this case, the average distance between common rational points of these two surfaces would serve as a measure for the resolution. Pinary cutoff could be defined as the smallest number of pinary digits in expansions of functions involved above which the resolution does not improve. Physically the optimal resolution would mean that p-adic space-time surface, 'cognitive space-time sheet', has a maximal number of intersections with the real space-time surface for which it provides a self-representation. This purely algebraic notion of faithfullness does not respect continuity: two rational points very near in real sense could be arbitrary far from each other with respect to the p-adic norm.
2. One could base the notion of faithfulness on the idea that p-adic space-time sheet provides almost continuous map of the real space-time sheet belonging to the external world by the basic properties of the canonical identification. The real canonical image of the p-adic space-time sheet and real space-time sheet could be compared and some geometric measure for the nearness of these surfaces could define the resolution of the cognitive map and pinary cutoff could be defined in the same manner as above.

1.5 Program

These ideas lead to a rather well defined p-adicization program. Define precisely the concepts of the p-adic space-time and reduced configuration space, formulate the finite-p p-adic versions of quantum TGD and construct the p-adic variants of TGD. Of course, the aim is not to just construct p-adic version of the real quantum TGD but to understand how real and p-adic quantum TGD:s fuse together to form the full theory of physics and cognition.

The construction of the p-adic TGD necessitates the generalization of the basic tools of standard physics such as differential and integral calculus, the concept of Hilbert space, Riemannian geometry, group theory, action principles, probability and unitary concepts to p-adic context. Also new physical thinking and philosophy is needed and this long chapter is devoted to the description of the new elements. Before going to the detailed exposition it is appropriate to give a brief overall view of the basic mathematical tools.

2 p-Adic numbers and consciousness

The idea that p-adic physics provides the physics of cognition and intentionality has become more and more attractive during the 12 years or so that I have spent with p-adic numbers and I feel that it is good to add a summary about these ideas here.

2.1 p-Adic physics as physics of cognition

p-Adic physics began from p-adic mass calculations. The next step in the progress was the idea that p-adic physics serves as a correlate for cognition and this thread gradually led to the recent view requiring the generalization of the number concept.

2.1.1 Decomposition of space-time surface into p-adic and real regions as representation for matter-mind duality

Space-time surfaces contain genuinely p-adic and possibly even rational-adic regions so that no p-adicization is performed by Nature itself at at this level and it is enough to mimic the Nature. One manner to end up with the idea about p-adic space-time sheets is following.

Number theoretic vision leads to the idea that space-time surfaces can be associated with a hierarchy of polynomials to which infinite primes are mapped. It can happen that the components of quaternion are not always in algebraic extension of rationals but become complex. In this case the equations might however allow smooth solutions in some algebraic extension of p-adics for some values of prime p. It could also happen that real and p-adic roots exist simultaneously. In both cases the interpretation would be that the p-adic space-time sheets resulting as roots of the rational function provide self-representations for the real space-time sheets represented by real roots. This p-adicization would occur in the regions where some roots of the rational polynomial is complex or real roots exist also in the p-adic sense.

The dynamically generated p-adic space-time sheets could have a common boundary with the real surface in the following sense. At this surface a real root is transformed to a p-adic root and this surface corresponds to a boundary of catastrophe region in catastrophe theory. This boundary provides information about external real world very much in accordance with how nervous system receives information about the external world and makes possible cognitive representations about external world. Since the conditions defining the space-time surface expresses the vanishing of a derivative, the solution involves p-adic pseudo constants so that the cognitive representations are not unique and system can have more or less faithful cognitive representations about itself and about external world.

Rational points of the imbedding space and thus also of space-time surfaces are common to p-adics and reals and p-adic and real space-time surfaces differ only in that completion is different. This fixes the geometric interpretation of the cognitive maps involved with the p-adicization.

2.1.2 Different kinds of cognitive representations

At the level of the space-time surfaces and imbedding space p-adicization boils down to the task of finding a map mapping real space-time region to a p-adic space-time region. These regions correspond to definite regions of the rational imbedding space so that the map has a clear geometric interpretation at the level of rational physics.

The basic constraint on the map is that both real and p-adic space-time regions satisfy field equations: p-adic field equations make sense even if the integral defining the Kähler action does not exist p-adically. p-Adic nondeterminism makes possible this map when one allows finite pinary cutoff characterizing the resolution of the cognitive representation.

There are three basic types of cognitive representations which might be called self-representations and representations of the external world and the the map mediating p-adicization is different for these two maps.

1. The correspondence induced by the common rational points respects algebraic structures and defines self-representation. Real and p-adic space-time surfaces have a subset of rational points (defined by the resolution of the cognitive map) as common. The quality of the representation is defined by the resolution of the map and pinary cutoff for the rationals in pinary expansion is a natural measure for the resolution just as decimal cutoff is a natural measure for the resolution of a numerical model.

2. Canonical identification maps rationals to rationals since the periodic pinary expansion of a rational is mapped to a periodic expansion in the canonical identification. The rationals $q = m/n$ for which n is not divisible by p are mapped to rationals with p-adic norm not larger

than unity. Canonical identification respects continuity. Real numbers with real norm larger than p are mapped to real numbers with norm smaller than one in canonical identification whereas reals with real norm in the interval $[1, p)$ are mapped to p-adics with p-adic norm equal to one. Obviously the generalization of the canonical identification can map the world external to a given space-time region into the interior of this region and provides an example of an abstract cognitive representation of the external world. Also now pinary cutoff serves as a natural measure for the quality of the cognitive map.

3. The basic problems of canonical identification is that it does not respect unitarity. For this reason it is not well suited for relating p-adic and real scattering amplitudes. The problem of the correspondence via direct rationals is that it does not respect continuity. A compromise between algebra and topology is achieved by using a modification of canonical identification $I_{R_p \rightarrow R}$ defined as $I_1(r/s) = I(r)/I(s)$. If the conditions $r \ll p$ and $s \ll p$ hold true, the map respects algebraic operations and also unitarity and various symmetries.

   This variant of canonical identification is not equivalent with the original one using the infinite expansion of q in powers of p since canonical identification does not commute with product and division. The variant is however unique in the recent context when r and s in $q = r/s$ have no common factors. For integers $n < p$ it reduces to direct correspondence.

   It seems that this option, the discovery of which took almost a decade, must be used to relate p-adic transition amplitudes to real ones and vice versa [F5]. In particular, real and p-adic coupling constants are related by this map. Also some problems related to p-adic mass calculations find a nice resolution when I_1 is used.

A fascinating possibility is that cognitive self-maps and maps of the external world at the level of human brain are basically realized by using these two basic types of mappings. Obviously canonical identification performed separately for all coordinates is the only possibility if this map is required to be maximally continuous.

2.1.3 p-Adic physics as a mimicry of p-adic cognitive representations

The success of the p-adic mass calculations suggests that one could apply the idea of p-adic cognitive representation even at the level of quantum TGD to build models which have maximal simplicity and calculational effectiveness. p-Adic mass calculations represent this kind of model: now canonical identification is performed for the p-adic mass squared values and can be interpreted as a map from cognitive representation back to real world.

The basic task is the construction of the cognitive self-map or a cognitive map of external world: the laws of p-adic physics define the cognitive model itself automatically. For the cognitive representations of external world involving some variant of canonical identification mapping the exterior of the imbedding space region inside this region. For self-representations situation is much more simpler. In practice, the direct modelling of p-adic physics without explicit construction of the cognitive map could give valuable information about real physics.

In the earlier approach based on phase preserving canonical identification to the mapping of real space-time surface to its p-adic counterpart led to the requirement about existence of unique (almost) imbedding space coordinates. In present case the selection of the quaternionic coordinates for the imbedding space is unique only apart from quaternion-analytic change of coordinates. This does not seem however pose any problems now. One must also remember that only cognitive representations are in question. These representations are not unique and selection of quaternionic coordinates might be even differentiate between different cognitive representations.

Since infinite primes serve as a bridge between classical and quantum, this map also assigns to a real Fock state associated with infinite prime its p-adic version identifiable as the ground state of a superconformal representation. Thus the map respects quantum symmetries automatically.

If the construction of the states of the representation is a completely algebraic process, there are hopes of constructing the p-adic counterpart of S-matrix. If S-matrix is complex rational it can be mapped to its real counterpart. If the localization in zero modes occurs in each quantum jump the predictions of the theory could reduce to the integration in fiber degrees of freedom of CH reducible in turn to purely algebraic expressions making sense also p-adically.

2.2 Zero energy ontology, cognition, and intentionality

One could argue that conservation laws forbid p-adic-real phase transitions in practice so that cognitions (intentions) realized as real-to-padic (p-adic-to-real) transitions would not be possible. The situation changes if one accepts what might be called zero energy ontology [C1, C2].

2.2.1 Zero energy ontology classically

In TGD inspired cosmology [D5] the imbeddings of Robertson-Walker cosmologies are vacuum extremals. Same applies to the imbeddings of Reissner-Nordström solution [D3] and in practice to all solutions of Einstein's equations imbeddable as extremals of Kähler action. Since four-momentum currents define a collection of vector fields rather than a tensor in TGD, both positive and negative signs for energy corresponding to two possible assignments of the arrow of the geometric time to a given space-time surface are possible. This leads to the view that all physical states have vanishing net energy classically and that physically acceptable universes are creatable from vacuum.

The result is highly desirable since one can avoid unpleasant questions such as "What are the net values of conserved quantities like rest mass, baryon number, lepton number, and electric charge for the entire universe?", "What were the initial conditions in the big bang?", "If only single solution of field equations is selected, isn't the notion of physical theory meaningless since in principle it is not possible to compare solutions of the theory?". This picture fits also nicely with the view that entire universe understood as quantum counterpart 4-D space-time is recreated in each quantum jump and allows to understand evolution as a process of continual re-creation.

2.2.2 Zero energy ontology at quantum level

Also the construction of S-matrix [C2] leads to the conclusion that all physical states possess vanishing conserved quantum numbers. Furthermore, the entanglement coefficients between positive and negative energy components of the state define a unitary S-matrix. S-matrix thus becomes a property of the zero energy state and physical states code by their structure what is usually identified as quantum dynamics.

Also the transitions between zero energy states are possible but general arguments lead to the conclusion that the corresponding S-matrix is almost trivial. This finding, which actually forced the new view about S-matrix, is highly desirable since it explains why positive energy ontology works so well if one forgets effects related to intentional action.

At space-time level this would mean that positive energy component and negative energy component are at a temporal distance characterized by an appropriate p-adic time scale and the integer characterizing the value of Planck constant for the state in question. The scale in question would also characterize the geometric duration of quantum jump and the size scale of space-time region contributing to the contents of conscious experience. The interpretation in terms of a mini bang followed by a mini crunch suggests itself also.

2.2.3 Hyper-finite factors of type II_1 and new view about S-matrix

The representation of S-matrix as unitary entanglement coefficients would not make sense in ordinary quantum theory but in TGD the von Neumann algebra in question is not a type I factor as for quantum mechanics or a type III factor as for quantum field theories, but what is called hyper-finite factor of type II_1 [A8]. This algebra is an infinite-dimensional algebra with the almost defining, and at the first look very strange, property that the infinite-dimensional unit matrix has unit trace. The infinite dimensional Clifford algebra spanned by the configuration space gamma matrices (configuration space understood as the space of 3-surfaces, the "world of classical worlds") is indeed very naturally algebra of this kind since infinite-dimensional Clifford algebras provide a canonical representations for hyper-finite factors of type II_1.

2.2.4 The new view about quantum measurement theory

This mathematical framework leads to a new kind of quantum measurement theory. The basic assumption is that only a finite number of degrees of freedom can be quantum measured in a given measurement and the rest remain untouched. What is known as Jones inclusions $\mathcal{N} \subset \mathcal{M}$ of von Neumann algebras allow to realize mathematically this idea [A8]. $\mathcal{N}$ characterizes measurement resolution and quantum measurement reduces the entanglement in the non-commutative quantum space $\mathcal{M}/\mathcal{N}$. The outcome of the quantum measurement is still represented by a unitary S-matrix but in the space characterized by $\mathcal{N}$. It is not possible to end up with a pure state with a finite sequence of quantum measurements.

The obvious objection is that the replacement of a universal S-matrix coding entire physics with a state dependent unitary entanglement matrix is too heavy a price to be paid for the resolution of the above mentioned paradoxes. Situation could be saved if the S-matrices have fractal structure. The quantum criticality of TGD Universe indeed implies fractality. The possibility of an infinite sequence of Jones inclusions for hyperfinite type II_1 factors isomorphic as von Neumann algebras expresses this fractal character algebraically. Thus one can hope that the S-matrix appearing as entanglement coefficients is more or less universal in the same manner as Mandelbrot fractal looks more or less the same in all length scales and for all resolutions. Whether this kind of universality must be posed as an additional condition on entanglement coefficients or is an automatic consequence of unitarity in type II_1 sense is an open question.

2.2.5 The S-matrix for p-adic-real transitions makes sense

In zero energy ontology conservation laws do not forbid p-adic-real transitions and one can develop a relatively concrete vision about what happens in these kind of transitions. The starting point is the generalization of the number concept obtained by gluing p-adic number fields and real numbers along common rationals (expressing it very roughly). At the level of the imbedding space this means that p-adic and real space-time sheets intersect only along common rational points of the imbedding space and transcendental p-adic space-time points are infinite as real numbers so that they can be said to be infinite distant points so that intentionality and cognition become cosmic phenomena.

In this framework the long range correlations characterizing p-adic fractality can be interpreted as being due to a large number of common rational points of imbedding space for real space-time sheet and p-adic space-time sheet from which it resulted in the realization of intention in quantum jump. Thus real physics would carry direct signatures about the presence of intentionality. Intentional behavior is indeed characterized by short range randomness and long range correlations.

One can even develop a general vision about how to construct the S-matrix elements characterizing the process [C2]. The basic guideline is the vision that real and various p-adic physics as well as their hybrids are continuable from the rational physics. This means that these S-matrix elements must be characterizable using data at rational points of the imbedding space shared by p-adic and real space-time sheets so that more or less same formulas describe all these S-matrix elements. Note that also $p_1 \rightarrow p_2$ p-adic transitions are possible.

3 An overall view about p-adicization of TGD

In real context the coordinatization of manifold is regarded as a trivial problem. It took long time to realize that in p-adic context the proper treatment of coordinatization problem leads to deep insights about p-adic symmetries and about the origin og the p-adic length scales hypothesis. There are several approaches to the construction of the p-adic Riemann geometry. The most simple minded approach relies on a direct generalization of the real line element and to the proposed integral for p-adically analytic functions. A more refined approach relies on the general physical consistency conditions provided by quantum TGD and by the proposed definition of the Riemann integral.

3.1 p-Adic Riemannian geometry

p-Adic Riemann geometry is a direct formal generalization of the ordinary Riemann geometry. In the minimal generalization one does not try to define concepts like arch length and volume involving definite integrals but simply defines the p-adic geometry via the metric identified as a quadratic form in the tangent space of the p-adic manifold. One could also formally calculate p-adic arc lengths, areas, etc.. since canonical identification makes it possible to define p-adic Riemann integral. Also p-adic Fourier analysis could make possible to define the integrals in question. It seems however that these concepts are not needed in the formulation of QFT limit.

Group theoretical considerations dictate the p-adic counterpart of the Riemann geometry for $M^4_+ \times CP_2$ essentially uniquely. The most natural looking manner to define the p-adic counterpart of M^4_+ and CP_2 is by using a p-adic completion for a subset of rational points in coordinates which are preferred on physical basis. In case of M^4_+ linear Minkowski coordinates are an obvious choice. Rational CP_2 could be defined as a coset space $SU(3,Q)/U(2,Q)$ associated with complex rational unitary 3×3-matrices. CP_2 could be defined as coset space of complex rational matrices by choosing one point in each coset $SU(3,Q)/U(2,Q)$ as a complex rational 3×3-matrix representable in terms of Pythagorean phases and performing a completion for the elements of this matrix by multiplying the elements with the p-adic exponentials $exp(iu)$, $|u|_p < 1$ such that one obtains p-adically unitary matrix.

3.2 p-Adic imbedding space

It has become clear that the construction of both quantum TGD and p-adic QFT limit requires p-adicization of the imbedding space geometry. Also the fact that p-adic Poincare invariance throws considerable light to the p-adic length scale hypothesis suggests that p-adic geometry is really needed. The construction of the p-adic version of the imbedding space geometry and spinor structure relies on the symmetry arguments and to the generalization of the analytic formulas of the real case almost as such by requiring that the real counterpart for the length of the infinitesimal geodesic line segment is in the lowest order same as the corresponding real length. This approach leads to a highly nontrivial generalization of the symmetry concept and p-adic Poincare invariance throws light to the p-adic length scale hypothesis. An important delicacy is related to the identification of the fundamental p-adic length scale, which corresponds to the unit element of the p-adic number field and is mapped to the unit element of the real number field in the canonical identification.

3.2.1 The identification of the fundamental p-adic length scale

The fundamental p-adic length scale correponds to the p-adic unit $e = 1$ and is mapped to the unit of the real numbers in the canonical identification. The correct physical identification of the fundamental p-adic length scale is of crucial importance since the predictions of the theory for p-adic masses depend on the choice of this scale.

In TGD the 'radius' R of CP_2 is the fundamental length scale ($2\pi R$ is by definition the length of the CP_2 geodesics). In accordance with the idea that p-adic QFT limit makes sense only above length scales larger than the radius of CP_2 R is of same order of magnitude as the p-adic length scale defined as $l = \pi/m_0$, where m_0 is the fundamental mass scale and related to the 'cosmological constant' Λ ($R_{ij} = \Lambda s_{ij}$) of CP_2 by

$$m_0^2 = 2\Lambda \ . \tag{1}$$

The relationship between R and l is uniquely fixed:

$$R^2 = \frac{3}{m_0^3} = \frac{3}{2\Lambda} = \frac{3l^2}{\pi^2} \ . \tag{2}$$

Consider now the identification of the fundamental length scale.

1. One must use R^2 or its integer multiple, rather than l^2, as the fundamental p-adic length scale squared in order to avoid the appearance of the p-adically ill defined π:s in various formulas of CP_2 geometry.

2. The identification for the fundamental length scale as $1/m_0$ leads to difficulties.

    (a) The p-adic length for the CP_2 geodesic is proportional to $\sqrt{3}/m_0$. For the physically most interesting p-adic primes satisfying $p \ mod \ 4 = 3$ so that $\sqrt{-1}$ does not exist as an ordinary p-adic number, $\sqrt{3} = i\sqrt{-3}$ belongs to the complex extension of the p-adic numbers. Hence one has troubles in getting real length for the CP_2 geodesic.

    (b) If m_0^2 is the fundamental mass squared scale then general quark states have mass squared, which is integer multiple of $1/3$ rather than integer valued as in string models.

3. These arguments suggest that the correct choice for the fundamental length scale is as $1/R$ so that $M^2 = 3/R^2$ appearing in the mass squared formulas is p-adically real and all values of the mass squared are integer multiples of $1/R^2$. This does not affect the real counterparts of the thermal expectation values of the mass squared in the lowest p-adic order but the effects, which are due to the modulo arithmetics, are seen in the higher order contributions to the mass squared. As a consequence, one must identify the p-adic length scale l as

$$l \equiv \pi R \ ,$$

    rather than $l = \pi/m_0$. This is indeed a very natural identification. What is especially nice is that this identification also leads to a solution of some longstanding problems related to the p-adic mass calculations. It would be highly desirable to have the same p-adic temperature $T_p = 1$ for both the bosons and fermions rather than $T_p = 1/2$ for bosons and $T_p = 1$ for fermions. For instance, black hole elementary particle analogy as well as the need to get rid of light boson exotics suggests this strongly. It indeed turns out possible to achieve this with the proposed identification of the fundamental mass squared scale.

3.2.2 p-Adic counterpart of M^4_+

The construction of the p-adic counterpart of M^4_+ seems a relatively straightforward task and should reduce to the construction of the p-adic counter part of the real axis with the standard metric. As already noticed, linear Minkowksi coordinates are physically and mathematically preferred coordinates and it is natural to construct the metric in these coordinates.

There are some quite interesting delicacies related to the p-adic version of the Poincare invariance. Consider first translations. In order to have imaginary unit needed in the construction of the ordinary representations of the Poincare group one must have $p \ mod \ 4 = 3$ to guarantee that $\sqrt{-1}$ does not exist as an ordinary p-adic number. It however seems that the construction of the representations is at least formally possible by replacing imaginary unit with the square root of some other p-adic number not existing as a p-adic number.

It seems that only the discrete group of translations allows representations consisting of orthogonal planewaves. p-Adic planewaves can be defined in the lattice consisting of the multiples of $x_0 = m/n$ consisting of points with p-adic norm not larger that $|x_0|_p$ and the points $p^n x_0$ define fractally scaled-down versions of this set. In canonical identification these sets corresponds to volumes scaled by factors p^{-n}.

A physically interesting question is whether the Lorentz group should contain only the elements obtained by exponentiating the Lie-algebra generators of the Lorentz group or whether also large Lorentz transformations, containing as a subgroup the group of the rational Lorentz transformations, should be allowed. If the group contains only small Lorentz transformations, the quantization volume of M^4_+ (say the points with coordinates m^k having p-adic norm not larger than one) is also invariant under Lorentz transformations. This means that the quantization of the theory in the p-adic cube $|m^k| < p^n$ is a Poincare invariant procedure unlike in the real case.

The appearance of the square root of p, rather than the naively expected p, in the expression of the p-adic length scale can be undertood if the p-adic version of M^4 metric contains p as a scaling factor:

$$\begin{aligned} ds^2 &= pR^2 m_{kl} dm^k dm^l \ , \\ R &\leftrightarrow 1 \ , \end{aligned} \tag{3}$$

where m_{kl} is the standard M^4 metric $(1, -1, -1, -1)$. The p-adic distance function is obtained by integrating the line element using p-adic integral calculus and this gives for the distance along the k:th coordinate axis the expression

$$s = R\sqrt{p} m^k \ . \tag{4}$$

The map from p-adic M^4 to real M^4 is canonical identification plus a scaling determined from the requirement that the real counterpart of an infinitesimal p-adic geodesic segment is same as the length of the corresponding real geodesic segment:

$$m^k \rightarrow \pi(m^k)_R \ . \tag{5}$$

The p-adic distance along the k:th coordinate axis from the origin to the point $m^k = (p-1)(1 + p + p^2 + ...) = -1$ on the boundary of the set of the p-adic numbers with norm not larger than one, corresponds to the fundamental p-adic length scale $L_p = \sqrt{p} l = \sqrt{p} \pi R$:

$$\sqrt{p}((p-1)(1+p+...))R \rightarrow \pi R \frac{(p-1)(1+p^{-1}+p^{-2}+...)}{\sqrt{p}} = L_p \ . \tag{6}$$

What is remarkable is that the shortest distance in the range $m^k = 1, ..m-1$ is actually $L/\sqrt{p}$ rather than l so that p-adic numbers in range span the entire R_+ at the limit $p \rightarrow \infty$. Hence p-adic topology approaches real topology in the limit $p \rightarrow \infty$ in the sense that the length of the discretization step approaches to zero.

3.2.3 CP_2 as a p-adic coset space

In case of CP_2 one can proceed by defining the p-adic counterparts of $SU(3)$ and $U(2)$ and using the identification $CP_2 = SU(3)/U(2)$. The p-adic counterpart of $SU(3)$ consists of all 3×3 unitary matrices satisfying p-adic unitarity conditions (rows/colums are mutually orthogonal unit vectors) or its suitable subgroup: the minimal subgroup corresponds to the exponentials of the Lie-algebra generators. If one allows algebraic extensions of the p-adic numbers, one obtains several extensions of the group. The extension allowing the square root of a p-adically real number is the most interesting one in this respect since the general solution of the unitarity conditions involves square roots.

The subgroup of $SU(3)$ obtained by exponentiating the Lie-algebra generators of $SU(3)$ normalized so that their nonvanishing elements have unit p-adic norm, is of the form

$$SU(3)_0 = \{x = exp(\sum_k it_k X_k) \ ; \ |t_k|_p < 1\} = \{x = 1 + iy \ ; \ |y|_p < 1\} \ . \quad (7)$$

The diagonal elements of the matrices in this group are of the form $1 + O(p)$. In order $O(p)$ these matrices reduce to unit matrices.

Rational $SU(3)$ matrices do not in general allow a representation as an exponential. In the real case all $SU(3)$ matrices can be obtained from diagonalized matrices of the form

$$h = diag\{exp(i\phi_1), exp(i\phi_2), exp(exp(-i(\phi_1 + \phi_2)\} \ . \quad (8)$$

The exponentials are well defined provided that one has $|\phi_i|_p < 1$ and in this case the diagonal elements are of form $1 + O(p)$. For $p \ mod \ 4 = 3$ one can however consider much more general diagonal matrices

$$h = diag\{z_1, z_2, z_3\} \ ,$$

for which the diagonal elements are rational complex numbers

$$z_i = \frac{(m_i + in_i)}{\sqrt{m_i^2 + n_i^2}} \ ,$$

satisfying $z_1 z_2 z_3 = 1$ such that the components of z_i are integers in the range $(0, p-1)$ and the square roots appearing in the denominators exist as ordinary p-adic numbers. These matrices indeed form a group as is easy to see. By acting with $SU(3)_0$ to each element of this group and by applying all possible automorphisms $h \rightarrow ghg^{-1}$ using rational $SU(3)$ matrices one obtains entire $SU(3)$ as a union of an infinite number of disjoint components.

The simplest (unfortunately not physical) possibility is that the 'physical' $SU(3)$ corresponds to the connected component of $SU(3)$ represented by the matrices, which are unit matrices in order $O(p)$. In this case the construction of CP_2 is relatively straightforward and the real formalism should generalize as such. In particular, for $p \ mod \ 4 = 3$ it is possible to introduce complex coordinates ξ_1, ξ_2 using the complexification for the Lie-algebra complement of $su(2) \times u(1)$. The real counterparts of these coordinates vary in the range $[0, 1)$ and the end points correspond to the values of t_i equal to $t_i = 0$ and $t_i = -p$. The p-adic sphere S^2 appearing in the definition of the p-adic light cone is obtained as a geodesic submanifold of CP_2 ($\xi_1 = \xi_2$ is one possibility). From the requirement that real CP_2 can be mapped to its p-adic counterpart it is clear that one must allow all connected components of CP_2 obtained by applying discrete unitary matrices having no exponential representation to the basic connected component. In practice this corresponds to the allowance of all possible values of the p-adic norm for the components of the complex coordinates ξ_i of CP_2.

The simplest approach to the definition of the CP_2 metric is to replace the expression of the Kähler function in the real context with its p-adic counterpart. In standard complex coordinates for which the action of $U(2)$ subgroup is linear, the expression of the Kähler function reads as

$$\begin{aligned} K &= log(1+r^2) \ , \\ r^2 &= \sum_i \bar{\xi}_i \xi_i \ . \end{aligned} \tag{9}$$

p-Adic logarithm exists provided r^2 is of order $O(p)$. This is the case when ξ_i is of order $O(p)$. The definition of the Kähler function in a more general case, when all possible values of the p-adic norm are allowed for r, is based on the introduction of a p-adic pseudo constant C to the argument of the Kähler function

$$K = log(\frac{1+r^2}{C}) \ .$$

C guarantees that the argument is of the form $\frac{1+r^2}{C} = 1 + O(p)$ allowing a well-defined p-adic logarithm. This modification of the Kähler function leaves the definition of Kähler metric, Kähler form and spinor connection invariant.

A more elegant manner to avoid the difficulty is to use the exponent $\Omega = exp(K) = 1 + r^2$ of the Kähler function instead of Kähler function, which indeed well defined for all coordinate values. In terms of Ω one can express the Kähler metric as

$$g_{k\bar{l}} = \frac{\partial_k \partial_{\bar{l}} \Omega}{\Omega} - \frac{\partial_k \Omega \partial_{\bar{l}} \Omega}{\Omega^2} \ . \tag{10}$$

The p-adic metric can be defined as

$$s_{i\bar{j}} = R^2 \partial_i \partial_{\bar{j}} K = R^2 \frac{(\delta_{i\bar{j}} r^2 - \bar{\xi}_i \xi_j)}{(1+r^2)^2} \ . \tag{11}$$

The expression for the Kähler form is the same as in the real case and the components of the Kähler form in the complex coordinates are numerically equal to those of the metric apart from the factor of i. The components in arbitrary coordinates can be deduced from these by the standard transformation formulas.

It is of considerable interest to find whether and how the concept of the geodesic line could generalize to p-adic context. This need not to be the case since the Riemannian metric could be regarded in p-adic context as a mere bilinear form defined in tangent space: this is how metric is understood also in case of Hilbert spaces. The simplest solutions of the geodesic equations do not contain any pseudo-constants and the analytic expressions for the geodesic lines are the same as in the real context. The length of a geodesic line could be defined by using p-adic integration and canonical identification. One can restrict the consideration to the geodesic submanifold S^2 with the induced metric

$$ds^2 = R^2 \frac{dz d\bar{z}}{(1+r^2)^2} \ . \quad (12)$$

Under these assumptions the length of the geodesic segment $z = x \in (0, x_{max})$ extending from the North Pole to the Equator is

$$\frac{s}{R} = \int_0^{x_{max}} \frac{dx}{1+x^2} \ , \quad (13)$$

$$\int \frac{dx}{1+x^2} = \arctan(x) = \frac{1}{2i} log(\frac{x-i}{x+i}) \ . \quad (14)$$

$arctan(x)$ is well defined for $|x|_p < 1$. At the equator one has however $|x|_p \rightarrow \infty$ and one encounters the problem of defining the integral function properly. One possibility to proceed is by decomposing the integration interval two subintervals $[0, -p)$ and $[1, -p^{-N})$, $N \rightarrow \infty$ using ordering induced by canonical identification and to use the proposed integral formula. The first interval gives automatically a well defined result equal to $arctan(-p)$. The second integral gives zero on the lower boundary also zero on the upper boundary at the limit $N \rightarrow \infty$. Hence one has

$$\frac{s}{R} = arctan(-p) \ . \quad (15)$$

The real counterpart of the geodesic line length is

$$(s_{tot})_R = (4s)_R = R(4arctan(-p))_R \leq R < 2\pi R \ . \quad (16)$$

For the full geodesic line the length is smaller by a factor of order 2π for large values of p. In particular, the length of the full geodesic is shorter than the distance from the North Pole to the Equator! This is essentially due to the typical cancellation effects taking place in the p-adic summation.

3.3 Topological condensate as a generalized manifold

The ideas about how p-adic topology emerges from quantum TGD have varied. The first belief was that p-adic topology is only an effective topology of real space-time sheets. This belief turned out to be not quite correct. p-Adic topology emerges also as a genuine topology of the space-time and p-adic regions could be identified as correlates for cognition and intentionality. The vision about quantum TGD as a generalized number theory provides possible solutions to the basic problems associated with the precise definition of topological condensate.

3.3.1 Generalization of number concept and fusion of real and p-adic physics

The unification of real physics of material work and p-adic physics of cognition and intentionality leads to the generalization of the notion of number field. Reals and various p-adic number fields are glued along their common rationals (and common algebraic numbers too) to form a fractal book like structure. Allowing all possible finite-dimensional extensions of p-adic numbers brings additional pages to this "Big Book".

This generalization leads to a generalization of the notion of manifold as a collection of a real manifold and its p-adic variants glued together along common rationals. The precise formulation involves of course several technical problems. For instance, should one glue along common algebraic numbers and Should one glue along common transcendentals such as e^p? Are algebraic extensions of p-adic number fields glued together along the algebraics too?

This notion of manifold implies a generalization of the notion of imbedding space. p-Adic transcendentals can be regarded as infinite numbers in the real sense and thus most points of the p-adic space-time sheets would be at infinite distance and real and p-adic space-time sheets would intersect in a discrete set consisting of rational points. This view in which cognition and intentionality would be literally cosmic phenomena is in a sharp contrast with the often held belief that p-adic topology emerges below Planck length scale.

It took some time to end up with this vision. The first picture was based on the notion of real and p-adic space-time sheets glued together by using canonical identification or some of its variants but led to insurmountable difficulties since p-adic topology is so different from real topology. One can of course ask whether one can speak about p-adic counterparts of notions like boundary of 3-surface or genus of 2-surface crucial for TGD based model of family replication phenomenon. It seems that these notions generalize as purely algebraically defined concepts which supports the view that p-adicization of real physics must be a purely algebraic procedure.

3.3.2 How large p-adic space-time sheets can be?

Space-time region having finite size in the real sense can have arbitrarily large size in p-adic sense and vice versa. This raises a rather thought provoking questions. Could the p-adic space-time sheets have cosmological or even infinite size with respect to the real metric but have be p-adically finite? How large space-time surface is responsible for the p-adic representation of my body? Could the large or even infinite size of the cognitive space-time sheets explain why creatures of a finite physical size can invent the notion of infinity and construct cosmological theories? Could it be that pinary cutoff $O(p^n)$ defining the resolution of a p-adic cognitive representation would define the size of the space-time region needed to realize the cognitive representation?

In fact, the mere requirement that the neighborhood of a point of the p-adic space-time sheet contains points, which are p-adically infinitesimally near to it can mean that points infinitely distant from this point in the real sense are involved. A good example is provided by an integer valued point $x = n < p$ and the point $y = x + p^m$, $m > 0$: the p-adic distance of these points is p^{-m} whereas at the limit $m \to \infty$ the real distance goes as p^m and becomes infinite for infinitesimally near points. The points $n+y$, $y = \sum_{k>0} x_k p^k$, $0 < n < p$, form a p-adically continuous set around $x = n$. In the real topology this point set is discrete set with a minimum distance $\Delta x = p$ between neighboring points whereas in the p-adic topology every point has arbitrary nearby points. There are also rationals, which are arbitrarily near to each other both p-adically and in the real sense. Consider points $x = m/n$, m and n not divisible by p, and $y = (m/n) \times (1 + p^k r)/(1 + p^k s)$, $s = r + 1$ such that neither r or s is divisible by p and $k >> 1$ and $r >> p$. The p-adic and real distances are $|x - y|_p = p^{-k}$ and $|x - y| \simeq (m/n)/(r + 1)$ respectively. By choosing k and r large enough the points can be made arbitrarily close to each other both in the real and p-adic senses.

The idea about infinite size of the p-adic cognitive space-time sheets providing representation of body and brain is consistent with TGD inspired theory of consciousness, which forces to take very seriously the idea that even human consciousness involves cosmic length scales.

3.3.3 What determines the p-adic primes assignable to a given real space-time sheet?

The p-adic realization of the Slaving Principle suggests that various levels of the topological condensate correspond to real matter like regions and p-adic mind like regions labelled by p-adic primes p. The larger the length scale, the larger the value of p and the course the induced real topology. If the most interesting values of p indeed correspond Mersenne primes, the number of most interesting levels is finite: at most 12 levels below electron length scale: actually also primes near prime powers of two seem to be physically important.

The intuitive expectation is that the p-adic prime associated with a given real space-time sheet characterizes its effective p-adic topology. As a matter fact, several p-adic effective topologies can be considered and the attractive hypothesis is that elementary particles are characterized by integers defined by the product of these p-adic primes and the integers for particles which can have direct interactions possess common prime factors.

The intuitive view is that those primes are favored for with the p-adic space-time sheet obtained by an algebraic continuation has as many rational or algebraic space-time points as possible in common with the real space-time sheet. The rationale is that if the real space-time sheet is generated in a quantum jump in which p-adic space-time sheet is transformed to a real one, it must have a large number of points in common with the real space-time sheet if the probability amplitude for this process involves a sum over the values of an n-point function of a conformal field theory over all common n-tuples and vanishes when the number of common points is smaller than n.

3.4 Infinite primes, cognition and intentionality

Somehow it is obvious that infinite primes must have some very deep role to play in quantum TGD and TGD inspired theory of consciousness. What this role precisely is has remained an enigma although I have considered several detailed interpretations, one of them above.

In the following an interpretation allowing to unify the views about fermionic Fock states as a representation of Boolean cognition and p-adic space-time sheets as correlates of cognition is discussed. Very briefly, real and p-adic partonic 3-surfaces serve as space-time correlates for the bosonic super algebra generators, and pairs of real partonic 3-surfaces and their algebraically continued p-adic variants as space-time correlates for the fermionic super generators. Intentions/actions are represented by p-adic/real bosonic partons and cognitions by pairs of real partons and their p-adic variants and the geometric form of Fermi statistics guarantees the stability of cognitions against intentional action. It must be emphasized that this interpretation is not identical with the one discussed above since it introduces different identification of the space-time correlates of infinite primes.

3.4.1 Infinite primes very briefly

Infinite primes have a decomposition to infinite and finite parts allowing an interpretation as a many-particle state of a super-symmetric arithmetic quantum field theory for which fermions and bosons are labelled by primes. There is actually an infinite hierarchy for which infinite primes of a given level define the building blocks of the infinite primes of the next level. One can map infinite primes to polynomials and these polynomials in turn could define space-time surfaces or at least light-like partonic 3-surfaces appearing as solutions of Chern-Simons action so that the classical dynamics would not pose too strong constraints.

The simplest infinite primes at the lowest level are of form $m_B X/s_F + n_B s_F$, $X = \prod_i p_i$ (product of all finite primes). The simplest interpretation is that X represents Dirac sea with all states filled and $X/s_F + s_F$ represents a state obtained by creating holes in the Dirac sea. m_B, n_B, and s_F are defined as $m_B = \prod_i p_i^{m_i}$, $n_B = \prod_i q_i^{n_i}$, and $s_F = \prod_i q_i$, m_B and n_B have no common prime factors. The integers m_B and n_B characterize the occupation numbers of bosons in modes labelled by p_i and q_i and $s_F = \prod_i q_i$ characterizes the non-vanishing occupation numbers of fermions.

The simplest infinite primes at all levels of the hierarchy have this form. The notion of infinite prime generalizes to hyper-quaternionic and even hyper-octonionic context and one can consider the possibility that the quaternionic components represent some quantum numbers at least in the sense that one can map these quantum numbers to the quaternionic primes.

The obvious question is whether configuration space degrees of freedom and configuration space spinor (Fock state) of the quantum state could somehow correspond to the bosonic and fermionic parts of the hyper-quaternionic generalization of the infinite prime. That hyper-quaternionic (or possibly hyper-octonionic) primes would define as such the quantum numbers of fermionic super generators does not make sense. It is however possible to have a map from the quantum numbers labelling super-generators to the finite primes. One must also remember that the infinite primes considered are only the simplest ones at the given level of the hierarchy and that the number of levels is infinite.

3.4.2 Precise space-time correlates of cognition and intention

The best manner to end up with the proposal about how p-adic cognitive representations relate bosonic representations of intentions and actions and to fermionic cognitive representations is through the following arguments.

1. In TGD inspired theory of consciousness Boolean cognition is assigned with fermionic states. Cognition is also assigned with p-adic space-time sheets. Hence quantum classical correspondence suggets that the decomposition of the space-time into p-adic and real space-time sheets should relate to the decomposition of the infinite prime to bosonic and fermionic parts in turn relating to the above mention decomposition of physical states to bosonic and fermionic parts.

   If infinite prime defines an association of real and p-adic space-time sheets this association could serve as a space-time correlate for the Fock state defined by configuration space spinor for given 3-surface. Also spinor field as a map from real partonic 3-surface would have as a space-time correlate a cognitive representation mapping real partonic 3-surfaces to p-adic 3-surfaces obtained by algebraic continuation.

2. Consider first the concrete interpretation of integers m_B and n_B. The most natural guess is that the primes dividing $m_B = \prod_i p^{m_i}$ characterize the effective p-adicities possible for the real 3-surface. m_i could define the numbers of disjoint partonic 3-surfaces with effective p_i-adic topology and associated with with the same real space-time sheet. These boundary conditions would force the corresponding real 4-surface to have all these effective p-adicities implying multi-p-adic fractality so that particle and wave pictures about multi-p-adic fractality would be mutually consistent. It seems natural to assume that also the integer n_i appearing in $m_B = \prod_i q_i^{n_i}$ code for the number of real partonic 3-surfaces with effective q_i-adic topology.

3. Fermionic statistics allows only single genuinely q_i-adic 3-surface possibly forming a pair with its real counterpart from which it is obtained by algebraic continuation. Pairing would conform with the fact that n_F appears both in the finite and infinite parts of the infinite prime (something absolutely essential concerning the consistency of interpretation!).

   The interpretation could be as follows.

   (a) Cognitive representations must be stable against intentional action and fermionic statistics guarantees this. At space-time level this means that fermionic generators correspond to pairs of real effectively q_i-adic 3-surface and its algebraically continued q_i-adic counterpart. The quantum jump in which q_i-adic 3-surface is transformed to a real 3-surface is impossible since one would obtain two identical real 3-surfaces lying on top of each other, something very singular and not allowed by geometric exclusion principle for surfaces. The pairs of boson and fermion surfaces would thus form cognitive representations stable against intentional action.

(b) Physical states are created by products of super algebra generators Bosonic generators can have both real or p-adic partonic 3-surfaces as space-time correlates depending on whether they correspond to intention or action. More precisely, m_B and n_B code for collections of real and p-adic partonic 3-surfaces. What remains to be interpreted is why m_B and n_B cannot have common prime factors (this is possible if one allows also infinite integers obtained as products of finite integer and infinite primes).

(c) Fermionic generators to the pairs of a real partonic 3-surface and its p-adic counterpart obtained by algebraic continuation and the pictorial interpretation is as fermion hole pair.

(d) This picture makes sense if the partonic 3-surfaces containing a state created by a product of super algebra generators are unstable against decay to this kind of 3-surfaces so that one could regard partonic 3-surfaces as a space-time representations for a configuration space spinor field.

4. Are alternative interpretations possible? For instance, could $q = m_B/m_B$ code for the effective q-adic topology assignable to the space-time sheet. That q-adic numbers form a ring but not a number field casts however doubts on this interpretation as does also the general physical picture.

3.4.3 Number theoretical universality of S-matrix

The discreteness of the intersection of the real space-time sheet and its p-adic variant obtained by algebraic continuation would be a completely universal phenomenon associated with all fermionic states. This suggests that also real-to-real S-matrix elements involve instead of an integral a sum with the arguments of an n-point function running over all possible combinations of the points in the intersection. S-matrix elements would have a universal form which does not depend on the number field at all and the algebraic continuation of the real S-matrix to its p-adic counterpart would trivialize. Note that also fermionic statistics favors strongly discretization unless one allows Dirac delta functions.

3.5 p-Adicization of second quantized induced spinor fields

Induction procedure makes it possible to geometrize the concept of a classical gauge field and also of the spinor field with internal quantum numbers. In the case of the electro-weak gauge fields induction means the projection of the H-spinor connection to a spinor connection on the space-time surface.

In the most recent formulation induced spinor fields appear only at the 3-dimensional light-like partonic 3-surfaces and the solutions of the modified Dirac equation can be written explicitly [C1, C2] as simple algebraic functions involving powers of the preferred coordinate variables very much like various operators in conformal field theory can be expressed as Laurent series in powers of a complex variable z with operator valued coefficients. This means that the continuation of the second quantized induced spinor fields to various p-adic number fields is a straightforward procedure. The second quantization of these induced spinor fields as free fields is needed to construct configuration space geometry and anti-commutation relation between spinor fields are fixed from the requirement that configuration space gamma matrices correspond to super-canonical generators.

The idea about rational physics as the intersection of the physics associated with various number fields inspires the hypothesis that induced spinor fields have only modes labelled by rational valued quantum numbers. Quaternion conformal invariance indeed implies that zero modes are characterized by integers. This means that same oscillator operators can define oscillator operators are universal. Powers of the quaternionic coordinate are indeed well-define in any number field provided the components of quaternion are rational numbers since p-adic quaternions have in this case always inverse.

3.6 Should one p-adicize at the level of configuration space?

If Duistermaat-Heckman theorem [20] holds true in TGD context, one could express configuration space functional integral in terms of exactly calculable Gaussian integrals around the maxima of the Kähler function defining what might be called reduced configuration space CH_{red}. The huge super-conformal symmetries raise the hope that the rest of S-matrix elements could be deduced using group theoretical considerations so that everything would become algebraic. If this optimistic scenario is realized, the p-adicization of CH_{red} might be enough to p-adicize all operations needed to construct the p-adic variant of S-matrix.

The optimal situation would be that S-matrix elements reduce to algebraic numbers for rational valued incoming momenta and that p-adicization trivializes in the sense that it corresponds only to different interpretations for the imbedding space coordinates (interpretation as real or p-adic numbers) appearing in the equations defining the 4-surfaces. For instance, space-time coordinates would correspond to preferred imbedding space coordinates and the remaining imbedding space coordinates could be rational functions of the latter with algebraic coefficients. Algebraic points in a given extension of rationals would thus be common to real and p-adic surfaces. It could also happen that there are no or very few common algebraic points. For instance, Fermat's theorem says that the surface $x^n + y^n = z^n$ has no rational points for $n > 2$.

This picture is probably too simple. The intuitive expectation is that ordinary S-matrix elements are proportional to a factor which in the real case involves an integration over the arguments of an n-point function of a conformal field theory defined at a partonic 2-surface. For p-adic-real transitions the integration should reduce to a sum over the common rational or algebraic points of the p-adic and real surface. Same applies to $p_1 \rightarrow p_2$ type transitions.

If this picture is correct, the p-adicization of the configuration space would mean p-adicization of CH_{red} consisting of the maxima of the Kähler function with respect to both fiber degrees of freedom and zero modes acting effectively as control parameters of the quantum dynamics. If CH_{red} is a discrete subset of CH ultrametric topology induced from finite-p p-adic norm is indeed natural for it. 'Discrete set in CH' need not mean a discrete set in the usual sense and the reduced configuration space could be even finite-dimensional continuum. Finite-p p-adicization as a cognitive model would suggest that p-adicization in given point of CH_{red} is possible for all p-adic primes associated with the corresponding space-time surface (maximum of Kähler function) and represents a particular cognitive representation about CH_{red}.

A basic technical problem is, whether the integral defining the Kähler action appearing in the exponent of Kähler function exists p-adically. Here the hypothesis that the exponent of the Kähler function is identifiable as a Dirac determinant of the modified Dirac operator defined at the light-like partonic 3-surfaces [B4] suggests a solution to the problem. By restricting the generalized eigen values of the modified Dirac operator to an appropriate algebraic extension of rationals one could obtain an algebraic number existing both in the real and p-adic sense if the number of the contributing eigenvalues is finite. The resulting hierarchy of algebraic extensions of R_p would have interpretation as a cognitive hierarchy. If the maxima of Kähler function assignable to the functional integral are such that the number of eigenvalues in a given algebraic extension is finite this hypothesis works.

If Duistermaat-Heckman theorem generalizes, the p-adicization of the entire configuration space would be un-necessary and it certainly does not look a good idea in the light of preceding considerations.

1. For a generic 3-surface the number of the eigenvalues in a given algebraic extension of rationals

need not be finite so that their product can fail to be an algebraic number.

2. The algebraic continuation of the exponent of the Kähler function from CH_{red} to the entire CH would be analogous to a continuation of a rational valued function from a discrete set to a real or p-adic valued function in a continuous set. It is difficult to see how the continuation could be unique in the p-adic case.

4 p-Adic probabilities

p-Adic Super Virasoro representations necessitate p-adic QM based on the p-adic unitarity and p-adic probability concepts. The concept of a p-adic probability indeed makes sense as shown by [16]. p-Adic probabilities can be defined as relative frequencies N_i/N in a long series consisting of total number N of observations and N_i outcomes of type i. Probability conservation corresponds to

$$\sum_i N_i = N \ , \tag{17}$$

and the only difference as compared to the usual probability is that the frequencies are interpreted as p-adic numbers.

The interpretation as p-adic numbers means that the relative frequencies converge to probabilities in a p-adic rather than real sense in the limit of a large number N of observations. If one requires that probabilities are limiting values of the frequency ratios in p-adic sense one must pose restrictions on the possible numbers of the observations N if N is larger than p. For N smaller than p, the situation is similar to the real case. This means that for $p = M_{127} \simeq 10^{38}$, appropriate for the particle physics experiments, p-adic probability differs in no observable manner from the ordinary probability.

If the number of observations is larger than p, the situation changes. If N_1 and N_2 are two numbers of observations they are near to each other in the p-adic sense if they differ by a large power of p. A possible interpretation of this restriction is that the observer at the p:th level of the condensate cannot choose the number of the observations freely. The restrictions to this freedom come from the requirement that the sensible statistical questions in a p-adically conformally invariant world must respect p-adic conformal invariance.

4.1 p-Adic probabilities and p-adic fractals

p-Adic probalities are natural in the statistical description of the fractal structures, which can contain same structural detail with all possible sizes.

1. The concept of a structural detail in a fractal seems to be reasonably well defined concept. The structural detail is clearly fixed by its topology and p-adic conformal invariants associated with it. Clearly, a finite resolution defined by some power of p of the p-adic cutoff scale must be present in the definition. For example, p-adic angles are conformal invariants in the p-adic case, too. The overall size of the detail doesn't matter. Let us therefore assume that it is possible to make a list, possibly infinite, of the structural details appearing in the p-adic fractal.

2. What kind of questions related to the structural details of the p-adic fractal one can ask? The first thing one can ask is how many times i:th structural detail appears in a finite region of the fractal structure: although this number is infinite as a real number it might possess (and probably does so!) finite norm as a p-adic number and provides a useful p-adic invariant

of the fractal. If a complete list about the structural details of the fractal is at use one can calculate also the total number of structural details defined as $N = \sum_i N_i$. This means that one can also define p-adic probability for the appearance of i:th structural detail as a relative frequency $p_i = N_i/N$.

3. One can consider conditional probabilities, too. It is natural to ask what is the probability for the occurrence of the structural detail subject to the condition that part of the structural detail is fixed (apart from the p-adic conformal transformations). In order to evaluate these probalities as relative frequencies one needs to look only for those structural details containing the substructure in question.

4. The evaluation of the p-adic probalities of occurrence can be done by evaluating the required numbers N_i and N in a given resolution. A better estimate is obtained by increasing the resolution and counting the numbers of the hitherto unobserved structural details. The increase in the resolution greatly increases the number of the observations in case of p-adic fractal and the fluctuations in the values of N_i and N increase with the resolution so that N_i/N has no well defined limit as a real number although one can define the probabilities of occurrence as a resolution dependent concept. In the p-adic sense the increase in the values of N_i and fluctuations are small and the procedure should converge rapidly so that reliable estimates should result with quite a reasonable resolution. Notice that the increase of the fluctuations in the real sense, when resolution is increased is in accordance with the criticality of the system.

5. p-Adic frequencies and probabilities define via the canonical correspondence real valued invariants of the fractal structure.

It must be emphasized that this picture can have practical applications only for small values of p, which could also be important in the macroscopic length scales. In elementary particle physics L_p is of the order of the Compton length associated with the particle and already in the first step CP_2 length scale is achieved and it is questionable whether it makes sense to continue the procedure below the length scale l. In particle physics context the renormalization is related to the the change of the reduction of the p-adic length scale L_p in the length scale hierarchy rather than p-adic fractality for a fixed value of p.

The most important application of the p-adic probability in this book is the description of the particle massivation based on p-adic thermodynamics. Instead of energy, Virasoro generator l is thermalized and in the low temperature phase temperature is quantized in the sense that the counterpart of the Boltzmann weight $exp(H/T)$ is $p^{L_0/T}$, where $T = 1/n$ from the requirement that Boltzmann weight exists (L_0 has integer spectrum). The surprising success of the mass calculations shows that p-adic probability theory is much more than a formal possibility.

4.2 Relationship between p-adic and real probabilities

There are uniqueness problems related to the mapping of p-adic probabilities to real ones. These problems find a nice resolution from the requirement that the map respects probability conservation. The implied modification of the original mapping does not change measurably the predictions for the masses of light particles.

4.2.1 How unique the map of p-adic probabilities and mass squared values are mapped to real numbers is?

The mapping of p-adic thermodynamical probabilities and mass squared values to real numbers is not completely unique.

1. Canonical identification $I : \sum x_n p^n \rightarrow \sum x_n p^{-n}$ takes care of this mapping but does not respect the sum of probabilities so that the real images $I(p_n)$ of the probabilities must be normalized. This is a somewhat alarming feature.

2. The modification of the canonical identification mapping rationals by the formula $I(r/s) = I(r)/I(s)$ has appeared naturally in various applications, in particular because it respects unitarity of unitary matrices with rational elements with $r < p, s < p$. In the case of p-adic thermodynamic the formula $I(g(n)p^n/Z) \rightarrow I(g(n)p^n)/I(Z)$ would be very natural although Z need not be rational anymore. For $g(n) < p$ the real counterparts of the p-adic probabilities would sum up to one automatically for this option. One cannot deny that this option is more convincing than the original one. The generalization of this formula to map p-adic mass squared to a real one is obvious.

3. Options 1) and 2) differ dramatically when the $n = 0$ massless ground state has ground state degeneracy $D > 1$. For option 1) the real mass is predicted to be of order CP_2 mass whereas for option 2) it would be by a factor $1/D$ smaller than the minimum mass predicted by the option 1). Thus option 2) would predict a large number of additional exotic states. For those states which are light for option 1), the two options make identical predictions as far as the significant two lowest order terms are considered. Hence this interpretation would not change the predictions of the p-adic mass calculations in this respect. Option 2) is definitely more in accord with the real physics based intuitions and the main role of p-adic thermodynamics would be to guarantee the quantization of the temperature and fix practically uniquely the spectrum of the "Hamiltonian".

4.2.2 Under what conditions the mapping of p-adic ensemble probabilities to real probabilities respects probability conservation?

One can consider also a more general situation. Assume that one has an ensemble consisting of independent elementary events such that the number of events of type i is N_i. The probabilities are given by $p_i = N_i/N$ and $N = \sum N_i$ is the total number of elementary events. Even in the case that N is infinite as a real number it is natural to map the p-adic probabilities to their real counterparts using the rational canonical identification $I(p_i) = I(N_i)/I(N)$. Of course, N_i and N exist as well defined p-adic numbers under very stringent conditions only.

The question is under what conditions this map respects probability conservation. The answer becomes obvious by looking at the pinary expansions of N_i and N. If the integers N_i (possibly infinite as real integers) have pinary expansions having no common pinary digits, the sum of probabilities is conserved in the map. Note that this condition can assign also to a finite ensemble with finite number of a unique value of p.

This means that the selection of a basis for independent events corresponds to a decomposition of the set of integers labelling pinary digits to disjoint sets and brings in mind the selection of orthonormalized basis of quantum states in quantum theory. What is physically highly non-trivial that this "orthogonalization" alone puts strong constraints on probabilities of the allowed elementary events. One can say that the probabilities define distributions of pinary digits analogous to non-negative probability amplitudes in the space of integers labelling pinary digits, and the probabilities of independent events must be orthogonal with respect to the inner product defined by point-wise multiplication in the space of pinary digits.

p-Adic thermodynamics for which Boltzman weights $g(E)exp(-E/T)$ are replaced by $g(E)p^{E/T}$ such that one has $g(E) < p$ and E/T is integer valued, satisfies this constraint. The quantization of E/T to integer values implies quantization of both T and "energy" spectrum and forces so called super conformal invariance in TGD applications, which is indeed a basic symmetry of the theory.

There are infinitely many ways to choose the elementary events and each choice corresponds to a decomposition of the infinite set of integers n labelling the powers of p to disjoint subsets.

These subsets can be also infinite. One can assign to this kind of decomposition a resolution which is the poorer the larger the subsets involved are. p-Adic thermodynamics would represent the situation in which the resolution is maximal since each set contains only single pinary digit. Note the analogy with the basis of completely localized wave functions in a lattice.

4.2.3 How to map p-adic transition probabilities to real ones?

p-Adic variants of TGD, if they exist, give rise to S-matrices and transition probabilities P_{ij}, which are p-adic numbers.

1. The p-adic probabilities defined by rows of S-matrix mapped to real numbers using canonical identification respecting the $q = r/s$ decomposition of rational number or its appropriate generalization should define real probabilities.

2. The simplest example would simple renormalization for the real counterparts of the p-adic probabilities $(P_{ij})_R$ obtained by canonical identification (or more probably its appropriate modification).

$$\begin{aligned}
P_{ij} &= \sum_{k\geq 0} P_{ij}^k p^k \ , \\
P_{ij} &\rightarrow \sum_{k\geq 0} P_{ij}^k p^{-k} \equiv (P_{ij})_R \ , \\
(P_{ij})_R &\rightarrow \frac{(P_{ij})_R}{\sum_j (P_{ij})_R} \equiv P_{ij}^R \ .
\end{aligned} \tag{18}$$

   The procedure converges rapidly in powers of p and resembles renormalization procedure of quantum field theories. The procedure automatically divides away one four-momentum delta function from the square of S-matrix element containing the square of delta function with no well defined mathematical meaning. Usually one gets rid of the delta function interpreting it as the inverse of the four-dimensional measurement volume so that transition rate instead of transition probability is obtained. Of course, also now same procedure should work either as a discrete or a continuous version.

3. Probability interpretation would suggest that the real counterparts of p-adic probabilities sum up to unity. This condition is rather strong since it would hold separately for each row and column of the S-matrix.

4. A further condition would be that the real counterparts of the p-adic probabilities for a given prime p are identical with the transition probabilities defined by the real S-matrix for real space-time sheets with effective p-adic topology characterized by p. This condition might allow to deduce all relevant phase information about real and corresponding p-adic S-matrices using as an input only the observable transition probabilities.

4.2.4 What it means that p-adically independent events are not independent in real sense?

A further condition would be that p-adic quantum transitions represent also in the real sense independent elementary events so that the real counterpart for a sum of the p-adic probabilities for a finite number of transitions equals to the sum of corresponding real probabilities. This condition is definitely too strong since only a single transition could correspond to a given p-adic norm of transition probability P_{ij} with i fixed.

The crucial question concerns the physical difference between the real counterpart for the sum of the p-adic transition probabilities and for the sum of the real counterparts of these probabilities, which are in general different:

$$(\sum_j P_{ij})_R \neq \sum_j (P_{ij})_R \quad . \tag{19}$$

The suggestion is that p-adic sum of the transition probabilities corresponds to the experimental situation, when one does not monitor individual transitions but using some common experimental signature only looks whether the transition leads to this set of the final states or not. When one looks each transition separately or effectively performs different experiment by considering only one transition channel in each experiment one must use the sum of the real probabilities. More precisely, the choice of the experimental signatures divides the set U of the final states to a disjoint union $U = \cup_i U_i$ and one must define the real counterparts for the transition probabilities P_{iU_k} as

$$\begin{aligned} P_{iU_k} &= \sum_{j \in U_k} P_{ij} \quad , \\ P_{iU_k} &\rightarrow (P_{iU_k})_R \quad , \\ (P_{iU_k})_R &\rightarrow \frac{(P_{iU_k})_R}{\sum_l (P_{iU_l})_R} \equiv P^R_{iU_k} \quad . \end{aligned} \tag{20}$$

The assumption means deep a departure from the ordinary probability theory. If p-adic physics is the physics of cognitive systems, there need not be anything mysterious in the dependence of the behavior of system on how it is monitored. At least half-jokingly one might argue that the behavior of an intelligent system indeed depends strongly on whether the boss is nearby or not. The precise definition for the monitoring could be based on the decomposition of the density matrix representing the entangled subsystem into a direct sum over the subspaces associated with the degenerate eigenvalues of the density matrix. This decomposition provides a natural definition for the notions of the monitoring and resolution.

The renormalization procedure is in fact familiar from standard physics. Assume that the labels j correspond to momenta. The division of momentum space to cells of a given size so that the individual momenta inside cells are not monitored separately means that momentum resolution is finite. Therefore one must perform p-adic summation over the cells and define the real probabilities in the proposed manner. p-Adic effects resulting from the difference between p-adic and real summations could be the counterpart of the renormalization effects in QFT. It should be added that similar resolution can be defined also for the initial states by decomposing them into a union of disjoint subsets.

4.3 p-Adic thermodynamics

The p-adic field theory limit as such is not expected to give a realistic theory at elementary particle physics level. The point is that particles are expected to be either massless or possess mass of order 10^{-4} Planck mass. The p-adic description of particle massivation described in the third part of the book shows that p-adic thermodynamics provides the proper formulation of the problem. What is thermalized is Virasoro generator L_0 (mass squared contribution is not included to L_0 so that states do not have fixed conformal weight). Temperature is quantized purely number theoretically in low temperature limit ($exp(H/kT) \rightarrow p^{L_0/T}$, $T = 1/n$): in fact, partition function does not even exist in high temperature phase. The extremely small mixing of massless states with Planck mass states implies massivation and predictions of the p-adic thermodynamics for the fermionic masses are in excellent agreement with experimental masses. Thermodynamic approach also explains the emergence of the length scale L_p for a given p-adic condensation level and one can develop arguments explaining why primes near prime powers of two are favored.

It should be noticed that rational p-adic temperatures $1/T = k/n$ are possible, if one poses the restriction that thermal probabilities are non-vanishing only for some subalgebra of the Super Virasoro algebra isomorphic to the Super Virasoro algebra itself. The generators L_{kn},G_{kn}, where k is a positive integer, indeed span this kind of a subalgebra by the fractality of the Super Virasoro algebra and $p^{L_0/T}$ is integer valued with this restriction.

One might apply thermodynamics approach should also in the calculation of S-matrix. What is is needed is thermodynamical expectation value for the transition amplitudes squared over incoming and outgoing states. In this expectation value 3-momenta are fixed and only mass squared varies.

4.4 Generalization of the notion of information

TGD inspired theory of consciousness, in particular the formulation of Negentropy Maximization Principle (NMP) in p-adic context, has forced to rethink the notion of the information concept. In TGD state preparation process is realized as a sequence of self measurements. Each self measurement means a decomposition of the sub-system involved to two unentangled parts. The decomposition is fixed highly uniquely from the requirement that the reduction of the entanglement entropy is maximal.

The additional assumption is that bound state entanglement is stable against self measurement. This assumption is somewhat ad hoc and it would be nice to get rid of it. The only manner to achieve this seems to be a generalized definition of entanglement entropy allowing to assign a negative value of entanglement entropy to the bound state entanglement, so that bound state entanglement would actually carry information, in fact conscious information (experience of understanding). This would be very natural since macro-temporal quantum coherence corresponds to a generation of bound state entanglement, and is indeed crucial for ability to have long lasting non-entropic mental images.

The generalization of the notion of number concept leads immediately to the basic problem. How to generalize the notion of entanglement entropy that it makes sense for a genuinely p-adic entanglement? What about the number-theoretically universal entanglement with entanglement probabilities, which correspond to finite extension of rational numbers? One can also ask whether the generalized notion of information could make sense at the level of the space-time as suggested by quantum-classical correspondence.

In the real context Shannon entropy is defined for an ensemble with probabilities p_n as

$$S = -\sum_n p_n log(p_n) \ . \tag{21}$$

As far as theory of consciousness is considered, the basic problem is that Shannon entropy is always non-negative so that as such it does not define a genuine information measure. One could define information as a change of Shannon entropy and this definition is indeed attractive in the sense that quantum jump is the basic element of conscious experience and involves a change. One can however argue that the mere ability to transfer entropy to environment (say by aggressive behavior) is not all that is involved with conscious information, and even less so with the experience of understanding or moment of heureka. One should somehow generalize the Shannon entropy without losing the fundamental additivity property.

4.4.1 p-Adic entropies

The key observation is that in the p-adic context the logarithm function $log(x)$ appearing in the Shannon entropy is not defined if the argument of logarithm has p-adic norm different from 1. Situation changes if one uses an extension of p-adic numbers containing $log(p)$: the conjecture is that this extension is finite-dimensional. One might however argue that Shannon entropy should be well defined even without the extension.

p-Adic thermodynamics inspires a manner to achieve this. One can replace $log(x)$ with the logarithm $log_p(|x|_p)$ of the p-adic norm of x, where log_p denotes p-based logarithm. This logarithm is integer valued ($log_p(p^n) = n$), and is interpreted as a p-adic integer. The resulting p-adic entropy

$$\begin{aligned} S_p &= \sum_n p_n k(p_n) \ , \\ k(p_n) &= -log_p(|p_n|) \ . \end{aligned} \tag{22}$$

is additive: that is the entropy for two non-interacting systems is the sum of the entropies of composites. Note that this definition differs from Shannon's entropy by the factor $log(p)$. This entropy vanishes identically in the case that the p-adic norms of the probabilities are equal to one. This means that it is possible to have non-entropic entanglement for this entropy.

One can consider a modification of S_p using p-adic logarithm if the extension of the p-adic numbers contains $log(p)$. In this case the entropy is formally identical with the Shannon entropy:

$$S_p = -\sum_n p_n log(p_n) = -\sum_n p_n \left[-k(p_n)log(p) + p^{k_n} log(p_n/p^{k_n}\right] \ . \tag{23}$$

It seems that this entropy cannot vanish.

One must map the p-adic value entropy to a real number and here canonical identification can be used:

$$\begin{aligned} S_{p,R} &= (S_p)_R \times log(p)) \ , \\ (\sum_n x_n p^n)_R &= \sum_n x_n p^{-n} \ . \end{aligned} \tag{24}$$

The real counterpart of the p-adic entropy is non-negative.

4.4.2 Number theoretic entropies and bound states

In the case that the probabilities are rational or belong to a finite-dimensional extension of rationals, it is possible to regard them as real numbers or p-adic numbers in some extension of p-adic numbers for any p. The visions that rationals and their finite extensions correspond to islands of order in the seas of chaos of real and p-adic transcendentals suggests that states having entanglement coefficients in finite-dimensional extensions of rational numbers are somehow very special. This is indeed the case. The p-adic entropy entropy $S_p = -\sum_n p_n log_p(|p_n|)log(p)$ can be interpreted in this case as an ordinary rational number in an extension containing $log(p)$.

What makes this entropy so interesting is that it can have also negative values in which case the interpretation as an information measure is natural. In the real context one can fix the value of the value of the prime p by requiring that S_p is maximally negative, so that the information content of the ensemble could be defined as

$$I \equiv Max\{-S_p, \ p \ \text{prime}\} \ . \tag{25}$$

This information measure is positive when the entanglement probabilities belong to a finite-dimensional extension of rational numbers. Thus kind of entanglement is stable against NMP, and has a natural interpretation as bound state entanglement. The prediction would be that the bound states of real systems form a number theoretical hierarchy according to the prime p and and dimension of algebraic extension characterizing the entanglement.

Number theoretically state function reduction and state preparation could be seen as information generating processes projecting the physical states from either real or p-adic sectors of the state space to their intersection. Later an argument that these processes have a purely number theoretical interpretation will be developed based on the generalized notion of unitarity allowing the U-matrix to have matrix elements between the sectors of the state space corresponding to different number fields.

4.4.3 Number theoretic information measures at the space-time level

Quantum classical correspondence suggests that the notion of entropy should have also space-time counterpart. Entropy requires ensemble and both the p-adic non-determinism and the non-determinism of Kähler action allow to define the required ensemble as the ensemble of strictly deterministic regions of the space-time sheet. One can measure various observables at these space-time regions, and the frequencies for the outcomes are rational numbers of form $p_k = n(k)/N$, where N is the number of strictly deterministic regions of the space-time sheet. The number theoretic entropies are well defined and negative if p divides the integer N. Maximum is expected to result for the largest prime power factor of N. This would mean the possibility to assign a unique prime to a given real space-time sheet.

The classical non-determinism resembles p-adic non-determinism in the sense that the space-time sheet obeys effective p-adic topology in some length and time scale range is consistent with this idea since p-adic fractality suggests that N is power of p.

5 p-Adic Quantum Mechanics

An interesting question is whether p-adic quantum mechanics might exist in some sense. The purely formal generalizations of the ordinary QM need not be very interesting physically and the following considerations describe p-adic QM as a limiting case of the p-adic field theory limit of TGD to be constructed later. This particular p-adic QM is based on the p-adic Hilbert-space, p-adic unitarity and p-adic probability concepts whereas the physical interpretation is based on the correspondence between the p-adic and real probabilities given by the canonical correspondence. p-Adic QM is expected to apply below the p-adic length scale $L_p = \sqrt{p}l$ and above L_p ordinary QM should work, when length scale resolution L_p is used. Although one can define p-adic Schrödinger equation formally without any difficulty it is not at all obvious whether, or even too plausible, that it emerges from the p-adic QFT limit of TGD.

5.1 p-Adic modifications of ordinary Quantum Mechanics

One can consider several modifications of the ordinary quantum mechanics depending on what kind of p-adicizations one is willing to make.

5.1.1 p-Adicization in dynamical degrees of freedom

The minimal alternative is to replace time- and spatial coordinates with their p-adic counterparts so that the space time is a Cartesian power of R_p. A more radical possibility is to replace the 3-space with a 3-dimensional algebraic extension of the p-adic numbers. This means that space time is replaced with a Cartesian product of R_p and its 3-dimensional extension. The most radical possibility, suggested by the relativistic considerations, is a four-dimensional algebraic extension treating space and time degrees of freedom in an equal position: this alternative is encountered in the formulation of the p-adic field theory limit of TGD.

In practice the formulation of the quantum theory involves an action principle defining the so called classical theory and this is defined by using the integral of the the action density. These integrals certainly exists as real quantities and are defined by the Haar measure for the p-adic numbers. Algebraic continuation of real integrals seems to be the only reasonable manner to defined these integrals.

5.1.2 p-Adicization at Hilbert space level

One can imagine essentially two different manners to p-adicize Hilbert space.

1. The first approach, followed in [17], is to keep Schrö-dinger amplitudes complex. In this case it is better to consider a Cartesian power of R_p instead of an algebraic extension as a coordinate space. The canonical identification allows to replace the expressions of the coordinate and momentum operators via their p-adic counterparts. For example, $x \times \Psi$ is replaced with $x \times_p \Psi$, where p-adic multiplication rule is used. Derivative corresponds to a p-adic derivative. It was the lack of the canonical identification replacement, which forced to give up the straightforward generalization of standard QM in the approach followed in [17, 18]. What this approach effects, is the replacement of the ordinary continuity and differentiability and concepts with the p-adic differentiability and the approach looks rather reasonable manner to construct a fractal quantum mechanics. This approach however is not applicable in the present context.

2. A more radical approach uses Schrödinger amplitude with values in some complex extension, say a square root allowing extension of the p-adic numbers. p-Adic inner product implies that the ordinary unitarity and probability concepts are replaced with there p-adic counterparts. This approach looks natural for various reasons. The representation theory for the Lie-groups generalizes to p-adic case and the replacement implies certain mathematical elegance since p-analyticity and the realization of the p-adic conformal invariance becomes possible. It will be found that p-adic valued inner product is the natural inner product for the quantized harmonic oscillator and for Super Virasoro representations. The concept of the p-adic probability makes sense as first shown by [16]. The physical interpretation of the theory is however always in terms of the real numbers and the canonical identification provides the needed tool to map the predictions of the theory to real numbers. That physical observables are always real numbers is suggested by the success of the p-adic mass calculations. p-Adic probabilities can be mapped to real probabilities and in the last chapter of the third part of the book it is shown that this correspondence predicts genuinely novel physical effects.

The p-adic representations of the Super Virasoro algebra to be used are defined in the p-adic Hilbert space and everything is well defined at algebraic level if 4- ($p > 2$) or 8- ($p = 2$) dimensional algebraic extension allowing square roots is used. Unitarity concept generalizes in a straightforward manner to the p-adic context and the elements of the S-matrix should have values in the same extension of the p-adic numbers. The requirement that the squares of S-matrix elements are p-adically real numbers gives strong constraints on the S-matrix elements since the quantities $S(m,n)\bar{S}(m,n)$ in general belong to the 4- (2-) dimensional real subspace $x + \theta y + \sqrt{p}z + \sqrt{p}\theta u$ of the 8- (4-) dimensional extension and p-adic reality implies the conditions:$y = z = .. = u = 0$. Reality conditions can be solved always since the solution involves only square roots of rational functions. What is exciting is that space time and imbedding space dimensions for the extension allowing square roots are forced by the quantum mechanical probability concept, by p-adic group theory and by the p-adic Riemannian Geometry.

The existence of the p-adic valued definite integral is crucial concerning the practical construction of the p-adic Quantum Mechanics.

1. In the ordinary wave mechanics the inner product involves an integration over the configuration space degrees of freedom. This inner product can be generalized to the p-adic integral of $\bar{\Psi}_1\Phi_2$ over the 3-space using p-adic valued integration defined in the first chapter, which works for all analytic functions and also for p-adic counterparts of the plane waves (nonanalytic functions).

2. The perturbative formulation QM in terms of the time development operator

$$U(t) = P(exp(i\int exp(\int dt\ V)) \ , \tag{26}$$

generalizes to the p-adic context. In particular, the concept of the time ordered product $P(...)$ appearing in the definition of the time development operator generalizes since the canonical identification induces ordering for the values of the p-adic time coordinate: $t_1 < t_2$ if $(t_1)_R < (t_2)_R$ holds true. Non-trivialities are related to the p-adic existence of the time development operator: for sufficiently larger values of the time coordinate, the exponent appearing in the time development operator does not exist p-adically and this implies infrared cutoff time and length scale in the p-adic QM.

One can define the action of the time development operator for longer time intervals only if one makes some restrictions on the physical states appearing in the matrix elements. This could explain color confinement number theoretically. For sufficiently long time intervals the color interaction part of the interaction Hamiltonian is so large for colored states that p-adic time development operator fails to exist number theoretically and one must restrict the physical states to be color singlets.

The generalization of the p-adic formula for Riemann integral [E4] suggests an exact formula for the time ordered product. The first guess is that one simply forms the product

$$\begin{aligned} Pexp(i\int_0^t Hdt) &\equiv P\prod_n exp\left[iV(t(n))\Delta t(n)\right] \ , \\ \Delta t(n) &= t_+(n) - t_-(n) = (1+p)p^{m(n)} \ , \end{aligned} \tag{27}$$

to obtain the value of the time ordered product for time values t having finite number of pinary digits. The product is over all points $t(n)$ having finite number of pinary digits and $m(n)$ is the highest pinary digit in the expansion of $t(n)$ and $t_\pm(n)$ denote the two p-adic images of the real coordinate $t(n)_R$ under canonical identification. $\Delta t(n)$ corresponds to the difference of the p-adic time coordinates, which are mapped to the same value of the real time coordinate in canonical identification so that one can regard the time ordered product as a limiting case in which real time coordinate differences are exactly zero in the time ordered product.

The time ordering of the product is induced by canonical identification from real time ordering. This time development operator is defined for time values with finite number of pinary digits only and defines p-adic pseudoconstant. The hope is that the inherent non-determinism of the p-adic differential equations, implied by the existence of the p-adic pseudo constants, makes it possible to continue this function to a p-adically differentiable function of the p-adic time coordinate satisfying the counterpart of the Schrödinger equation for the time development operator.

Not surprisingly, number theoretical problems are encountered also now: the exponential $exp\,[iV(t(n))\Delta t(n)]$ need not exist p-adically. The possibility of p-adic pseudo constants suggests that one could simply drop off the troublesome exponentials: this has far reaching physical consequences [F5].

5.2 p-Adic inner product and Hilbert spaces

Concerning the physical applications of algebraically extended p-adic numbers the problem is that p-adic norm is not in general bilinear in its arguments and therefore it does not define inner product and angle. One can however consider a generalization of the ordinary complex inner product $\bar{z}z$ to a p-adic valued inner product. It turns out that p-adic quantum mechanics in the sense as it is used in p-adic TGD can be based on this inner product.

The algebraic generalization of the ordinary Hilbert space inner product is bilinear and symmetric, defines p-adic valued norm. The norm can however for non-vanishing states. This inner product leads to p-adic generalization of unitarity and probability concept. The solution of the unitarity condition $\sum_k S_{mk}\bar{S}_{nk} = \delta(m,n)$ involves square root operations and therefore the minimal extension for the Hilbert space is 4-dimensional in $p > 2$ case and 8-dimensional in $p = 2$ case. Of course, extensions of arbitrary dimension are allowed.

The inner product associated with a minimal extension allowing square root near real axis provides a natural generalization of the real and complex Hilbert spaces respectively. Instead of real or complex numbers, a square root allowing algebraic extension extension appears as the multiplier field of the Hilbert space and one can understand the points of Hilbert space as infinite sequences $(Z_1, Z_2, ..., Z_n, ...)$, where Z_i belongs to the extension. The inner product $\sum_k \langle Z_k^1, Z_2^k\rangle$ is completely analogous to the ordinary Hilbert space inner product.

The generalization of the the Hilbert space of square integrable functions to a p-adic context is far from trivial since definite integral in in general ill defined procedure. Second problem is posed by the fact that p-adic counterparts of say oscillator operator wave functions do not exist in the entire p-adic variant of the configuration space. Algebraic definition of the inner product by using the rules of Gaussian integration provides a possible solution to the problem.

For Fock space generated by anti-commuting fermionic and commuting bosonic oscillator operators the p-adic counterpart exists naturally and it seems that Fock spaces can be seen as universal Hilbert spaces with rational coefficients identifiable as subspaces of both real Fock space and of all p-adic Fock spaces.

5.3 p-Adic unitarity and p-adic cohomology

p-Adic unitarity and probability concepts lead to highly nontrivial conclusions concerning the general structure of the p-adic S-matrix. The most general S-matrix is a product of a complex rational (extended rationals are also possible) unitary S-matrix S_Q and a genuinely p-adic S-matrix S_p which deviates only slightly from unity

$$\begin{aligned} S &= 1 + i\sqrt{p}T \ , \\ T &= O(p^0) \ . \end{aligned} \tag{28}$$

for $p \, mod \, 4 = 3$ allowing imaginary unit in its four-dimensional algebraic extension. In perturbative context one expects that the p-adic S-matrix differs only slightly from unity. Using the form $S = 1 + iT$, $T = O(p^0)$ one would obtain in general transition rates of order inverse of Planck mass and theory would have nothing to do with reality. Unitarity requirement implies iterative expansion of T in powers of p and the few lowest powers of p give excellent approximation for the physically most interesting values of p.

The unitarity condition implies that the moduli squared of the matrix T in $S = 1 + iT$ are of order $O(p^{1/2})$ if one assumes a four-dimensional p-adic extension allowing square root for the ordinary p-adic numbers and one can write

$$\begin{aligned} S &= 1 + i\sqrt{p}T \ , \\ i(T - T^{\dagger}) &+ \sqrt{p}TT^{\dagger} = 0 \ . \end{aligned} \tag{29}$$

This expression is completely analogous to the ordinary one since $i\sqrt{p}$ is one of the units of the four-dimensional algebraic extension. Unitarity condition in turn implies a recursive solution of the unitary condition in powers of p:

$$\begin{aligned} T &= \sum_{n\geq 0} T_n p^{n/2} \ , \\ T_n - T_n^{\dagger} &= \frac{1}{i}\sum_{k=0,..,n-1} T_{n-1-k}T_k^{\dagger} \ . \end{aligned} \tag{30}$$

If algebraic extension is not allowed then the expansion is in powers of p instead of $\sqrt{p}$. Note that the real counterpart of the series converges extremely rapidly for physically interesting primes (such as $M_{127} = 2^{127} - 1$).

In the p-adic context S-matrix $S = 1 + T$ satisfies the unitarity conditions

$$T + T^{\dagger} = -TT^{\dagger} \tag{31}$$

if the conditions

$$\begin{aligned} T &= T^{\dagger} \ , \\ T^2 &= 0 \ . \end{aligned} \tag{32}$$

defining what might be called p-adic cohomology, are satisfied [C3]. In the real context these conditions are not possible to satisfy as is clear from the fact that the total scattering rate from a given state, which is proportional to T^2_{mm} vanishes.

p-Adic cohomology defines a symmetry analogous to BRST symmetry: if T satisfies unitarity conditions and T_0 satisfies the conditions

$$\begin{array}{ll} T_0 = T_0^{\dagger} \ , & T_0^2 = 0 \ , \\ \{T_0, T\} = T_0T + TT_0 = 0 \ , & \end{array} \tag{33}$$

unitary conditions are satisfies also by the matrix $T_1 = T + T_0$. The total scattering rates are same for T and T_1.

5.4 The concept of monitoring

The relationship between p-adic and real probabilities involves the hypothesis that real transition probabilities depend on the cognitive resolution. Cognitive resolution is defined by the decomposition of the state space H into direct sum $H = \oplus H_i$ so that the experimental situation cannot differentiate between different states inside H_i. Each resolution defines different real transition probabilities unlike in ordinary quantum mechanics. Physically this means that the arrangement, where each state in H_i is monitored separately differs from the situation, when one only looks whether the state belongs to H_i. One can say that monitoring affects the behavior of a p-adic subsystem. Of course, these exotic effects relate to the physics of cognition rather than real physics.

Standard probability theory, which also lies at the root of the standard quantum theory, predicts that the probability for a certain outcome of experiment does not depend on how the system is monitored. For instance, if system has N outcomes $o_1, o_2, ...o_N$ with probabilities $p_1, ..., p_N$ then the probability that o_1 or o_2 occurs does not depend on whether common signature is used for o_1 and o_2 or whether observer also detects which of these outcomes occurs. The crucial signature of p-adic probability theory is that monitoring affects the behavior of the system.

Physically monitoring is represented by quantum entanglement [H2], and differentiates between two eigen states of the density matrix only provided the eigenvalues of the density matrix are different. If there are several degenerate eigenvalues, quantum jump occurs to any state in the eigen space and one can predict only the total probability for the quantum jump into this eigen space: the real probabilities for jumps into individual states are obtained by dividing total real probability by the degeneracy factor. Hence the p-adic probability for a quantum jump to a given eigenspace of density matrix is p-adic sum of probabilities over the eigen states belonging to this eigenspace:

$$P_i = \frac{(n(i)P(i))_R}{\sum_j (n(j)P(j))_R} \quad .$$

Here n_i are dimensions of various eigenspaces.

If the degeneracy of the eigenvalues is removed by an arbitrary small perturbation, the total probability for the transition to the same subspace of states becomes the sum for the real counterparts of probabilities and one has in good approximation:

$$P^R = \frac{n(i)P(i)_R}{[\sum_{j\neq i}\sum_j (n(j)P(j))_R + n(i)P(i)_R]} \quad .$$

Rather dramatic effects could occur. Suppose that that the entanglement probability $P(i)$ is of form $P(i) = np$, $n \in \{0, p-1\}$ and that n is large so that $(np)_R = n/p$ is a considerable fraction of unity. Suppose that this state becomes degenerate with a degeneracy m and $mn > p$ as integer. In this kind of situation modular arithmetics comes into play and $(mnp)_R$ appearing in the real probability $P(1 \; or \; 2)$ can become very small. The simplest example is $n = (p+1)/2$: if two states i and j have *very nearly equal but not identical* entanglement probabilities $P(i) = (p+1)p/2 + \epsilon$, $P(j) = (p+1)p/2 - \epsilon$, monitoring distinguishes between them for arbitrary small values of ϵ and the total probability for the quantum jump to this subspace is in a good approximation given by

$$\begin{aligned} P(1 \; or \; 2) &\simeq \frac{x}{\left[\sum_{k\neq i,j}(P_k)_R + x\right]} \quad , \\ x &= 2\left[(p+1)p/2\right]_R \quad . \end{aligned} \tag{34}$$

and is rather large. For instance, for Mersenne primes $x \simeq 1/2$ holds true. If the two states become degenerate then one has for the total probability

$$\begin{aligned} P(1 \; or \; 2) &\simeq \frac{x}{\left[\sum_{k\neq i,j}(P_k)_R + x\right]} \quad , \\ x &= \frac{1}{p} \quad . \end{aligned} \tag{35}$$

The order of magnitude for $P(1 \; or \; 2)$ is reduced by a factor of order $1/p$!

Since p-adicity is essential for the exotic effects related to monitoring, the exotic phenomena of monitoring should be related to the quantum physics of cognition rather than real quantum physics. A test for quantum TGD would be provided by the study of the dependence of the transition rates of quantum systems on the resolution of monitoring defined by the dimensions of the degenerate eigenspaces of the subsystem density matrix. One could even consider the possibility of measuring the value of the p-adic prime in this manner. The behavior of living systems is known to be sensitive to monitoring and an exciting possibility is that this sensitivity, if it really can be shown to have statistical nature, could be regarded as a direct evidence for TGD inspired theory of consciousness. Note that the mapping of the physical quantities to entanglement probabilities could provide an ideal manner to compare physical quantities with huge accuracy! Perhaps bio-systems have invented this possibility before physicists and this could explain the miraculous accuracy of biochemistry in realizing genetic code. The measurement of the monitoring effect could provide a manner to determine the value of p_i for each p-adic region of space-time.

5.5 p-Adic Schrödinger equation

5.5.1 The emergence of the p-adic infrared cutoff

The experience with the construction of the p-adic counterpart of the standard model shows that p-adic quantum theory involves in practice infrared cutoff length scale in both time and spatial directions. The cutoff length scale comes out purely number theoretically. In the time like direction the cutoff length scale comes out from the exponent of the time ordered integral: p-adic exponent function $exp(x)$ does not exist unless the p-adic norm of the argument is smaller than one and this in turn means that $P(exp(i\int_0^t Vdt))$ does not exist for too larger values of time argument. A more concrete manner to see this is to consider time dependence for the eigenstates of Hamiltonian: the exponent $exp(iEt)$ exists only for $|Et|_p < 1$. The necessity of the spatial cutoff length scale is seen by considering concrete examples. For instance, the p-adic counterparts of the harmonic oscillator Gaussian wavefunctions are defined only in a finite range of the argument. As far as the definition of exponent function is considered one must keep in mind that the formal exponent function does not have the usual periodicity properties. The definition as a p-adic plane wave gives the needed periodicity properties but also in this case the infrared cutoff is necessary.

One should be able to construct also global solutions of the p-adic Schrödinger equation. The concept of p-adic integration constant might make this possible: by multiplying the solution of thhe Schrödinger equation with a constant depending on a finite number of the pinary digits, one can extend the solution to an arbitrary large region of the space time. What one cannot however avoid is the decomposition of the space time into disjoint quantization volumes.

One of the original motivation to introduce p-adic numbers was to introduce ultraviolet cutoff as a p-adic cutoff but, as the considerations of the second part of the book show, UV divergences are absent in the p-adic case and short distance contributions to the loops are negligibly small so that the mere p-adicization eliminates automatically UV divergences. Rather, it seems that the length scale L_p serves as an infrared cutoff and, if a length scale resolution rougher than L_p is used, ordinary real theory should work. Only in the length scales $L \leq L_p$ should the p-adic field theory and Quantum Mechanics be useful. The applicability of the real QM for length scale resolution $L \geq L_p$ is in accordance with the fact that the real continuity implies p-adic contintuity.

5.5.2 Formal p-adicization of the Schrödinger equation

The formal p-adic generalization of the Schrödinger equation is of the following general form

$$\theta\frac{d\Psi}{dt} = H\Psi \ , \qquad (36)$$

where H is in some sense Hermitian operator. If Schödinger amplitudes are complex values θ can be taken to be imaginary unit i. The same identification is possible if Ψ possesses values in the extension of p-adic allowing square root and the condition $p \ mod \ 4 = 3$ or $p = 2$ guaranteing that $\sqrt{-1}$ does not exist as an ordinary p-adic number, is satisfied. For $p \ mod \ 4 = 1$ the situation is more complicated since imaginary unit i does not in general belong to the generators of the minimal extension allowing a square root. An open problem is whether one could replace θ appearing in the quadratic extension and define complex conjugation as the operation $\theta \rightarrow -\theta$. The analogy with the ordinary quantum mechanics suggests the form

$$H = -\frac{\nabla^2}{2m} + V \ , \tag{37}$$

for the Hamiltonian in $p \ mod \ 4 = 3$ case. In the complex case ∇^2 is obtained by replacing the ordinary derivatives with the p-adic derivatives and V is a p-adically differentiable function of the coordinates typically obtained from a p-analytic function via the canonical identication.

Although the formal p-adicization is possible, it is not at all obvious whether one can get anything physically interesting from the straightforward p-adicization of the Schrödinger equation. The study of the the p-adic hydrogen atom shows that formal p-adicization need not have anything to do with physics. For instance, Coulomb potential contains a factor $1/4\pi$ not existing p-adically, the energy eigenvalues depend on π and the straightforward p-adic counterparts of the exponentially decreasing wave functions are not exponentially decreasing functions p-adically and do not even exist for sufficiently large values of the argument r. It seems that a more realistic manner to define the p-adic Schrödinger equation is as limiting case of the p-adic field theory. Of course, it might also be that p-adic Schrödinger equation does not make sense. A more radical solution of the problems is the allowance of finite-dimensional extensions of p-adic numbers allowing also transcendental numbers.

5.5.3 p-Adic harmonic oscillator

The formal treatment of the p-adic oscillator using oscillator operator formalism is completely analogous to that of the ordinary harmonic oscillator. The only natural inner product is the p-adic valued one. That the treatment is correct is suggested by the fact that it is purely algebraic involving only the p-adic counter part of the oscillator algebra. The matrix elements of the oscillator operators $a^\dagger$ and a involve square roots and they exist provided the minimal extension allowing square roots appears as a coefficient ring of the Hilbert space. If two-dimensional quadratic extension not containing $\sqrt{p}$ is used occupation number must be restricted to the range $[0, p-1]$. If the Hilbert space inner product based on non-degenerate p-adic inner product $Z_c Z + \hat{Z}_c \hat{Z}$ the extension implies a characteristic degeneracy of states with complex amplitudes related to the conjugation $\sqrt{p} \rightarrow -\sqrt{p}$. 2-adic and p-adic cases differ in radical manner since the dimensions of the extension are 4 for $p > 2$ and 8 for $p = 2$. Since the representations of the Kac Moody and Super Virasoro algebras are based on oscillator operators this means that there is deep difference between $p = 2$ and $p > 2$ p-adic conformal field theories.

The p-adic energy eigen values are $E_n = (n + 1/2)\omega_0$ and their real counterparts form a quasi-continuous spectrum in the interval $(2, 4)$ for $p = 2$ and $(1, p)$ for $p > 2$! If p is very large (of order 10^{38} in the TGD:eish applications) the small quantum number limit $n < p$ gives the quantum number spectrum of the ordinary quantum mechanics. The occupation numbers $n > p$ have no counterpart in the conventional quantum theory and it seems that the classical theory with a quasi-continuous spectrum but with energy cutoff $p\omega_0$ is obtained at the limit of the arbitrarily large occupation numbers. The limit $p \rightarrow \infty$ gives essentially the classical theory with no upper bound for the energy.

The results suggests the idea that p-adic QM might be somewhere halfway between ordinary QM and classical mechanics. This need not however be the case as the study of the p-adic thermodynamics suggests. p-Adic thermodynamics allows a low temperature phase $exp(E_n/T) \equiv p^{n/T_k}$, $T_k = 1/k$, with quantized value of temperature. In this phase the probabilities for the energy eigenstates E_n, $n = \sum_k n_k p^k$ are extremely small except for the smallest values of n so that low temperature thermodynamics does not allow the effective energy continuum. One might argue that situation changes in the high temperature phase. The problem is that p-adic thermodynamics for the harmonic oscillator allows only formally high temperature phase $T = t_0\omega_0/p^k$, $k = 1, 2, ...,$ $|t_0| = 1$. The reason is that Boltzmann weights $exp(-E_n/T) = exp(np^k/t_0)$ have p-adic norm equal to 1 so that the sum of probabilities giving free energy converges only formally. If one accepts the formal definition of the free energy as $exp(F) \equiv 1/(1 - exp(-E_0/T))$ then the real counterpart of the energy spectrum indeed becomes continuum also in the thermodynamic sense.

Consider next what a more concrete treatment using Schrödinger equation gives. The p-adic counterpart of the Schrödinger equation is formally the same as the ordinary Schrödinger equation. Ψ is assumed to have values in a minimal extension of p-adic numbers allowing square root and possessing imaginary unit so that the condition $p\ mod\ 4 = 3$ or $p = 2, 3$ must hold true. For the energy momentum eigenstates the equation reduces to

$$(-\frac{d^2}{dy^2} + y^2)\Psi \quad = \quad 2e\Psi \ , \tag{38}$$

where the dimensionless variables $y = \sqrt{\omega}x$ and $e = \frac{E}{\omega}$ have been introduced. This transformation makes sense provided ω possesses p-adic square root.

The solution ansatz to this equation can be written in the general form $\Psi = exp(-y^2/2)H_{e-1/2}(y)$, where H is the p-adic counter part of a Hermite polynomial. The first thing to notice is that vacuum wave function does not converge in a p-adic sense for all values of y. A typical term in series is of the form $X_n = \frac{y^{2n}}{2^n n!}$. In ordinary situation the factors, in particular $n!$, in the numerator imply convergence but in present case the situation is exactly the opposite.

In 2-adic case both the factor 2^n and the factor $n!$ in the denominator cause troubles whereas for $p > 2$ the p-adic norm of 2^n is equal to one. $n!$ gives at worst the power 2^{n-1} to the 2-adic norm. Therefore the 2-adic norm of X_n behaves as $N(X_n) \simeq |y_2|^{2n}2^n2^{n-1}$. The convergence is therefore achieved for $|y|_2 \leq 1/4$ only. For $p > 2$ the convergence is achieved for $|y|_p \leq 1/p$. One can continue the oscillator Gaussian to a globally defined function of y by observing that the scaling $y \rightarrow y/\sqrt{2}$ corresponds to taking a square root of the oscillator Gaussian and this square root exists if minimal quadratic extension allowing square root is used. In the usual situation the function $H_e(y)$ must be polynomial since otherwise it behaves as $exp(y^2)$ and does not converge: this implies the quantization of energy also now.

The inner product, which should orthogonalize the states is the p-adic valued inner product based on the p-adic generalization of the definite integral. The generalizations of the analytic formulas encountered in the real case should hold true also now. The guess motivated by the formal treatment is that p-adic energies are quantized according to the usual formula and classical energies form a continuum below the upper bound $e_R \leq 4$ in 2-adic case and $e_R =\leq p$ in p-adic case. In fact, the mere requirement $|e|_p \leq 1$ implies that energy is quantized according to the formula $e = n + 1/2$ in p-adic case.

5.5.4 p-Adic fractality in the temporal domain

The assumption that p-adic physics gives faithful cognitive representation of the real physics leads to highly nontrivial predictions, the most important prediction being p-adic fractality with long range temporal correlations and microtemporal chaos.

In p-adic context the diagonalization of the Hamiltonian for N-dimensional state space in general requires N-dimensional algebraic extension of p-adic numbers even when the matrix elements of the Hamiltonian are complex rational numbers. TGD as a generalized number theory vision allows all algebraic extensions of p-adic numbers so that this is not a problem. The necessity to decompose p-adic Hamiltonian to a complex rational free part and p-adically small interaction part could provide the fundamental reason for why Hamiltonians have the characteristic decomposition into free and interaction parts. Of course, it might be that Hamiltonian formalism does not make sense in the p-adic context and should be replaced with the approach based on Lagrangian formalism: at least in case of p-adic QFT limit of TGD this approach seems to be more promising. One could also argue that the very fact that p-adic physics provides a cognitive representations of TGD based physics gives a valuable guide to the real physics itself, and that one should try to identify the constraints on real physics from the requirement that its p-adic counterpart exists. The following discussion is motivated by this kind of attitude.

The emergence of various dynamical time scales is a very general phenomenon. For instance, it seems that strong and weak interactions correspond to different time scales in well defined sense and that it is a good approximation to neglect strong interaction in weak time scales and vice versa. p-Adic framework gives hopes of finding a more precise formulation for this heuristics using number theoretical ideas. The basic observation is that the time ordered exponential of a given interaction Hamiltonian exists only over a finite time interval of length $T_p(n) = p^n L_p$. This suggests that one should distinguish between the time developments associated with various p-adic time scales $T_n = p^n L_p/c$: obviously temporal fractality would be in question.

More concretely, the p-adic exponential $exp(iH\Delta t)$ of the free Hamiltonian exists p-adically only if one assumes that Δt is a small rational proportional to a positive power of p: $\Delta t \propto p^n$. Of course, this restriction to the allowed values of Δt might be interpreted as a failure of the cognitive representation rather than a real physical effect. Alternatively, one might argue that the emergence of the p-adic time scales is a real physical effect and that one must define a separate S-matrix for each p-adic time scale $\Delta t \propto p^n$. Thus p-adic S-matrices for time intervals that differ from each other by arbitrarily long real time interval could be essentially identical. This would mean extremely precise fractal long range correlations and chaos in short time scales also at the level of real physics. This is certainly a testable and rather dramatic prediction in sharp contrast with standard physics views. $1/f$ noise could be seen as one manifestation of these long range correlations.

What would distinguish between different times scales would be different decomposition of the Hamiltonian to free and interaction parts to achieve interaction part which is p-adically small in the time scale involved. For instance, it could be possible to understand color confinement in this manner: in quark gluon plasma phase below the length scale L_p many quark states without any constraints on color are the natural state basis whereas above the length scale L_p physical states must be color singlets since otherwise time evolution operator does not exist.

In case of the cognitive representations of the external world canonical identification maps long external time and length scales to short internal time and length scales and vice versa. Thus p-adic fractality of the cognitive dynamics induces at the level of cognitive representation order in short length and time scales and chaos in long length and time scales: this is of course natural since sensory information comes mainly from the nearby spatiotemporal regions of the system. For self-representations there is chaos in short time scales and fractal long range correlations (so that our temptation to see our life as a coherent temporal pattern would not be self deception!). This kind of fractality is of course absolutely essential in order to understand bio-systems as intentional systems able to plan their future behavior. This prediction is about behaviorial patterns of cognitive systems and also testable.

One can get a more quantitative grasp on this idea by studying the time development operator associated with a diagonalizable Hamiltonian. If the eigenvalues E_n of the diagonalized

Hamiltonian have p-adic norms $|E_n|_p \leq p^{-m}$, the time evolution determined by this Hamiltonian is defined at most over a time interval of length norm $T_p(m) = p^{m-1}L_p$ since for time intervals longer than this the eigenvalues $exp(iE_nt)$ of $exp(iHt)$ do not exists as a p-adic numbers for all energy eigenstates. Thus one must restrict the time evolution to time scale $t \leq p^{m-1}L_p$: this is consistent with a p-adic hierarchy of interaction time scales.

An alternative approach is based on the requirement that the complex phase factors $exp(iET)$ for the eigenstates of the diagonal part of the Hamiltonian are complex rational phases forming a multiplicative group. This means that one can map the phase factors $exp(iET)$ directly to their p-adic counterparts as complex rational numbers. With suitable constraints on the energy spectrum this makes sense if the interaction time T is quantized so that it is proportional to a power of p. The decomposition of the Hamiltonian to free and interacting parts could be done in such a manner that the exponential of Hamiltonian decomposes to a product of diagonal part representable as complex rational phases and interaction part which is of higher order in p so that ordinary exponential exists for sufficiently small values of interaction time. This decomposition depends on the p-adic time scale.

5.5.5 How to define time ordered products?

In perturbation theory one must deal with the p-adic counterpart of the time ordered exponential $\prod_n Pexp\left[\int_0^t H_{int}(n)dt\right]$ appearing in the definition of the time development operator. In the case of a nondiagonal, time dependent interaction Hamiltonian the very definition of the p-adic counterpart of the time ordered integral is far from obvious since p-adic numbers do not allow natural ordering. Perhaps the simplest possibility is based on Fourier analysis based on the use of Pythagorean phases. This automically involves the introduction of a time resolution $\Delta t = q = m/n$ and discretization of the time coordinate. Depending on the p-adic norm of Δt one obtains a hierarchy of S-matrices corresponding to different p-adic fractalities. Time ordering would be naturally induced from the ordering of ordinary integers since only the integer multiples of Δt are involved in the discretized version of integral defined by the inner product for the Pythagorean plane waves. The requirement that all time values have same p-adic norm implies $T = n\Delta t$, $n = 0, ..., p-1$. If one assumes that long range fractal temporal order is present one can also allow time intervals $T = n\delta t + mp^k$ which correspond to arbitrarily long real time intervals.

5.5.6 p-Adic particle stability is not equivalent with real stability

It is natural to require that single hadron states are eigenstates for that part of the total Hamiltonian, which consists of the kinetic part of the Hamiltonian. If this the case, one can require that the effect of $exp(iH_0t)$ is just a multiplication by the factor $exp(iEt)$. The fact that particles are not stable against decay to many-particle states suggests that E must be complex. Generalizing the construction of the p-adic planewaves one could define this prefactor for all values of time even in this case. One can however criticize this approach: the introduction of the decay width as imaginary part of E is is category error since decay width characterizes the statistical aspects of the dynamics associated with quantum jumps rather than the dynamics of the Schrödinger equation.

p-Adic unitarity concept suggests a more elegant description. The truncated S-matrix describing the transitions $H_p \rightarrow H_p$ is unitary despite the fact that the transitions between different sectors are possible. This makes sense because the total p-adic transition probability from H_p to H_q, $q \neq p$, vanishes by generalized unitarity conditions. Generalizing, the p-adic representations of elementary particles and even hadrons would p-adically stable in the sense that the total p-adic decay probability would vanishes for them. One could also say that in absence of monitoring p-adic cognitive representation of particle would be stable. This picture is consistent with the notion of p-adic cohomology reducing unitarity conditions for S-matrix $S = 1 + iT$ to the conditions $T = T^\dagger$ and $T^2 = 0$. Of course, it would apply only at the level of cognitive physics.

6 Generalized Quantum Mechanics

One can consider two generalizations of quantum mechanics to a fusion of p-adic and real quantum mechanics.

1. For the first generalization the guiding principle for the generalization of quantum mechanics is that quantum mechanics in a given number field is obtained as an algebraic continuation of the quantum mechanics in the field of rational numbers common to all number fields or in finite-dimensional extensions of rational numbers. This means that U-matrices U_F for transitions from H_Q to H_F, where F refers to various completions of rationals, are obtained as algebraic continuations of the unitary U-matrix U_Q for H_Q.

   The variant of the canonical identification I mapping rationals as $r/s \rightarrow I(r)/I(s)$ is the most natural relationship between real and p-adic U-matrices since it is a compromise between topology and algebra mapping rationals to rationals in a continuous manner and respecting rational unitarity assuming that the matrix elements of U do not involve integers $n > p-1$. At the limit $p \rightarrow \infty$ unitarity is possible for all rational matrices U. This argument applies also for the extensions of rationals. The generalization means enormously strong algebraic constraints on the form of the U-matrix, especially so for small values of p.

2. A more radical option is that transitions from rational Hilbert space H_Q to the Hilbert spaces H_F associated with different number fields occur. This requires that U-process is followed by a process analogous to a state function reduction and preparation takes care that the resulting states become states in H_Q: this is what makes this generalization of a special interest. In this case one can speak about total scattering probability from H_Q to H_F. The U-matrices U_F are not anymore mere analytic continuations of U_Q. A possible interpretation of the unitary process $H_Q \rightarrow H_F$ is as generation of intention whereas the reduction and preparation means the transformation of the intention to action.

   The assumption that H_Q allows an algebraic continuation to the spaces H_F is probably too strong an idealization in p-adic and even in the real case. For instance, one cannot allow all rational valued momenta in p-adic case for the simple reason that the continuation to the p-adic case involves always some momentum cutoff if the extension of p-adics remains finite. Even in the real case the summation over all rational momenta in the unitarity conditions of U-matrix fails to make sense and cutoff is needed. A hierarchy of cutoffs suggests itself and has a natural interpretation as number theoretical hierarchy of extensions of p-adics.

In order to avoid un-necessary complications the following formal discussion however uses H_Q as a universal Hilbert space contained by the various state spaces H_F.

6.1 Quantum mechanics in H_F as a algebraic continuation of quantum mechanics in H_Q

The rational Hilbert space H_Q is representable as the set of sequences of real or complex rationals of which only finite number are non-vanishing. Real and p-adic Hilbert spaces are obtained as the numbers in the sequences to become real or p-adic numbers and no limitations are posed to the number of non-vanishing elements. All these Hilbert spaces have rational Hilbert space H_Q as a common sub-space. Also momenta and other continuous quantum numbers are replaced by a discrete value set. Superposition principle holds true only in a restricted sense, and state function reduction and preparation leads always to a final state which corresponds to a state in H_Q. This picture differs from the earlier one in which p-adic and real Hilbert spaces were assumed to form a direct sum.

The notion of unitarity generalizes. Contrary to the earlier beliefs, U-matrix does not possess matrix elements between different number fields but between rational Hilbert space and Hilbert spaces associated with various completions of rationals. This makes sense since the final state of the quantum jump (and thus the initial state of the unitary process, is always in H_Q.

The U-matrix is a collection of matrices U_F having matrix elements in the number field F. U_F maps H_Q to H_F. Each of these U-matrices is unitary. Also U_Q is unitary and U_F is obtained by algebraic continuation in the quantum numbers labelling the states of U_Q to U_F.

Hermitian conjugation makes sense since the defining condition

$$\langle \alpha_F | U n_Q \rangle = \langle U^\dagger \alpha_F | n_Q \rangle \ . \quad (39)$$

allows to interpret $|n_Q\rangle$ also as an element of H_F. If U would map different completed number fields to each other, hermiticity conditions would not make sense.

The hermitian conjugate of U-matrix maps H_F to H_Q so that $UU^\dagger$ *resp.* $U^\dagger U$ maps H_F *resp.* H_Q to itself. This means that there are two independent unitarity conditions

$$\begin{aligned} U_F U_F^\dagger &= Id_F \ , \\ U_F^\dagger U_F &= Id_Q \ . \end{aligned} \quad (40)$$

One can write $U = P_Q + T_F$ and $U^\dagger = P_Q + T_F^\dagger$, where P_Q refers to the projection operator to H_Q.
This gives

$$\begin{aligned} T_F + T_F^\dagger &= -T_F T_F^\dagger \ , \\ P_Q T_F + T_F^\dagger P_Q &= -T_F^\dagger T_F \ . \end{aligned} \quad (41)$$

It is convenient to introduce the notations $T_Q = P_Q T_F$ and $T_Q^\dagger = T_F^\dagger P_Q$ with analogous notations for U and $U^\dagger$. The first condition, when multiplied from both sides by P_Q, gives together with the second equation unitarity conditions for T_Q

$$\begin{aligned} T_Q + T_Q^\dagger &= -T_Q T_Q^\dagger \ , \\ T_Q + T_Q^\dagger &= -T_F^\dagger T_F \ . \end{aligned} \quad (42)$$

This means that the restriction of the U-matrix to H_Q is unitary.

The difference between the right hand sides of the equation should vanish. The understanding of how this happens requires more delicate considerations. For instance, in the case of $F = C$ continuous sum over indices appears at the right hand side coming from four-momenta labelling the states. The restrictions of quantum numbers to Q and its subsets could be a process analogous to the momentum cutoff of quantum field theories. The continuation from discrete integer valued labels of, say discrete momenta, to continuous values is performed routinely in various physical models routinely, and it would seem that this process has cognitive and physical counterparts. This picture conforms with the vision that the rational (or extended rational) U-matrix U_Q gives the U-matrices U_F by an algebraic continuation in the quantum numbers labelling the states (say 4-momenta).

6.2 Could U_F describe dispersion from H_Q to the spaces H_F?

One can also consider a more general situation in which the states in H_Q can be said to disperse to the sectors H_F. In this case one can write

$$T = \text{"}\sum_F\text{"} T_F \ . \tag{43}$$

Here the sum has only a symbolic meaning since different number fields are in question and an actual summation is not possible. The T-matrix T_Q is the sum of the restrictions of T_F to H_Q and is the sum of rational valued T-matrices: $T_Q = \sum_F P_Q T_F$.

The T-matrices T_F are not anymore obtainable by algebraic continuation from same T-matrix T_Q. The unitarity conditions

$$\sum_F (P_Q T_F + T_F^\dagger P_Q) = -\sum_F T_F^\dagger T_F \tag{44}$$

make sense only if they are satisfied separately for each T_F, exactly as in the previous case. T

The diagonal elements

$$T_F^{mm} + \overline{T}_F^{mm} = \sum_\alpha T_F^{m\alpha} \overline{T}_F^{m\alpha} = \sum_r T_F^{mr} \overline{T}_F^{mr}$$

give essentially total scattering probabilities from the state $|m\rangle$ of H_Q to the sector H_F, and must be rational (or extended rational) numbers. One can therefore say that each U-process leads with a definite probability to a particular sector of the state space.

The fact that states which are superpositions of states in different spaces H_F does not make sense mathematically, forces the occurrence of a process, which might be regarded as a number theoretical counterpart of state function reduction and preparation. First a sector H_F is selected with probability p_F. Then F-valued (in particular complex valued) entanglement in H_F is reduced by state reduction and preparation type processes to a rational or extended rational entanglement having interpretation as bound state entanglement. It would be natural to assume that Negentropy Maximization Principle governs this process. Obviously the possibility to reduce state function reduction to number theory forces to consider quite seriously the proposed option.

6.3 Do state function reduction and state-preparation have number theoretical origin?

The foregoing considerations support the view that state function reduction and state preparation are number theoretical necessities so that there would be a deep connection between number theory and free will. One could even say that free will is a number theoretic necessity. The resulting more unified view provides the reason why for state function reduction, and preparation and allows to generalize previous views developed gradually by physics and consciousness inspired educated guess work.

6.3.1 Negentropy Maximization Principle as variational principle of cognition

It is useful to discuss the original view about Negentropy Maximization Principle (NMP) before considering the possible generalization of NMP inspired by the number theoretic vision.

NMP was originally motivated by the need to construct a TGD based quantum measurement theory. Gradually it however became clear that standard quantum measurement theory more or less follows from the assumption that the world of conscious experience is classical: this meant that NMP became a principle governing only state preparation.

State function reduction is achieved if a localization in zero modes occurs in each quantum jump, and if U matrix in zero modes corresponds to a flow in some orthogonal basis for the configuration space spinor fields in the quantum fluctuating fiber degrees of freedom of the configuration space. The requirement that U-matrix induces effectively a flow in zero modes is consistent with the effective classicality of the zero modes requiring that quantum evolution causes no dispersion. The one-one correlation between preferred quantum state basis in quantum fluctuating degrees of freedom and zero modes implies nothing but a one-one correspondence between quantum states and classical variables crucial for the interpretation of quantum theory. It seems that number theoretical vision forces to generalize this view, and to raise NMP to a completely general principle applying also to the state function reduction as the original proposal indeed was.

In its original form NMP governs the dynamics of self measurements and thus applies to the quantum jumps reducing the entanglement between quantum fluctuating degrees of freedom for given values of zero modes. Self measurements reduce the entanglement only between subsystems in quantum fluctuating degrees of freedom since they occur after the localization in the zero modes. Self measurement is repeated again and again for the unentangled subsystems resulting in each self measurement. This cascade of self measurements leads to a state possessing only extended rational entanglement identifiable as bound state entanglement and having negative number theoretic entanglement entropy. This process should be equivalent with the state preparation process assumed to be performed by a conscious observer in standard quantum measurement theory.

NMP states that the self measurement can be regarded as a quantum measurement of the subsystem's density matrix reducing the counterpart of the entanglement entropy of some subsystem to a smaller value, and that this occurs for the subsystem for which the reduction of the entanglement entropy is largest among all subsystems of the p-adic self. Inside each self NMP fixes some subsystem which is quantum measured in the quantum jump. One could perhaps say that self measurements make possible quantum level self repair since they allow the system in self state to fight against thermalization which results from the generation of unbound entanglement between subsystem-complement pairs.

6.3.2 NMP and number theory

The requirement the universe of conscious experience is classical is one manner to justify quantum jump. This hypothesis could be replaced by a postulate that state function reduction and preparation project quantum states to a definite number field and that only extended rational entanglement identifiable as bound state entanglement is stable. This is consistent with NMP since it is possible to assign to an extended rational entanglement a non-negative number theoretic negentropy as the maximum over entropies defined by various p-adic entropies $S_p = -\sum p_k log(|p_k|_p)$.

The unitary process U would thus start from a product ofbound states for which entanglement coefficient are extended rationals, and would lead to a formal superposition of states belonging to different number fields. Both state function reduction and state preparation would begin with a localization to a definite number field. This localization would be followed by a self measurement cascade reducing the entanglement to extended rational entanglement.

This vision forces to challenge the earlier views about state function reduction.

1. There is no good reason for why NMP could not be applied to both state function reduction and preparation.

2. If the entanglement between zero modes and quantum fluctuating degrees of freedom involves only discrete values of zero modes, the problems caused by the fact that no well-defined functional integral measure over zero modes exists, find an automatic resolution. Since

extended rational entanglement possesses negative entanglement entropy, it is stable also against reduction if NMP applies completely generally. A discrete entanglement involving transcendentals not contained to any *finite* extension of any p-adic number field is unstable and reduced.

3. The quantum measurement lasts for a time determined by the life-time of the bound state entanglement between zero modes and quantum fluctuating degrees of freedom. Physical considerations of course support the view that it takes more than single quantum jump (10^{-39} seconds of psychological time) for the state function reduction to take place. The notion of zero mode-zero mode bound state entanglement seems however to be self-contradictory. If join along boundaries bonds are space-time correlates for the bound state entanglement, their formation should transform roughly half of the zero modes associated with the two space-time sheets to quantum fluctuating degrees of freedom.

4. If p-adic length scale hierarchy has as its counterpart a hierarchy of state function reduction and preparation cascades, one must accept the quantum parallel occurrence of state function reduction and preparation processes in the parallel quantum universes corresponding to different p-adic length scales. This picture provides a justification for the modelling of hadron as a quantum system in long length and time scales and as a dissipative system consisting of quarks and gluons in shorter length and time scales. The bound state entanglement between subsystems of entangled systems having as a space-time correlate join along boundaries bonds connecting subsystem space-time sheets, is a second important implication of the new sub-system concept, and plays a central role in TGD inspired theory of consciousness.

7 Generalization of the notion of configuration space

The only manner to possibly p-adicize the notion of the configuration space is provided by the algebraic continuation from a subset of rational configuration space consisting of points for which a finite number of coordinates are non-vanishing and rational values. The representability of the configuration space as a union of symmetric spaces means an enormous simplification since everything reduces to a single point, most naturally the maximum of Kähler function for given zero modes, but there are still several challenges involved.

1. One must construct p-adic counterparts of Kähler function, Kähler metric and Kähler form. There are hopes to achieve this if it is possible to assign to each real space-time sheet a p-adic space-time sheet and identifye the value of p-adic Kähler function as that of the real Kähler function in the case that the values of the real Kählr function $K(X^3)$ values belongs to a finite-dimensional extension of rationals for rational argument. This assignment need not be unique and an entire hierarchy of assignments labelled by the dimension of p-adic numbers involved. The higher the dimension the shorter the pinary cutoff.

2. If Kähler action is rational function in a generalized sense the continuation at rational points is in principle trivial. Also the exponent of Kähler function defining vacuum functional should have continuation to the p-adic context.

3. The continuation of the configuration space Kähler metric and Kähler form reduce to the algebraic continuation of the configuration space Hamiltonians and corresponding super charges. If these define rational or algebraic functions in generalized sense also this continuation might be possible.

4. Also the p-adic variant of the configuration space functional integral must be constructed. Here symmetric space structure gives hopes that Gaussian integral of free field theories

generalizes to a functional integral around maxima of Kähler function. It is essential that free field theory situation prevails since only in this case one has control over the extended rationality of the resulting expressions for S-matrix elements.

7.1 p-Adic counterparts of configuration space Hamiltonians

One must continue the δM^4_+ local CP_2 Hamiltonians appearing in the integrals defining configuration space Hamiltonians to various p-adic sectors. CP_2 harmonics are homogeneous polynomials with rational coefficients and do not therefore produce any trouble since normalization factors involve only square roots. The p-adicization of δM^4_+ function basis defining representations of Lorentz group involves more interesting aspects.

7.1.1 p-Adicization of representations of Lorentz group

In the light cone geometry Poincare invariance is strictly speaking broken to Lorentz invariance with respect to the dip of the light cone and at least cosmologically a more natural basis is characterized by the eigenvalues of angular momentum and boost operator in a given direction. The eigenvalue spectrum of the boost operator is continuous without further conditions. One can study these conditions in the realization of the unitary representations of Lorenz group as left translations in the Lorentz group itself by utilizing homogenous functions of four complex variables z^1, z^2, z^3, z^4 satisfying the constraint $z_1z_4 - z_2z_3 = 1$ expressing the fact that they correspond to the homogenous coordinates of the Lorentz group defined by that matrix elements of the SL(2,C) matrix

$$\begin{pmatrix} z_1 & z_3 \\ z_2 & z_4 \end{pmatrix} .$$

The function basis consists of

$$\begin{aligned}
f^{a_1,a_2,a_3,a_4}(z_1,z_2,z_3,z_4) &= z_1^{a_1}z_2^{a_2}z_3^{a_3}z_4^{a_4} \quad , \\
a_1 = m_1 + i\alpha, \qquad & a_2 = m_2 - i\alpha \quad , \\
a_3 = m_3 - i\alpha, \qquad & a_4 = m_4 + i\alpha \quad , \\
m_1 + m_2 = M \quad , \qquad & m_3 + m_4 = M \quad .
\end{aligned}$$

The action of Lorentz transformation is given by

$$\begin{pmatrix} z_1 & z_3 \\ z_2 & z_4 \end{pmatrix} \rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z_1 & z_3 \\ z_2 & z_4 \end{pmatrix} . \tag{45}$$

and unimodular ($ad - bc = 1$). Lorentz transformation preserves the imaginary parts $i\alpha$ of the complex degrees $d_i = m \pm i\alpha$ of $z_k^{\pm i\alpha + m_k}$ (as can be seen by using binomial series representations for the transformed coordinates). Also the sums $m_1 + m_2 = M$ and $m_3 + m_4 = M$ are Lorentz invariants. Hence the representation is characterized by the the pair (α, M). M corresponds to the minimum angular momentum for the $SU(2)$ decomposition of the representation.

The imaginary parts $i\alpha$ of the complex degrees correspond to the eigen values of Lorentz boost in the direction of the quantization axis of angular momentum. The eigen functions are proportional to the factor

$$\begin{aligned}
&\rho_1^{i2\alpha}\rho_2^{-i2\alpha}\rho_3^{-i2\alpha}\rho_4^{i2\alpha} \quad , \\
&\rho_i = \sqrt{z_i\overline{z}_i} \quad .
\end{aligned}$$

Since one can write $\rho^{i2\alpha} = e^{i2log(\rho)\alpha}$, these are nothing but the logarithmic plane waves. The value set of $\alpha \geq 0$ is continuous in the real context.

The requirement that the logarithmic plane waves are continuable to p-adic number fields and exist p-adically for rational values of $\rho_i = m/n$, quantizes the values of α. This condition is satisfied if the quantities $p^{i2\alpha_i} = e^{i2log(p)\alpha_i}$ exist p-adically for any prime. As shown in [E8], there seems to be no number theoretical obstructions for the simplest hypothesis $log(p) = q_1(p)exp\left[q_2(p)\right]/\pi$, with $q_2(p_1) \neq q_2(p_2)$ for all pairs of primes. The existence of p^{iy} in a finite-dimensional extension would require that α_i is proportional to π by a coefficient which for a given prime p_1 has sufficiently small p-adic norm so that the exponent can be expanded in powers series.

Obviously p-adicization gives strong quantization conditions. There is also a second possibility. As discussed in the same chapter, the allowance of infinite primes changes the situation. Let $X = \prod p_i$ be the product of all finite primes. $1+X$ is the simplest infinite prime and the ratio $Y = X/(1+X)$ equals to 1 in real sense and has p-adic norm $1/p$ for all finite primes. If one allows α to be proportional to a power Y, then the p-adic norm of α can be so small for all primes that the expansion converges without further conditions. Infinite primes will be discussed later in more detail.

Exactly similar exponents (p^{iy}) appear in the partition function decomposition of the Riemann Zeta, and the requirement that these quantities exist in a finite algebraic extension of p-adic numbers for the zeros $z = 1/2 + iy$ of ζ requires that $e^{ilog(p)y}$ is in a finite-dimensional extension involving algebraic numbers and e. One could argue that for the extensions of p-adics the zeros of Zeta define a universal spectrum of the eigen values of the Lorentz boost generator. This might have implications in hadron physics, where the so called rapidity distribution correspond to the distributions of the particles with respect to the variable characterizing finite Lorentz boosts.

Although the realization of the using the functions in Lorentz group differs from the discussed one, the conclusion is same also for them, in particular for the representation realized at the boundary of the light cone which is one of the homogenous spaces associated with Lorentz group.

7.1.2 Function basis of δM^4_+

One can consider two function basis for δM^4_+ and both function basis allow continuation to p-adic values under similar conditions.

1. Spherical harmonic basis

The first basis consists of functions $Y^l_m \times (r_M/r_0)^{n/2+i\rho}$, $n = -2, -1, 0, ...$ For $n = -2$ these functions define a unitary representation of Lorentz group. The spherical harmonics Y^l_m require a finite-dimensional algebraic extension of p-adic numbers. Radial part defines a logarithmic wave $exp[i\rho log(r_M/r_0)]$ and the existence of this for finite-dimensional extension of p-adic numbers for rational values ρ and r_M is guaranteed by $log(p) = q_1 exp(q_2)/\pi$ ansatz under the conditions already discussed.

2. Basis consisting of eigen functions of angular momentum and boost

Another function basis of δM^4_+ defining a non-unitary representation of Lorentz group and of conformal algebra consists of eigen states of rotation generator and Lorentz boost and is given by

$$f_{m,n,k} = e^{im\phi}\frac{\rho^{n-k}}{(1+\rho^2)^k} \times (\frac{r_M}{r_0})^k \ . \qquad (46)$$

$n = n_1 + in_2$ and $k = k_1 + ik_2$ are in general complex numbers. The condition

$$n_1 - k_1 \geq 0$$

is required by regularity at the origin of S^2. The requirement that the integral over S^2 defining norm exists (the expression for the differential solid angle is $d\Omega = \frac{\rho}{1+\rho^2)^2} d\rho d\phi$) implies

$$n_1 < 3k_1 + 2 \ .$$

From the relationship $(cos(\theta), sin(\theta)) = (\rho^2 - 1)/(\rho^2 + 1), 2\rho/(\rho^2 + 1))$ one can conclude that for $n_2 = k_2 = 0$ the representation functions are proportional to f $sin(\theta)^{n-k}(cos(\theta) - 1)^{n-k}$. Therefore they have in their decomposition to spherical harmonics only spherical harmonics with angular momentum $l < 2(n - k)$. This suggests that the condition

$$|m| \leq 2(n - k) \tag{47}$$

is satisfied quite generally.

The emergence of the three quantum numbers (m, n, k) can be understood. Light cone boundary can be regarded as a coset space $SO(3, 1)/E^2 \times SO(2)$, where $E^2 \times SO(2)$ is the group leaving the light like vector defined by a particular point of the light cone invariant. The natural choice of the Cartan group is therefore $E^2 \times SO(2)$. The three quantum numbers (m, n, k) have interpretation as quantum numbers associated with this Cartan algebra. The representations of the Lorentz group are characterized by half-integer valued parameter $l_0 = m/2$ and complex parameter l_1. Thus k_2 and n_2, which are Lorentz invariants, might not be independent parameters, and the simplest option is $k_2 = n_2$.

It is interesting to compare the representations in question to the unitary representations of Lorentz group discussed in [19].

1. The unitary representations discussed in [19] are characterized by are constructed by deducing the explicit representations for matrix elements of the rotation generators J_x, J_y, J_z and boost generators L_x, L_y, L_z by decomposing the representation into series of representations of $SU(2)$ defining the isotropy subgroup of a time like momentum. Therefore the states are labelled by eigenvalues of J_z. In the recent case the isotropy group is $E^2 \times SO(2)$ leaving light like point invariant. States are therefore labelled by three different quantum numbers.

2. The representations of [19] are realized the space of complex valued functions of complex coordinates ξ and $\overline{\xi}$ labelling points of complex plane. These functions have complex degrees $n_+ = m/2 - 1 + l_1$ with respect to ξ and $n_- = -m/2 - 1 + l_1$ with respect to $\overline{\xi}$. l_0 is complex number in the general case but for unitary representations of main series it is given by $l_1 = i\rho$ and for the representations of supplementary series l_1 is real and satisfies $0 < |l_1| < 1$. The main series representation is derived from a representation space consisting of homogenous functions of variables z^0, z^1 of degree n_+ and of $\overline{z}^0$ and $\overline{z}^1$ of degrees $n_\pm$. One can separate express these functions as product of $(z^1)^{n^+}$ $(\overline{z}^1)^{n_-}$ and a polynomial of $\xi = z^1/z^2$ and $\overline{\xi}$ with degrees n_+ and n_-. Unitarity reduces to the requirement that the integration measure of complex plane is invariant under the Lorentz transformations acting as Moebius transformations of the complex plane. Unitarity implies $l_1 = -1 + i\rho$.

3. For the representations at δM^4_+ unitarity reduces to the requirement that the integration measure of $r_M^2 d\Omega dr_M/r_M$ of δM^4_+ remains invariant under Lorentz transformations. The action of Lorentz transformation on the complex coordinates of S^2 induces a conformal scaling which can be compensated by an S^2 local radial scaling. At least formally the function space of δM^4_+ thus defines a unitary representation. For the function basis f_{mnk} $k = -1 + i\rho$ defines a candidate for a unitary representation since the logarithmic waves in the radial coordinate are completely analogous to plane waves. This condition would be completely analogous to the vanishing of conformal weight for the physical states of super

conformal representations. The problem is that for $k_1 = -1$ guaranteing square integrability in S^2 implies $-2 < n_1 < -2$ so that unitarity in this sense is not possible.

There is no deep reason against non-unitary representations and symmetric space structure indeed requires that k_1 is half-integer valued. First of all, configuration space spinor fields are analogous to ordinary spinor fields in M^4, which also define non-unitary representations of Lorentz group. Secondly, if 3-surfaces at the light cone boundary are finite-sized, the integrals defined by f_{mnk} over 3-surfaces Y^3 are always well-defined. Thirdly, the continuous spectrum of k_2 could be transformed to a discrete spectrum when k_1 becomes half-integer valued.

7.1.3 Logarithmic waves and possible connections with number theory and fundamental physics

Logarithmic plane waves labelled by eigenvalues of the scaling momenta appear also in the definition of the Riemann Zeta defined as $\zeta(z) = \sum_n n^{-z}$, n positive integer [E8]. Riemann Zeta is expressible as a product of partition function factors $1/(1+p^{-x-iy})$, p prime and the powers n^{-x-iy} appear as summands in Riemann Zeta. Riemann hypothesis states that the non-trivial zeros of Zeta reside at the line $x = 1/2$. There are indeed intriguing connections. $Log(p)$ corresponds now to the $log(r_M/r_{min})$ and -x-iy corresponds to the scaling momentum $k_1 + ik_2$ so that the special physical role of the conformal weights $k_1 = 1/2 + iy$ corresponds to Riemann hypothesis. The appearance of powers of p in the definition of the Riemann Zeta corresponds to p-adic length scale hypothesis, ($r_M/r_0 = p$ in ζ and corresponds to a secondary p-adic length scale).

The assumption that the logarithmic plane waves are algebraically continuable from the rational points $r_M/r_{min} = m/n$ to p-adic plane waves using a finite-dimensional extension of p-adic numbers leads to the $log(p) = q_1 exp(q_2)/\pi$ ansatz. Similar hypothesis is inspired by the hypothesis that Riemann Zeta is a universal function existing simultaneously in all number fields. This inspires several interesting observations.

1. p-adic length scale hypothesis stating that $r_{max}/r_{min} = p^n$ is consistent with the number theoretical universality of the logarithmic waves. The universality of Riemann Zeta inspires the hypothesis that the zeros of Riemann Zeta correspond to rational numbers and to preferred values $k_1 + ik_2$ of the scaling momenta appearing in the logarithmic plane waves. In the recent context the most general hypothesis would be that the allowed momenta k_2 correspond to the linear combinations of the zeros of Riemann Zeta with integer coefficients.

2. Hardmuth Mueller [22] claims on basis of his observations that gravitational interaction involves logarithmic radial waves for which the nodes come as $r/r_{min} = e^n$. This is true if the the scaling momenta k_2 satisfy the condition $k_2/\pi \in Z$. Perhaps Mueller's logarithmic waves really could be seen as a direct signature of the fundamental symmetries of the configuration space. In particular, this would require $r_{max}/r_{min} = e^m$.

3. The special role of Golden Mean $\Phi = (1+\sqrt{5})/2$ in Nature could be understood if also $log(\Phi) = q_1 exp(q_2)/\pi$ or more general ansatz holds true. This would imply that the nodes of logarithmic waves can correspond also to the powers of Φ.

   One could of course argue that the number theory at the moment of Big Bang cannot have strong effects on what is observed in laboratory. This might be the case. On the other hand, the non-determinism of the Kähler action however strongly suggests that the construction of the configuration space geometry involves all possible light like 3-surfaces of the future light cone so that logarithmic waves would appear in all length scales. Be as it may, it would be amazing if such an abstract mathematical structure as configuration space geometry would have direct implications to cosmology and to the physics of living systems.

7.2 Configuration space integration

Assuming that U-matrix exists simultaneously in all number fields (allowing finite-dimensional extensions of p-adics), the immediate question is whether also the construction procedure of the real S-matrix could have a p-adic counterpart for each p, and whether the mere requirement that this is the case could provide non-trivial intuitions about the general structure of the theory. Not only the configuration space but also Kähler function and its exponent, Kähler metric, and configuration space functional integral should have p-adic variants. In the following this challenge is discussed in a rather optimistic number theoretic mood using the ideas stimulated by the connections between number theory and cognition.

7.2.1 Does symmetric space structure allow algebraization of configuration space integration?

The basic structure is the rational configuration space whose points have rational valued coordinates. This space is common to both real and p-adic variants of the configuration space. Therefore the construction of the generalized configuration space as such is not a problem.

The assumption that configuration space decomposes into a union of symmetric spaces labelled by zero modes means that the left invariant metric for each space in the union is dictated by isometries. It should be possible to interpret the matrix elements of the configuration space metric in the basis of properly normalized isometry currents as p-adic numbers in some finite extension of p-adic numbers allowing perhaps also some transcendentals. Note that the Kähler function is proportional to the inverse of Kähler coupling strength α_K which depends on p-adic prime p, and does seem to be a rational number if one takes seriously various arguments leading to the hypothesis $1/\alpha_K = klog(K^2)$, $K^2 = p \times 2 \times 3 \times 5.. \times 23$, and $k = \pi/4$ or $k = 137/107$ for the two alternative options discussed in [E8]. If so then the most general transcendentals required and allowed in the extensions used correspond to roots of polynomials with coefficients in an extension of rationals by e and algebraic numbers. As already discussed, infinite primes might provide the ultimate solution to the problem of continuation.

The continuation of the exponent of Kähler function and of configuration space spinor fields to p-adic sectors would require some selection of a subset of points of the rational configuration space. On the other hand, the minimum requirement is that it is possible to define configuration space integration in the p-adic context. The only manner to achieve this is by defining configuration space integration purely algebraically by perturbative expansion. For free field theory Gaussian integrals are in question and one can calculate them trivially. The Gaussian can be regarded as a Kähler function of a flat Kähler manifold having maximal translational and rotational symmetries. Physically infinite number of harmonic oscillators are in question. The origin of the symmetric space is preferred point as far as Kähler function is considered: metric itself is invariant under isometries.

7.2.2 Algebraization of the configuration space functional integral

Configuration space is a union of infinite-dimensional symmetric spaces labelled by zero modes. One can hope that the functional integral could be performed perturbatively around the maxima of the Kähler function. In the case of CP_2 Kähler function has only single maximum and is a monotonically decreasing function of the radial variable r of CP_2 and thus defines a Morse function. This suggests that a similar situation is true for all symmetric spaces and this might indeed be the case. The point is that the presence of several maxima implies also saddle points at which the matrix defined by the second derivatives of the Kähler function is not positive definite. If the derivatives of type $\partial_K \partial_L K$ and $\partial_{\overline{K}} \partial_{\overline{L}} K$ vanish at the saddle point (this is the crucial assumption) in some complex coordinates holomorphically related to those in which the same holds true at maximum, the Kähler metric is not positive definite at this point. On the other hand, by symmetric space property the metric should be isometric with the positive define metric at maxima so that a contradiction results.

If this argument holds true, for given values of zero modes Kähler function has only one maximum, whose value depends on the values zero modes. Staying in the optimistic mood, one could go on to guess that the Duistermaat-Heckman theorem [20] generalizes and the functional integral is simply the exponent of the Kähler function at the maximum (due to the compensation of Gaussian and metric determinants). Even more, one could bravely guess that for configuration space spinor fields belonging to the representations of symmetries the inner products reduces to the generalization of correlation functions of Gaussian free field theory. Each configuration space spinor field would define a vertex from which lines representing the propagators defined by the contravariant configuration space metric in isometry basis emanate.

If this optimistic line of reasoning makes sense, the definition of the p-adic configuration space integral reduces to a purely algebraic one. What is needed is that the contravariant Kähler metric fixed by the symmetric space-property exists and that the exponent of the maximum of the Kähler function exists for rational values of zero modes or subset of them if finite-dimensional algebraic extension is allowed. This would give could hopes that the U-matrix elements resulting from the configuration space integrals would exist also in the p-adic sense.

7.3 Are the exponential of the Kähler function and reduced Kähler action rational functions?

The simplest possibilities one can imagine are that the exponent e^{2K} of Kähler function appearing in the configuration space inner products is a rational or at most a simple algebraic function existing in a finite-dimensional algebraic extension of p-adic numbers. One could also require that the reduced Kähler action without the $1/4\pi\alpha_K$ factor, which affects in no manner the dynamics of the absolute minimization, is a rational function.

1. Is e^{2K} a rational function?

The exponent of the CP_2 Kähler function is a rational function of the standard complex coordinates and thus rational-valued for all rational values of complex CP_2 coordinates. Therefore one is lead to ask whether this property holds true quite generally for symmetric spaces and even in the infinite-dimensional context. If so, then the continuation of the vacuum functional to the p-adic sectors of the configuration space would be possible in the entire configuration space. Also the spherical harmonics of CP_2 are rational functions containing square roots in normalization constants. That also configuration space spinor fields could use rational functions containing square roots as normalization constant as basic building blocks would conform with general number theoretical ideas as well as with the general features of harmonic oscillator wave functions.

This idea is supported by the earlier work. Various arguments for the p-adic evolution of the Kähler coupling strength imply that the exponent of the Kähler function for CP_2 type extremal is a rational number being the product $K^2 = p \times 2 \times 3 \times 5... \times 23$. In this case the Kähler coupling strength is $1/\alpha_K = (4/\pi) \times log(K^2)$. α_K. The general number theoretical conjectures implied by p-adic physics and physics of cognition and intention state that this is the case. Although one must take these arguments with a big grain of salt, the general idea might be correct. Also the elements of the configuration space metric would be rational functions as is clear from the fact that one can express the second derivatives of the Kähler function in terms of $F = exp(K)$ as

$$\partial_K \partial_{\overline{L}} K = \frac{\partial_K \partial_{\overline{L}} F}{F} - \frac{\partial_K F \partial_{\overline{L}} F}{F^2} \quad .$$

2. Is reduced Kähler action a rational function?

Kähler coupling strength does not appear at all in the field equations for the extremals of the Kähler action. Therefore one could argue that also the reduced Kähler action $S_R(X^4(X^3))$ defined as $S_R = \int J^{\mu\nu}J_{\mu\nu}\sqrt{g}d^4x$ should be rational valued, when X^3 corresponds to a rational point of the configuration space (including zero modes). On the other hand, also the exponent $e^{2K} = e^{S_R/8\pi\alpha_K}$ appearing in configuration space inner products should be rational valued for rational points of the configuration space.

If one takes seriously the conjecture $1/\alpha_K = (4/\pi) \times log(K^2)$, $K^2 = p \times \prod_{q=2,3,...23} q$, one one can write the exponent of Kähler function as

$$e^K = \left[p \times \prod_{q=2,3,...23} q\right]^{S_R/2\pi^2} .$$

This corresponds numerically to $1\alpha_K \simeq 136.5585$ giving $\alpha/\alpha_K \simeq .9965$.

The rational-valuedness for the e^{2K} appearing in the configuration space inner products would require a quantization of the absolute minimum of the Kähler action as integer multiples of CP_2 action

$$S_R = n \times S_R(CP_2) = n \times 2\pi^2 \ , \tag{48}$$

where n is integer. Needless to say, the quantization of the absolute minimum of Kähler action as multiples of CP_2 Kähler action would be quite a dramatic implication and would correspond to the basic idea of the first days of the quantum theory about quantization of action. Probably something like this is too much to hope for and would probably have un-physical consequences.

The milder assumption that the exponent of Kähler function has values in a finite-dimensional algebraic extension requires that S_R is rational multiple of CP_2 reduced Kähler action $S_R(CP_2) = 2\pi^2$: this would require an extension generated by a finite root of p. That CP_2 action would serve as a universal unit for the absolute minima associated with the rational valued zero modes, sounds reasonable.

An alternative option is based on the fact that e^p is always an ordinary p-adic number in R_p so that powers of $e^{m/n}$ for a given value of n exist always in a finite-dimensional transcendental extension of p-adic numbers. In this case not only rational values of Kähler function but all values which are rationally proportional to an n:th root of e would be possible. The proportionality of the Kähler function to a power of e requires much less than the proportionality to a rational number.

One can consider also an alternative ansatz based on the requirement that Kähler function is rational number rather than a logarithm of a power of integer K^2. This requires an extension of p-adic numbers involving some root of e and a finite number of its powers. The reduced action S_R must be rational valued using Kähler action $S_K(CP_2) = 2\pi^2$ of CP_2 type extremal as a basic unit. In fact, not only rational values of Kähler function but all values which differ from a rational value by a perturbation with a p-adic norm smaller than one and rationally proportional to a power of e or even its root exist p-adically in this case if they have small enough p-adic norm. The most general perturbation of the action is in the field defined by the extension of rationals defined by the root of e and algebraic numbers.

Since CP_2 action is rationally proportional to π^2, the exponent is rational if $g_K^2 = 4\pi\alpha_K$ is also proportional to π^2. If the $log(p) = q_1exp(q_2)/\pi$ ansatz holds true for every prime, then the earlier ansatz $1/\alpha_K(p) = (4/\pi)log(K^2)$ does not guarantee this, and $4/\pi$ must be replaced with a rational number $q \simeq 4/\pi$. The presence of $log(K^2)$, K^2 product of primes, is well motivated also in this case because it gives the desired $1/\pi$ factor. The replacement is also supported by the proposal that Kähler action can be defined as a fermionic effective action using ζ function regularization [B4].

Since $k = 137$ (atomic length scale) and $k = 107$ (hadronic length scale) are the most important nearest p-adic neighbors of electron, one could make a free fall into number mysticism and try the replacement $4/\pi \rightarrow 137/107$. This would give $\alpha_K = 137.3237$ to be compared with $\alpha = 137.0360$: the deviation from α is .2 per cent (of course, α_K need not equal to α and the evolutions of these couplings are quite different). Thus it seems that $log(p) = q_1 exp(q_2)/\pi$ hypothesis is supported also by the properties of Kähler action and leads to an improved understanding of the origin of the mystery prime $k = 137$.

7.3.1 Could infinite primes appear in the p-adicization of the exponent of Kähler action?

The difficulties related to the p-adic continuation of Kähler function to an arbitrary p-adic number field and the fact that infinities are every day life in quantum field theory bring in mind infinite primes discussed in [E3].

Infinite primes are not divisible by any finite prime. The simplest infinite prime is of form $\Pi = 1 + X$, $X = \prod_i p_i$, where product is over all finite primes. The factor $Y = X/(1 + X)$ is in the real sense equivalent with 1. In p-adic sense it has norm $1/p$ for every prime. Thus one could multiply Kähler function by Y or its positive power in order to guarantee that the continuation to p-adic number fields exists for all primes. Of course, these states might differ physically in p-adic sense from the states having $Y = 1$. Thus it would seem that the physics of cognition could differentiate between states which are in real sense equivalent.

More general infinite primes are of form $\Pi = mX/s + ns$, $s = \prod_i q_i$, $n = \prod_i q_i^{m_i}$ such that $m = \prod_i p_i^{n_i}$ and s have no common factors. The interpretation could be as a counterpart for a state of a super-symmetric theory containing fermion and m_i bosons in the mode labelled by q_i and n_i bosons in the mode labelled by p_i. Also positive powers of the ratio $Y = X/\Pi$, Π some infinite prime, are possible as a multiplier of the Kähler function. In the real sense this ratio would correspond to the ratio m/n.

If this picture is correct, infinite primes would emerge naturally in the p-adicization of the theory. Since octonionic infinite primes could correspond to the states of a super-symmetric quantum field theory more or less equivalent with TGD, the presence of infinite primes could make it possible to code the quantum physical state to the vacuum functional via coupling constant renormalization.

One could also consider the possibility of defining functions like $exp(x)$ and $log(1+x)$ p-adically by replacing x with Yx without introducing the algebraic extension. The series would converge for all values of x also p-adically and would be in real sense equivalent with the function. This trick would apply to a very general class of Taylor series having rational coefficients. One could also say that p-adic physics allowing infinite primes would be very similar to real physics.

The fascination of infinite primes is that the ratios of infinite primes which are ordinary rational numbers in the real sense could code the particle number content of a super-symmetric arithmetic quantum field theory. For the octonic version of the theory natural in the TGD framework these states could represent the states of a real Universe. Universe would be an algebraic hologram in the sense that space-time points, something devoid of any structure in the standard view, could code for the quantum states of possible Universes!

The simplest manner to realize this scenario is to consider an extension of rational numbers by the multiplicative group of real units obtained from infinite primes and powers of X. Real number 1 would code everything in its structure! This group is generated as products of powers of $Y(m/n) = (m/n) \times [X/\Pi(m/n)]$ which is a unit in the real sense. Each $Y(m/n)$ would define a subgroup of units and the power of $Y(m/n)$ would code for the number of factors of a given integer with unit counted as a factor. This would give a hierarchy of integers with their p-adic norms coming as powers of p with the prime factors of m and n forming an exception and being reflected in p-adic physics of cognition, Universe would "feel" its real or imagined state with its every point, be it a point of space-time surface, of imbedding space, or of configuration space.

7.3.2 Coupling constant evolution and number theory

The coupling constant evolution associated with the Kähler action might be at least partially understood number-theoretically.

A given space-time sheet is connected by wormhole contacts to the larger space-time sheets. The induced metric within the wormhole contact has an Euclidian signature so that the wormhole contact is surrounded by elementary particle horizons at which the metric is degenerate so that the horizons are metrically effectively 2-dimensional giving rise to quaternion conformal invariance. Because of the causal horizon it would seem that Kähler coupling strength can depend on the space-time sheet via the p-adic prime characterizing it. If so the exponent of the Kähler function would be simply the product of the exponents for the space-time sheets and one would have finite-dimensional extension as required.

If the exponent of the Kähler function is rational function, also the components of the contravariant Kähler metric are rational functions. This would suggest that one function of the coupling constant evolution is to keep the exponent rational.

From the point of view of p-adicization the ideal situation results if Kähler coupling strength is invariant under the p-adic coupling constant evolution as I believed originally. For a long time it however seemed that this option cannot be realized since the prediction $G = L_p^2 exp(-2S_K(CP_2))$ for the gravitational coupling constant following from dimensional considerations alone implies that G increases without limit as a function of p-adic length scale if α_K is RG invariant. If one however assumes that bosonic space-time sheets correspond to Mersenne primes, situation changes since M_{127} defining electron length scale is the largest Mersenne prime for which p-adic length scale is not super-astronomical and thus excellent candidate for characterizing gravitonic space-time sheets. There is much stronger motivation for this hypothesis coming from the fact that a nice picture about evolution of electro-weak and color coupling strengths emerges just from the physical interpretation of the fact that classical color action and electro-weak $U(1)$ action are proportional to Kähler action [A8].

7.3.3 Consistency check in the case of CP_2

It is interesting to look whether this vision works or fails in a simple finite-dimensional case. For CP_2 the Kähler function is given by $K = -log(1 + r^2)$. This function exists if an extension containing the logarithms of primes is used. $log(1 + x)$, $x = O(p)$ exists as an ordinary p-adic number and a logarithm of $log(m)$, $m < p$ such that the powers of m span the numbers $1, ..., p-1$ besides $log(p)$ should be introduced to the extension in order that logarithm of any integer and in fact of any rational number exists p-adically. Also logarithms of roots of integers and their products would exist. The problem is however that the powers of $log(m)$ and $log(p)$ would generate an infinite-dimensional extension since finite-dimensional extension leads to a contradiction as shown in [E8].

The exponent of Kähler function as well as Kähler metric and Kähler form have rational-valued elements for rational values of the standard complex coordinates for CP_2. The exponent of the Kähler function is $1/(1+r^2)$ and exists as a rational number at 3-spheres of rational valued radius. The negative of the Kähler function has a single maximum at $r = 0$ and vanishes at the coordinate singularity $r \to \infty$, which corresponds to the geodesic sphere S^2.

If one wants to cognize about geodesic length, areas of geodesic spheres, and about volume of CP_2, π must be introduced to the extension of p-adics and means infinite-dimensional extension by the arguments of [E8]. The introduction of π is not however necessary for introducing of spherical coordinates if one expresses everything in terms of trigonometric functions. For ordinary spherical coordinates this means effectively replacing θ and ϕ by $u = \theta/\pi$ and $v = \phi/2\pi$ as coordinates. By allowing u and v to have a finite number of rational values requires only the introduction of a finite-dimensional algebraic extension in order to define cosines and sines of the angle variables at these values. What seems clear is that the evolution of cognition as the emergence of higher-dimensional extensions corresponds quite concretely to the emergence of finer discretizations.

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