TGD as a Generalized Number Theory III: Infinite Primes

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Abstract

Infinite primes are besides p-adicization and the representation of space-time surface as a hyper-quaternionic sub-manifold of hyper-octonionic space, basic pillars of the vision about TGD as a generalized number theory and will be discussed in the third part of the multi-chapter devoted to the attempt to articulate this vision as clearly as possible.

1. Why infinite primes are unavoidable

Suppose that 3-surfaces could be characterized by p-adic primes characterizing their effective p-adic topology. p-Adic unitarity implies that each quantum jump involves unitarity evolution U followed by a quantum jump. Simple arguments show that the p-adic prime characterizing the 3-surface representing the entire universe increases in a statistical sense. This leads to a peculiar paradox: if the number of quantum jumps already occurred is infinite, this prime is most naturally infinite. On the other hand, if one assumes that only finite number of quantum jumps have occurred, one encounters the problem of understanding why the initial quantum history was what it was. Furthermore, since the size of the 3-surface representing the entire Universe is infinite, p-adic length scale hypothesis suggest also that the p-adic prime associated with the entire universe is infinite.

These arguments motivate the attempt to construct a theory of infinite primes and to extend quantum TGD so that also infinite primes are possible. Rather surprisingly, one can construct what might be called generating infinite primes by repeating a procedure analogous to a quantization of a super symmetric quantum field theory. At given level of hierarchy one can identify the decomposition of space-time surface to p-adic regions with the corresponding decomposition of the infinite prime to primes at a lower level of infinity: at the basic level are finite primes for which one cannot find any formula.

2. Two views about the role of infinite primes and physics in TGD Universe

Two different views about how infinite primes, integers, and rationals might be relevant in TGD Universe have emerged.

a) The first view is based on the idea that infinite primes characterize quantum states of the entire Universe. 8-D hyper-octonions make this correspondence very concrete since 8-D hyper-octonions have interpretation as 8-momenta. By quantum-classical correspondence also the decomposition of space-time surfaces to p-adic space-time sheets should be coded by infinite hyper-octonionic primes. Infinite primes could even have a representation as hyper-quaternionic 4-surfaces of 8-D hyper-octonionic imbedding space.

b) The second view is based on the idea that infinitely structured space-time points define space-time correlates of mathematical cognition. The mathematical analog of Brahman=Atman identity would however suggest that both views deserve to be taken seriously.

3. Infinite primes and infinite hierarchy of second quantizations

The discovery of infinite primes suggested strongly the possibility to reduce physics to number theory. The construction of infinite primes can be regarded as a repeated second quantization of a super-symmetric arithmetic quantum field theory. Later it became clear that the process generalizes so that it applies in the case of quaternionic and octonionic primes and their hyper counterparts. This hierarchy of second quantizations means an enormous generalization of physics to what might be regarded a physical counterpart for a hierarchy of abstractions about abstractions about.... The ordinary second quantized quantum physics corresponds only to the lowest level infinite primes. This hierarchy can be identified with the corresponding hierarchy of space-time sheets of the many-sheeted space-time.

One can even try to understand the quantum numbers of physical particles in terms of infinite primes. In particular, the hyper-quaternionic primes correspond four-momenta and mass squared is prime valued for them. The properties of 8-D hyper-octonionic primes motivate the attempt to identify the quantum numbers associated with CP_2 degrees of freedom in terms of these primes. The representations of color group $SU(3)$ are indeed labelled by two integers and the states inside given representation by color hyper-charge and iso-spin.

It turns out that associativity constraint allows only rational infinite primes. One can however decompose rational infinite primes to hyper-octonionic infinite primes at lower level of the hierarchy. Physically this would mean that the number theoretic 8-momenta have only time-component. This decomposition is completely analogous to the decomposition of hadrons to its colored constituents and might be even interpreted in terms of color confinement. The interpretation of the decomposition of rational primes to primes in the algebraic extensions of rationals, hyper-quaternions, and hyper-octonions would have an interpretation as an increase of number theoretical resolution and the principle of number theoretic confinement could be seen as a fundamental physical principle implied by associativity condition.

4. Infinite primes as a bridge between quantum and classical

An important stimulus came from the observation stimulated by algebraic number theory. Infinite primes can be mapped to polynomial primes and this observation allows to identify completely generally the spectrum of infinite primes whereas hitherto it was possible to construct explicitly only what might be called generating infinite primes.

This in turn led to the idea that it might be possible represent infinite primes (integers) geometrically as surfaces defined by the polynomials associated with infinite primes (integers).

Obviously, infinite primes would serve as a bridge between Fock-space descriptions and geometric descriptions of physics: quantum and classical. Geometric objects could be seen as concrete representations of infinite numbers providing amplification of infinitesimals to macroscopic deformations of space-time surface. We see the infinitesimals as concrete geometric shapes!

5. Conjecture about various equivalent characterizations of space-times as surfaces

One can imagine several number-theoretic characterizations of the space-time surface.

a) The approach based on octonions and quaternions suggests that space-time surfaces correspond to associative, or equivalently, hyper-quaternionic surfaces of hyper-octonionic imbedding space HO. Also co-associative, or equivalently, co-hyper-quaternionic surfaces are possible. These foliations can be mapped in a natural manner to the foliations of $H = M^4 \times CP_2$ by space-time surfaces which are identified as preferred extremals of the Kähler action (absolute minima or maxima for regions of space-time surface in which action density has definite sign). These views are consistent if hyper-quaternionic space-time surfaces correspond to so called Kähler calibrations [E2].

b) Hyper-octonion real-analytic surfaces define foliations of the imbedding space to hyper-quaternionic 4-surfaces and their duals to co-hyper-quaternionic 4-surfaces representing space-time surfaces.

c) Rational infinite primes can be mapped to rational functions of n arguments. For hyper-octonionic arguments non-associativity makes these functions poorly defined unless one assumes that arguments are related by hyper-octonion real-analytic maps so that only single independent variable remains. These hyper-octonion real-analytic functions define foliations of HO to space-time surfaces if b) holds true.

The challenge of optimist is to prove that these characterizations are equivalent.

6. The representation of infinite hyper-octonionic primes as 4-surfaces

The difficulties caused by the Euclidian metric signature of the number theoretical norm forced to give up the idea that space-time surfaces could be regarded as quaternionic sub-manifolds of octonionic space, and to introduce complexified octonions and quaternions resulting by extending quaternionic and octonionic algebra by adding imaginary units multiplied with $\sqrt{-1}$. This spoils the number field property but the notion of prime is not lost. The sub-space of hyper-quaternions *resp.* -octonions is obtained from the algebra of ordinary quaternions and octonions by multiplying the imaginary part with $\sqrt{-1}$. The transition is the number theoretical counterpart for the transition from Riemannian to pseudo-Riemannin geometry performed already in Special Relativity.

The commutative $\sqrt{-1}$ relates naturally to the algebraic extension of rationals generalized to an algebraic extension of rational quaternions and octonions and conforms with the vision about how quantum TGD could emerge from infinite dimensional Clifford algebra identifiable as a hyper-finite factor of type II_1 [C6, A9].

The notions of hyper-quaternionic and octonionic manifold make sense but it is implausible that $H = M^4 \times CP_2$ could be endowed with a hyper-octonionic manifold structure. Indeed, space-time surfaces can be assumed to be hyper-quaternionic or co-hyper-quaternionic 4-surfaces of 8-dimensional Minkowski space M^8 identifiable as the hyper-octonionic space HO. Since the hyper-quaternionic sub-spaces of HO with a locally fixed complex structure (preferred imaginary unit contained by tangent space at each point of HO) are labelled by CP_2, each (co)-hyper-quaternionic four-surface of HO defines a 4-surface of $M^4 \times CP_2$. One can say that the number-theoretic analog of spontaneous compactification occurs.

Any hyper-octonion analytic function $HO \rightarrow HO$ defines a function $g : OH \rightarrow SU(3)$ acting as the group of octonion automorphisms leaving a preferred imaginary unit invariant, and g in turn defines a foliation of HO and $H = M^4 \times CP_2$ by space-time surfaces. The selection can be local which means that G_2 appears as a local gauge group.

Since the notion of prime makes sense for the complexified octonions, it makes sense also for the hyper-octonions. It is possible to assign to infinite prime of this kind a hyper-octonion analytic polynomial $P : HO \rightarrow HO$ and hence also a foliation of HO and $H = M^4 \times CP_2$ by 4-surfaces. Therefore space-time surface can be seen as a geometric counterpart of a Fock state. The assignment is not unique but determined only up to an element of the local octonionic automorphism group G_2 acting in HO and fixing the local choices of the preferred imaginary unit of the hyper-octonionic tangent plane. In fact, a map $HO \rightarrow S^6$ characterizes the choice since $SO(6)$ acts effectively as a local gauge group.

The construction generalizes to all levels of the hierarchy of infinite primes if one poses the associativity requirement implying that hyper-octonionic variables are related by hyper-octonion real-analytic maps, and produces also representations for integers and rationals associated with hyper-octonionic numbers as space-time surfaces. A close relationship with algebraic geometry results and the polynomials define a natural hierarchical structure in the space of 3-surfaces. By the effective 2-dimensionality naturally associated with infinite primes represented by real polynomials 4-surfaces are determined by data given at partonic 2-surfaces defined by the intersections of 3-D and 7-D light-like causal determinants. In particular, the notions of genus and degree serve as classifiers of the algebraic geometry of the 4-surfaces. The great dream is to prove that this construction yields the solutions to the absolute minimization of Kähler action.

7. Generalization of ordinary number fields: infinite primes and cognition

Both fermions and p-adic space-time sheets are identified as correlates of cognition in TGD Universe. The attempt to relate these two identifications leads to a rather concrete model for how bosonic generators of super-algebras correspond to either real or p-adic space-time sheets (actions and intentions) and fermionic generators to pairs of real space-time sheets and their p-adic variants obtained by algebraic continuation (note the analogy with fermion hole pairs).

The introduction of infinite primes, integers, and rationals leads also to a generalization of real numbers since an infinite algebra of real units defined by finite ratios of infinite rationals multiplied by ordinary rationals which are their inverses becomes possible. These units are not units in the p-adic sense and have a finite p-adic norm which can be differ from one. This construction generalizes also to the case of hyper- quaternions and -octonions although non-commutativity and in case of octonions also non-associativity pose technical problems to which the reduction to ordinary rational is simplest cure which would however allow interpretation as decomposition of infinite prime to hyper-octonionic lower level constituents. Obviously this approach differs from the standard introduction of infinitesimals in the sense that sum is replaced by multiplication meaning that the set of real units becomes infinitely degenerate.

Infinite primes form an infinite hierarchy so that the points of space-time and imbedding space can be seen as infinitely structured and able to represent all imaginable algebraic structures. Certainly counter-intuitively, single space-time point is even capable of representing the quantum state of the entire physical Universe in its structure. For instance, in the real sense surfaces in the space of units correspond to the same real number 1, and single point, which is structure-less in the real sense could represent arbitrarily high-dimensional spaces as unions of real units.

One might argue that for the real physics this structure is completely invisible and is relevant only for the physics of cognition. On the other hand, one can consider the possibility of mapping the configuration space and configuration space spinor fields to the number theoretical anatomies of a single point of imbedding space so that the structure of this point would code for the world of classical worlds and for the quantum states of the Universe. Quantum jumps would induce changes of configuration space spinor fields interpreted as wave functions in the set of number theoretical anatomies of single point of imbedding space in the ordinary sense of the word, and evolution would reduce to the evolution of the structure of a typical space-time point in the system. Physics would reduce to space-time level but in a generalized sense. Universe would be an algebraic hologram, and there is an obvious connection both with Brahman=Atman identity of Eastern philosophies and Leibniz's notion of monad.

1 Introduction

The third part of the multi-chapter discussing the idea about physics as a generalized number theory is devoted to the possible role of infinite primes in TGD.

The notion of prime seems to capture something very essential about what it is to be elementary building block of matter and has become a fundamental conceptual element of TGD. The notion of prime gains it generality from its reducibility to the notion of prime ideal of an algebra. Thus the notion of prime is well-defined, not only in case of quaternions and octonions, but also in the case of hyper-quaternions and -octonions, which are especially natural physically and for which numbers having zero norm correspond physically to light-like 8-vectors. Many interpretations for infinite primes have been competing for survival but it seems that the recent state of TGD allows to exclude some of them from consideration.

1.1 The notion of infinite prime

p-Adic unitarity implies that each quantum jump involves unitarity evolution U followed by a quantum jump to some sector D_p of the configuration space labelled by a p-adic prime. Simple arguments show that the p-adic prime characterizing the 3-surface representing the entire universe increases in a statistical sense. This leads to a peculiar paradox: if the number of quantum jumps already occurred is infinite, this prime is most naturally infinite. On the other hand, if one assumes that only finite number of quantum jumps have occurred, one encounters the problem of understanding why the initial quantum history was what it was. Furthermore, since the size of the 3-surface representing the entire Universe is infinite, p-adic length scale hypothesis suggest also that the p-adic prime associated with the entire universe is infinite.

These arguments motivate the attempt to construct a theory of infinite primes and to extend quantum TGD so that also infinite primes are possible. Rather surprisingly, one can construct infinite primes by repeating a procedure analogous to a quantization of a super symmetric quantum field theory. At given level of hierarchy one can identify the decomposition of space-time surface to p-adic regions representing selves with the corresponding decomposition of the infinite prime to primes at lower level of infinity: at the basic level are finite primes for which one cannot find any formula.

This and other observations suggest that the Universe of quantum TGD might basically provide a physical representation of number theory allowing also infinite primes. The proposal suggests also a possible generalization of real numbers to a number system akin to hyper-reals introduced by Robinson in his non-standard calculus [40] providing rigorous mathematical basis for calculus. In fact, some rather natural requirements lead to a unique generalization for the concepts of integer, rational and real. Somewhat surprisingly, infinite integers and reals can be regarded as infinite-dimensional vector spaces with integer and real valued coefficients respectively and this raises the question whether the tangent space for the configuration space of 3-surfaces could be regarded as the space of generalized 8-D hyper-octonionic numbers.

1.2 Generalization of ordinary number fields

The introduction of infinite primes, integers, and rationals leads also to a generalization of real numbers since an infinite algebra of real units defined by finite ratios of infinite rationals multiplied by ordinary rationals which are their inverses becomes possible. These units are not units in the p-adic sense and have a finite p-adic norm which can be differ from one. This construction generalizes also to the case of hyper- quaternions and -octonions although non-commutativity and in case of hyper-octonions also non-associativity pose technical problems. Obviously this approach differs from the standard introduction of infinitesimals in the sense that sum is replaced by multiplication meaning that the set of real units becomes infinitely degenerate.

1.3 Infinite primes and physics in TGD Universe

Several different views about how infinite primes, integers, and rationals might be relevant in TGD Universe have emerged.

1.3.1 Infinite primes, cognition, and intentionality

The correlation of infinite primes with cognition is the first fascinating possibility and this possibility has stimulated several ideas.

1. The hierarchy of infinite primes associated with algebraic extensions of rationals leading gradually towards algebraic closure of rationals would in turn define cognitive hierarchy corresponding to algebraic extensions of p-adic numbers.

2. Infinite primes form an infinite hierarchy so that the points of space-time and imbedding space can be seen as infinitely structured and able to represent all imaginable algebraic structures. Certainly counter-intuitively, single space-time point might be even capable of representing the quantum state of the entire physical Universe in its structure. For instance, in the real sense surfaces in the space of units correspond to the same real number 1, and single point, which is structure-less in the real sense could represent arbitrarily high-dimensional spaces as unions of real units. For real physics this structure is completely invisible and is relevant only for the physics of cognition. One can say that Universe is an algebraic hologram, and there is an obvious connection both with Brahman=Atman identity of Eastern philosophies and Leibniz's notion of monad.

3. One can assign to infinite primes at n^{th} level of hierarchy rational functions of n rational arguments which form a natural hierarchical structure in that highest level corresponds to a polynomial with coefficients which are rational functions of the arguments at the lower level. One can solve one of the arguments in terms of lower ones to get a hierarchy of algebraic extensions. At the lowest level algebraic extensions of rationals emerge, at the next level algebraic extensions of space of rational functions of single variable, etc... This would suggest that infinite primes code for the correlation between quantum states and the algebraic extensions appearing in their their physical description and characterizing their cognitive correlates. The hierarchy of infinite primes would also correlate with a hierarchy of logics of various orders (hierarchy of statements about statements about...).

1.3.2 Infinite primes and super-symmetric quantum field theory

Consider next the physical interpretation.

1. The discovery of infinite primes suggested strongly the possibility to reduce physics to number theory. The construction of infinite primes can be regarded as a repeated second quantization of a super-symmetric arithmetic quantum field theory. This suggests that configuration space spinor fields or at least the ground states of associated super-conformal representations could be mapped to infinite primes in both bosonic and fermionic degrees of freedom. The process might generalize so that it applies in the case of quaternionic and octonionic primes and their hyper counterparts. This hierarchy of second quantizations means enormous generalization of physics to what might be regarded a physical counterpart for a hierarchy of abstractions about abstractions about.... The ordinary second quantized quantum physics corresponds only to the lowest level infinite primes.

2. The ordinary primes appearing as building blocks of infinite primes at the first level of the hierarchy could be identified as coding for p-adic primes assignable to fermionic and bosonic partons identified as 2-surfaces of a given space-time sheet. The hierarchy of infinite primes would correspond to hierarchy of space-time sheets defined by the topological condensate. This leads also to a precise identification of p-adic and real variants of bosonic partonic 2-surfaces as correlates of intention and action and pairs of p-adic and real fermionic partons as correlates for cognitive representations.

3. The idea that infinite primes characterize quantum states of the entire Universe, perhaps ground states of super-conformal representations, if not all states, could be taken further. Could 8-D hyper-octonions correspond to 8-momenta in the description of TGD in terms of 8-D hyper-octonion space M^8? Could 4-D hyper-quaternions have an interpretation as four-momenta? The problems caused by non-associativity and non-commutativity however suggests that it is perhaps wiser to restrict the consideration to infinite primes associated with rationals and their algebraic extensions.

Here however emerges the idea about the number theoretic analog of color confinement. Rational (infinite) primes allow not only a decomposition to (infinite) primes of algebraic extensions of rationals but also to algebraic extensions of quaternionic and octonionic (infinite) primes. The physical analog is the decomposition of a particle to its more elementary constituents. This fits nicely with the idea about number theoretic resolution represented as a hierarchy of Galois groups defined by the extensions of rationals and realized at the level of physics in terms of Jones inclusions [C6] defined by these groups having a natural action on space-time surfaces, induced spinor fields, and on configuration space spinor fields representing physical states [C1].

1.3.3 Infinite primes and physics as number theory

The hierarchy of algebraic extensions of rationals implying corresponding extensions of p-adic numbers suggests that Galois groups, which are the basic symmetry groups of number theory, should have concrete physical representations using induced spinor fields and configuration space spinor fields and also infinite primes and real units formed as infinite rationals. These groups permute zeros of polynomials and thus have a concrete physical interpretation both at the level of partonic 2-surfaces dictated by algebraic equations and at the level of braid hierarchy. The vision about the role of hyperfinite factors of II_1 and of Jones inclusions as descriptions of quantum measurements with finite measurement resolution leads to concrete ideas about how these groups are realized.

1.3.4 Space-time correlates of infinite primes

One can assign to infinite primes at the n^{th} level of hierarchy rational functions of n arguments with arguments ordered in a hierarchical manner. It would be nice to assign some concrete interpretation to the polynomials of n arguments in the extension of field of rationals.

1. Do infinite primes code for space-time surfaces?

Infinite primes code naturally for Fock states in a hierarchy of super-symmetric arithmetic quantum field theories. Quantum classical correspondence leads to ask whether infinite primes could also code for the space-time surfaces serving as symbolic representations of quantum states. This would a generalization of algebraic geometry would emerge and could reduce the dynamics of Kähler action to algebraic geometry and organize 4-surfaces to a physical hierarchy according to their algebraic complexity. Note that this conjecture which should be consistent with several conjectures about the dynamics of space-time surfaces (space-time surfaces as preferred extrema of Kähler action, as Kähler calibrations, as quaternionic or co-quaternionic (as associative or co-associative) 4-surfaces of hyper-octonion space M^8.

The most promising variant of this idea is based on the conjecture that hyper-octonion real-analytic maps define foliations of $HO = M^8$ by hyper-quaternionic space-time surfaces providing in turn preferred extremals of Kähler action. This would mean that lowest level infinite primes would define hyper-analytic maps $HO \rightarrow HO$ as polynomials. The intuitive expectation is that higher levels should give rise to more complex HO analytic maps.

The basic objections against the idea is the failure of associativity. The only manner to guarantee associativity is to assume that the arguments oh_n in the polynomial are not independent but that one has $h_i = f_i(h_{i-1},\ i = 2, ..., n$ where f_i is hyper-octonion real-analytic function. This assumption means that one indeed obtains foliation of HO by hyper-quaternionic surfaces also now and that these foliations become increasingly complex as n increases. One could of course consider also the possibility that the hierarchy of infinite primes is directly mapped to functions of single hyper-octonionic argument $h_n = ... = h_1 = h$.

2. What about the interpretation of zeros and poles of rational functions associated with infinite primes

If one accepts this interpretation of infinite primes, one must reconsider the interpretation of the zeros and also poles of the functions $f(o)$ defined by the infinite primes. The set of zeros and poles consists of discrete points and this suggests an interpretation in terms of preferred points of HO, which appear naturally in the quantization of quantum TGD [C1] if one accepts the ideas about hyper-finite factors of type II_1 [C6] and the generalization of the notion of imbedding space and quantization of Planck constant [A9].

The M^4 projection of the preferred point would code for the position tip of future or past light-cone $\delta M^4_\pm$ whereas E^4 projection would choose preferred origin for coordinates transforming linearly under $SO(4)$. At the level of CP_2 the preferred point would correspond to the origin of coordinates transforming linearly under $U(2) \subset SU(3)$. These preferred points would have interpretation as arguments of n-point function in the construction of S-matrix and theory would assign to each argument of n-point function (not necessarily so) "big bang" or "big crunch".

Also configuration space CH (the world of classical worlds) would decompose to a union CH_h of the classical world consisting of 3-surfaces inside $\delta M^4_\pm \times CP_2$ with CP_2 possessing also a preferred point. The necessity of this decomposition in M^4 degrees of freedom became clear long time ago.

3. Why effective 1-dimensionality in algebraic sense?

The identification of arguments (via hyper-octonion real-analytic map in the most general case) means that one consider essentially functions of single variable in the algebraic sense of the word. Rational functions of single variable defined on curve define the simplest function fields having many resemblances with ordinary number fields, and it is known that the dimension $D = 1$ is completely exceptional in algebraic sense [51].

1. Langlands program [50] is based on the idea that the representations of Galois groups can be constructed in terms of so called automorphic functions to which zeta functions relate via Mellin transform. The zeta functions associated with 1-dimensional algebraic curve on finite field F_q, $q = p^n$, code the numbers of solutions to the equations defining algebraic curve in extensions of F_q which form a hierarchy of finite fields F_{q^m} with $m = kn$ [48]: these conjectures have been proven. Algebraic 1-dimensionality is also responsible for the deep results related to the number theoretic Langlands program as far as 1-dimensional function fields on finite fields are considered [48, 50]. In fact, Langlands program is formulated only for algebraic extensions of 1-dimensional function fields.

2. The exceptional character of algebraically 1-dimensional surfaces is responsible the successes of conformal field theory inspired approach to the realization of the geometric Langlands program [51]. It is also responsible for the successes of string models.

3. Effective 1-dimensionality in the sense that the induced spinor fields anti-commute only along 1-D curve of partonic 2-surface is also crucial for the stringy aspects of quantum TGD [C1].

4. Associativity is a key axiom of conformal field theories and would dictate both classical and quantum dynamics of TGD in the approach based on hyper-finite factors of type II_1[C6]. Hence it is rather satisfactory outcome that the mere associativity for octonionic polynomials forces algebraic 1-dimensionality.

1.4 About literature

The reader not familiar with the basic algebra of quaternions and octonions is encouraged to study some background material: the home page of Tony Smith provides among other things an excellent introduction to quaternions and octonions [20]. String model builders are beginning to grasp the potential importance of octonions and quaternions and the articles about possible applications of octonions [21, 22, 23] provide an introduction to octonions using the language of physicist.

Personally I found quite frustrating to realize that I had neglected totally learning of the basic ideas of algebraic geometry, despite its obvious potential importance for TGD and its applications in string models. This kind of losses are the price one must pay for working outside the scientific community. It is not easy for a physicist to find readable texts about algebraic geometry and algebraic number theory from the bookshelves of mathematical libraries. The book "Algebraic Geometry for Scientists and Engineers" by Abhyankar [24], which is not so elementary as the name would suggest, introduces in enjoyable manner the basic concepts of algebraic geometry and binds the basic ideas with the more recent developments in the field. "Problems in Algebraic Number Theory" by Esmonde and Murty [19] in turn teaches algebraic number theory through exercises which concretize the abstract ideas. The book "Invitation to Algebraic Geometry" by K. E. Smith. L. Kahanpää, P. Kekäläinen and W. Traves is perhaps the easiest and most enjoyable introduction to the topic for a novice. It also contains references to the latest physics inspired work in the field.

2 Infinite primes, integers, and rationals

By the arguments of introduction p-adic evolution leads to a gradual increase of the p-adic prime p and at the limit $p \to \infty$ Omega Point is reached in the sense that the negentropy gain associated with quantum jump can become arbitrarily large. There several interesting questions to be answered. Does the topology R_P at the limit of infinite P indeed approximate real topology? Is it possible to generalize the concept of prime number and p-adic number field to include infinite primes? This is is possible is suggested by the fact that sheets of 3-surface are expected to have infinite size and thus to correspond to infinite p-adic length scale. Do p-adic numbers R_P for sufficiently large P give rise to reals by canonical identification? Do the number fields R_P provide an alternative formulation/generalization of the non-standard analysis based on the hyper-real numbers of Robinson [40]? Is it possible to generalize the adelic formula [E4] so that one could generalize quantum TGD so that it allows effective p-adic topology for infinite values of p-adic prime? It must be emphasized that the consideration of infinite primes need not be a purely academic exercise: for infinite values of p p-adic perturbation series contains only two terms and this limit, when properly formulated, could give excellent approximation of the finite p theory for large p.

It turns out that there is not any unique infinite prime nor even smallest infinite prime and that there is an entire hierarchy of infinite primes. Somewhat surprisingly, R_P is not mapped to entire set of reals nor even rationals in canonical identification: the image however forms a dense subset of reals. Furthermore, by introducing the corresponding p-adic number fields R_P, one necessarily obtains something more than reals: one might hope that for sufficiently large infinite values of P this something might be regarded as a generalization of real numbers to a number field containing both infinite numbers and infinitesimals.

The pleasant surprise is that one can find a general construction recipe for infinite primes and that this recipe can be characterized as a repeated second quantization procedure in which the many boson states of the previous level become single boson states of the next level of the hierarchy: this realizes Cantor's definition 'Set as Many allowing to regard itself as One' in terms of the basic concepts of quantum physics. Infinite prime allows decomposition to primes at lower level of infinity and these primes can be identified as primes labelling various space-time sheets which are in turn geometric correlates of selves in TGD inspired theory of consciousness. Furthermore, each infinite prime defines decomposition of a fictive many particle state to a purely bosonic part and to part for which fermion number is one in each mode. This decomposition corresponds to the decomposition of the space-time surface to cognitive and material space-time sheets. Thus the concept of infinite prime suggests completely unexpected connection between quantum field theory, TGD based theory of consciousness and number theory by providing in its structure nothing but a symbolic representation of mathematician and external world!

The definition of the infinite integers and rationals is a straightforward procedure. Infinite primes also allow generalization of the notion of ordinary number by allowing infinite-dimensional space of real units which are however non-equivalent in p-adic sense. This means that space-time points are infinitely structured. The fact that this structure completely invisible at the level of real physics suggests that it represents the space-time correlate for mathematical cognition.

2.1 The first level of hierarchy

In the following the concept of infinite prime is developed gradually by stepwise procedure rather than giving directly the basic definitions. The hope is that the development of the concept in the same manner as it actually occurred would make it easier to understand it.

Step 1

One could try to define infinite primes P by starting from the basic idea in the proof of Euclid for the existence of infinite number of primes. Take the product of all finite primes and add 1 to get a new prime:

$$\begin{array}{l} P = 1 + X \ , \\ X = \prod_p p \ . \end{array} \quad (1)$$

If P were divisible by finite prime then $P - X = 1$ would be divisible by finite prime and one would encounter contradiction. One could of course worry about the possible existence of infinite primes smaller than P and possibly dividing P. The numbers $N = P - k$, $k > 1$, are certainly not primes since k can be taken as a factor. The number $P' = P - 2 = -1 + X$ could however be prime. P is certainly not divisible by $P - 2$. It seems that one cannot express P and $P - 2$ as product of infinite integer and finite integer. Neither it seems possible to express these numbers as products of more general numbers of form $\prod_{p \in U} p + q$, where U is infinite subset of finite primes and q is finite integer.

Step 2

P and $P - 2$ are not the only possible candidates for infinite primes. Numbers of form

$$\begin{array}{l} P(\pm, n) = \pm 1 + nX \ , \\ k(p) = 0, 1, ... \ , \\ n = \prod_p p^{k(p)} \ , \\ X = \prod_p p \ , \end{array} \quad (2)$$

where $k(p) \neq 0$ holds true only in finite set of primes, are characterized by a integer n, and are also good prime candidates. The ratio of these primes to the prime candidate P is given by integer n. In general, the ratio of two prime candidates $P(m)$ and $P(n)$ is rational number m/n telling which of the prime candidates is larger. This number provides ordering of the prime candidates $P(n)$. The reason why these numbers are good canditates for infinite primes is the same as above. No finite prime p with $k(p) \neq 0$ appearing in the product can divide these numbers since, by the same arguments as appearing in Euclid's theorem, it would divide also 1. On the other hand it seems difficult to invent any decomposition of these numbers containing infinite numbers. Already at this stage one can notice the structural analogy with the construction of multiboson states in quantum field theory: the numbers $k(p)$ correspond to the occupation numbers of bosonic states of quantum field theory in one-dimensional box, which suggests that the basic structure of QFT might have number theoretic interpretation in some very general sense. It turns out that this analogy generalizes.

Step 3

All $P(n)$ satisfy $P(n) \geq P(1)$. One can however also the possibility that $P(1)$ is not the smallest infinite prime and consider even more general candidates for infinite primes, which are smaller than $P(1)$. The trick is to drop from the infinite product of primes $X = \prod_p p$ some primes away by dividing it by integer $s = \prod_{p_i} p_i$, multiply this number by an integer n not divisible by any prime dividing s and to add to/subtract from the resulting number nX/s natural number ms such that m expressible as a product of powers of only those primes which appear in s to get

$$\begin{array}{l} P(\pm, m, n, s) = n\frac{X}{s} \pm ms \ , \\ m = \prod_{p|s} p^{k(p)} \ , \\ n = \prod_{p|\frac{X}{s}} p^{k(p)}, \ \ k(p) \geq 0 \ . \end{array} \quad (3)$$

Here $x|y$ means 'x divides y'. To see that no prime p can divide this prime candidate it is enough to calculate $P(\pm, m, n, s)$ modulo p: depending on whether p divides s or not, the prime divides only the second term in the sum and the result is nonzero and finite (although its precise value is not known). The ratio of these prime candidates to $P(+,1,1,1)$ is given by the rational number n/s: the ratio does not depend on the value of the integer m. One can however order the prime candidates with given values of n and s using the difference of two prime candidates as ordering criterion. Therefore these primes can be ordered.

One could ask whether also more general numbers of the form $n\frac{X}{s} \pm m$ are primes. In this case one cannot prove the indivisibility of the prime candidate by p not appearing in m. Furthermore, for $s \ mod \ 2 = 0$ and $m \ mod \ 2 \neq 0$, the resulting prime candidate would be even integer so that it looks improbable that one could obtain primes in more general case either.

Step 4

An even more general series of candidates for infinite primes is obtained by using the following ansatz which in principle is contained in the original ansatz allowing infinite values of n

$$\begin{aligned}
&P(\pm, m, n, s|r) = nY^r \pm ms \ , \\
&Y = \frac{X}{s} \ , \\
&m = \prod_{p|s} p^{k(p)} \ , \\
&n = \prod_{p|\frac{X}{s}} p^{k(p)}, \ \ k(p) \geq 0 \ .
\end{aligned} \tag{4}$$

The proof that this number is not divisible by any finite prime is identical to that used in the previous case. It is not however clear whether the ansatz for given r is not divisible by infinite primes belonging to the lower level. A good example in $r = 2$ case is provided by the following unsuccessful ansatz

$$\begin{aligned}
&N = (n_1 Y + m_1 s)(n_2 Y + m_2 s) = \frac{n_1 n_2 X^2}{s^2} - m_1 m_2 s^2 \ , \\
&Y = \frac{X}{s} \ , \\
&n_1 m_2 - n_2 m_1 = 0 \ .
\end{aligned}$$

Note that the condition states that n_1/m_1 and $-n_2/m_2$ correspond to the same rational number or equivalently that (n_1, m_1) and (n_2, m_2) are linearly dependent as vectors. This encourages the guess that all other $r = 2$ prime candidates with finite values of n and m at least, are primes. For higher values of r one can deduce analogous conditions guaranteing that the ansatz does not reduce to a product of infinite primes having smaller value of r. In fact, the conditions for primality state that the polynomial $P(n, m, r)(Y) = nY^r + m$ with integer valued coefficients ($n > 0$) defined by the prime candidate is irreducible in the field of integers, which means that it does not reduce to a product of lower order polynomials of same type.

Step 5

A further generalization of this ansatz is obtained by allowing infinite values for m, which leads to the following ansatz:

$$\begin{aligned}
&P(\pm, m, n, s|r_1, r_2) = nY^{r_1} \pm ms \ , \\
&m = P_{r_2}(Y)Y + m_0 \ , \\
&Y = \frac{X}{s} \ , \\
&m_0 = \prod_{p|s} p^{k(p)} \ , \\
&n = \prod_{p|Y} p^{k(p)}, \ \ k(p) \geq 0 \ .
\end{aligned} \tag{5}$$

Here the polynomial $P_{r_2}(Y)$ has order r_2 is divisible by the primes belonging to the complement of s so that only the finite part m_0 of m is relevant for the divisibility by finite primes. Note that the part proportional to s can be infinite as compared to the part proportional to Y^{r_1}: in this case one must however be careful with the signs to get the sign of the infinite prime correctly. By using same arguments as earlier one finds that these prime candidates are not divisible by finite primes. One must also require that the ansatz is not divisible by lower order infinite primes of the same type. These conditions are equivalent to the conditions guaranteing the polynomial primeness for polynomials of form $P(Y) = nY^{r_1} \pm (P_{r_2}(Y)Y + m_0)s$ having integer-valued coefficients. The construction of these polynomials can be performed recursively by starting from the first order polynomials representing first level infinite primes: Y can be regarded as formal variable and one can forget that it is actually infinite number.

By finite-dimensional analogy, the infinite value of m means infinite occupation numbers for the modes represented by integer s in some sense. For finite values of m one can always write m as a product of powers of $p_i|s$. Introducing explicitly infinite powers of p_i is not in accordance with the idea that all exponents appearing in the formulas are finite and that the only infinite variables are X and possibly S (formulas are symmetric with respect to S and X/S). The proposed representation of m circumvents this difficulty in an elegant manner and allows to say that m is expressible as a product of infinite powers of p_i despite the fact that it is not possible to derive the infinite values of the exponents of p_i.

Summarizing, an infinite series of candidates for infinite primes has been found. The prime candidates $P(\pm, m, n, s)$ labelled by rational numbers n/s and integers m plus the primes $P(\pm, m, n, s|r_1, r_2)$ constructed as r_1:th or r_2:th order polynomials of $Y = X/s$: the latter ansatz reduces to the less general ansatz of infinite values of n are allowed.

One can ask whether the $p\ mod\ 4 = 3$ condition guaranteing that the square root of -1 does not exist as a p-adic number, is satisfied for $P(\pm, m, n, s)$. $P(\pm, 1, 1, 1)\ mod\ 4$ is either 3 or 1. The value of $P(\pm, m, n, s)\ mod\ 4$ for odd s on n only and is same for all states containing even/odd number of $p\ mod\ = 3$ excitations. For even s the value of $P(\pm, m, n, s)\ mod\ 4$ depends on m only and is same for all states containing even/odd number of $p\ mod\ = 3$ excitations. This condition resembles G-parity condition of Super Virasoro algebras. Note that either $P(+, m, n, s)$ or $P(-, m, n, s)$ but not both are physically interesting infinite primes ($2m\ mod\ 4 = 2$ for odd m) in the sense of allowing complex Hilbert space. Also the additional conditions satisfied by the states involving higher powers of X/s resemble to Virasoro conditions. An open problem is whether the analogy with the construction of the many-particle states in super-symmetric theory might be a hint about more deeper relationship with the representation of Super Virasoro algebras and related algebras.

It is not clear whether even more general prime candidates exist. An attractive hypothesis is that one could write explicit formulas for all infinite primes so that generalized theory of primes would reduce to the theory of finite primes.

2.2 Infinite primes form a hierarchy

By generalizing using general construction recipe, one can introduce the second level prime candidates as primes not divisible by any finite prime p or infinite prime candidate of type $P(\pm, m, n, s)$ (or more general prime at the first level: in the following we assume for simplicity that these are the only infinite primes at the first level). The general form of these prime candidates is exactly the same as at the first level. Particle-analogy makes it easy to express the construction receipe. In present case 'vacuum primes' at the lowest level are of the form

$$\begin{array}{l} \frac{X_1}{S} \pm S \ , \\ X_1 = X \prod_{P(\pm,m,n,s)} P(\pm, m, n, s) \ , \\ S = s \prod_{P_i} P_i \ , \\ s = \prod_{p_i} p_i \ . \end{array} \tag{6}$$

S is product or ordinary primes p and infinite primes $P_i(\pm, m, n, s)$. Primes correspond to physical states created by multiplying X_1/S (S) by integers not divisible by primes appearing S (X_1/S). The integer valued functions $k(p)$ and $K(p)$ of prime argument give the occupation numbers associated with X/s and s type 'bosons' respectively. The non-negative integer-valued function $K(P) = K(\pm, m, n, s)$ gives the occupation numbers associated with the infinite primes associated with X_1/S and S type 'bosons'. More general primes can be constructed by mimicking the previous procedure.

One can classify these primes by the value of the integer $K_{tot} = \sum_{P|X/S} K(P)$: for a given value of K_{tot} the ratio of these prime candidates is clearly finite and given by a rational number. At given level the ratio P_1/P_2 of two primes is given by the expression

$$\frac{P_1(\pm,m_1,n_1,s_1K_1,S_1}{P_2(\pm,m_2,n_2,s_2,K,S_2)} = \frac{n_1s_2}{n_2s_1}\prod_{\pm,m,n,s}(\frac{n}{s})^{K_1^+(\pm,n,m,s)-K_2^+(\pm,n,m,s)} \ . \tag{7}$$

Here K_i^+ denotes the restriction of $K_i(P)$ to the set of primes dividing X/S. This ratio must be smaller than 1 if it is to appear as the first order term $P_1P_2 \rightarrow P_1/P_2$ in the canonical identification and again it seems that it is not possible to get all rationals for a fixed value of P_2 unless one allows infinite values of N expressed neatly using the more general ansatz involving higher power of S.

2.3 Construction of infinite primes as a repeated quantization of a super-symmetric arithmetic quantum field theory

The procedure for constructing infinite primes is very much reminiscent of the second quantization of an super-symetric arithmetic quantum field theory in which single particle fermion and boson states are labelled by primes. In particular, there is nothing especially frightening in the particle representation of infinite primes: theoretical physicists actally use these kind of representations quite routinely.

1. The binary-valued function telling whether a given prime divides s can be interpreted as a fermion number associated with the fermion mode labelled by p. Therefore infinite prime is characterized by bosonic and fermionic occupation numbers as functions of the prime labelling various modes and situation is super-symmetric. X can be interpreted as the counterpart of Dirac sea in which every negative energy state state is occupied and $X/s \pm s$ corresponds to the state containing fermions understood as holes of Dirac sea associated with the modes labelled by primes dividing s.

2. The multiplication of the 'vacuum' X/s with $n = \prod_{p|X/s} p^{k(p)}$ creates $k(p)$ 'p-bosons' in mode of type X/s and multiplication of the 'vacuum' s with $m = \prod_{p|s} p^{k(p)}$ creates $k(p)$ 'p-bosons'. in mode of type s (mode occupied by fermion). The vacuum states in which bosonic creation operators act, are tensor products of two vacuums with tensor product represented as sum

$$|vac(\pm)\rangle = |vac(\frac{X}{s})\rangle \otimes |vac(\pm s)\rangle \leftrightarrow \frac{X}{s} \pm s \tag{8}$$

obtained by shifting the prime powers dividing s from the vacuum $|vac(X)\rangle = X$ to the vacuum ± 1. One can also interpret various vacuums as many fermion states. Prime property follows directly from the fact that any prime of the previous level divides either the first or second factor in the decomposition $NX/S \pm MS$.

3. This picture applies at each level of infinity. At a given level of hierarchy primes P correspond to all the Fock state basis of all possible many-particle states of second quantized super-symmetric theory. At the next level these many-particle states are regarded as single particle states and further second quantization is performed so that the primes become analogous to the momentum labels characterizing various single-particle states at the new level of hierarchy.

4. There are two nonequivalent quantizations for each value of S due to the presence of $\pm$ sign factor. Two primes differing only by sign factor are like G-parity $+1$ and -1 states in the sense that these primes satisfy $P \ mod \ 4 = 3$ and $P \ mod \ 4 = 1$ respectively. The requirement that -1 does not have p-adic square root so that Hilbert space is complex, fixes G-parity to say $+1$. This observation suggests that there exists a close analogy with the theory of Super Virasoro algebras so that quantum TGD might have interpretation as number theory in infinite context. An alternative interpretation for the $\pm$ degeneracy is as counterpart for the possibility to choose the fermionic vacuum to be a state in which either all positive or all negative energy fermion states are occupied.

5. One can also generalize the construction to include polynomials of $Y = X/S$ to get infinite hierarchy of primes labelled by the two integers r_1 and r_2 associated with the polynomials in question. An entire hierarchy of vacuums labelled by r_1 is obtained. A possible interpretation of these primes is as counterparts for the bound states of quantum field theory. The coefficient for the power $(X/s)^{r_1}$ appearing in the highest term of the general ansatz, codes the occupation numbers associated with vacuum $(X/s)^{r_1}$. All the remaining terms are proportional to s and combine to form, in general infinite, integer m characterizing various infinite occupation numbers for the subsystem characterized by s. The additional conditions guaranteeing prime number property are equivalent with the primality conditions for polynomials with integer valued coefficients and resemble Super Virasoro conditions. For $r_2 > 0$ bosonic occupation numbers associated with the modes with fermion number one are infinite and one cannot write explicit formula for the boson number.

6. One could argue that the analogy with super-symmetry is not complete. The modes of Super Virasoro algebra are labelled by natural number whereas now modes are labelled by prime. This need not be a problem since one can label primes using natural number n. Also 8-valued spin index associated with fermionic and bosonic single particle states in TGD world is lacking (space-time is surface in 8-dimensional space). This index labels the spin states of 8-dimensional spinor with fixed chirality. One could perhaps get also spin index by considering infinite octonionic primes, which correspond to vectors of 8-dimensional integer lattice such that the length squared of the lattice vector is ordinary prime:

$$\sum_{k=1,...,8} n_k^2 = \text{ prime } .$$

Thus one cannot exclude the possibility that TGD based physics might provide representation for octonions extended to include infinitely large octonions. The notion of prime octonion is well defined in the set of integer octonions and it is easy to show that the Euclidian norm squared for a prime octonion is prime. If this result generalizes then the construction of generalized prime octonions would generalize the construction of finite prime octonions. It would be interesting to know whether the results of finite-dimensional case might generalize to the infinite-dimensional context. One cannot exclude the possibility that prime octonions are in one-one correspondence with physical states in quantum TGD.

These observations suggest a close relationship between quantum TGD and the theory of infinite primes in some sense: even more, entire number theory and mathematics might be reducible to quantum physics understood properly or equivalently, physics might provide the representation of basic mathematics. Of course, already the uniqueness of the basic mathematical structure of quantum TGD points to this direction. Against this background the fact that 8-dimensionality of the imbedding space allows introduction of octonion structure (also p-adic algebraic extensions) acquires new meaning. Same is also suggested by the fact that the algebraic extensions of p-adic numbers allowing square root of real p-adic number are 4- and 8-dimensional.

What is especially interesting is that the core of number theory would be concentrated in finite primes since infinite primes are obtained by straightforward procedure providing explicit formulas for them. Repeated quantization provides also a model of abstraction process understood as construction of hierarchy of natural number valued functions about functions about ...... At the first level infinite primes are characterized by the integer valued function $k(p)$ giving occupation numbers plus subsystem-complement division (division to thinker and external world!). At the next level prime is characterized in a similar manner. One should also notice that infinite prime at given level is characterized by a pair $(R = MN, S)$ of integers at previous level. Equivalently, infinite prime at given level is characterized by fermionic and bosonic occupation numbers as functions in the set of primes at previous level.

2.4 Construction in the case of an arbitrary commutative number field

The basic construction recipe for infinite primes is simple and generalizes even to the case of algebraic extensions of rationals. Let $K = Q(\theta)$ be an algebraic number field (see the Appendix of [E1] for the basic definitions). In the general case the notion of prime must be replaced by the concept of irreducible defined as an algebraic integer with the property that all its decompositions to a product of two integers are such that second integer is always a unit (integer having unit algebraic norm, see Appendix of [E1]).

Assume that the irreducibles of $K = Q(\theta)$ are known. Define two irreducibles to be equivalent if they are related by a multiplication with a unit of K. Take one representative from each equivalence class of units. Define the irreducible to be positive if its first non-vanishing component in an ordered basis for the algebraic extension provided by the real unit and powers of θ, is positive. Form the counterpart of Fock vacuum as the product X of these representative irreducibles of K.

The unique factorization domain (UFD) property (see Appendix of [E1]) of infinite primes does not require the ring O_K of algebraic integers of K to be UFD although this property might be forced somehow. What is needed is to find the primes of K; to construct X as the product of all irreducibles of K but not counting units which are integers of K with unit norm; and to apply second quantization to get primes which are first order monomials. X is in general a product of powers of primes. Generating infinite primes at the first level correspond to generalized rationals for K having similar representation in terms of powers of primes as ordinary rational numbers using ordinary primes.

2.5 Mapping of infinite primes to polynomials and geometric objects

The mapping of the generating infinite primes to first order monomials labelled by their rational zeros is extremely simple at the first level of the hierarchy:

$$P_{\pm}(m,n,s) = \frac{mX}{s} \pm ns \rightarrow x_{\pm} \pm \frac{m}{sn} \quad . \tag{9}$$

Note that a monomial having zero as its root is not obtained. This mapping induces the mapping of all infinite primes to polynomials.

The simplest infinite primes are constructed using ordinary primes and second quantization of an arithmetic number theory corresponds in one-one manner to rationals. Indeed, the integer $s = \prod_i p_i^{k_i}$ defining the numbers k_i of bosons in modes k_i, where fermion number is one, and the integer r defining the numbers of bosons in modes where fermion number is zero, are co-prime. Moreover, the generating infinite primes can be written as $(n/s)X \pm ms$ corresponding to the two vacua $V = X \pm 1$ and the roots of corresponding monomials are positive *resp.* negative rationals.

More complex infinite primes correspond sums of powers of infinite primes with rational coefficients such that the corresponding polynomial has rational coefficients and roots which are not rational but belong to some algebraic extension of rationals. These infinite primes correspond simply to products of infinite primes associated with some algebraic extension of rationals. Obviously the construction of higher infinite primes gives rise to a hierarchy of higher algebraic extensions.

It is possible to continue the process indefinitely by constructing the Dirac vacuum at the n:th level as a product of primes of previous levels and applying the same procedure. At the second level Dirac vacuum $V = X \pm 1$ involves X which is the product of all primes at previous levels and in the polynomial correspondence X thus correspond to a new independent variable. At the n:th level one would have polynomials $P(q_1|q_2|...)$ of q_1 with coefficients which are rational functions of q_2 with coefficients which are.... The hierarchy of infinite primes would be thus mapped to the functional hierarchy in which polynomial coefficients depend on parameters depending on ....

At the second level one representation of infinite primes would be as algebraic curve resulting as a locus of $P(q_1|q_2) = 0$: this certainly makes sense if q_1 and q_2 commute. At higher levels the locus is a higher-dimensional surface.

2.6 How to order infinite primes?

One can order the infinite primes, integers and rationals. The ordering principle is simple: one can decompose infinite integers to two parts: the 'large' and the 'small' part such that the ratio of the small part with the large part vanishes. If the ratio of the large parts of two infinite integers is different from one or their sign is different, ordering is obvious. If the ratio of the large parts equals to one, one can perform same comparison for the small parts. This procedure can be continued indefinitely.

In case of infinite primes ordering procedure goes like follows. At given level the ratios are rational numbers. There exists infinite number of primes with ratio 1 at given level, namely the primes with same values of N and same S with MS infinitesimal as compared to NX/S. One can order these primes using either the relative sign or the ratio of $(M_1S_1)/(M_2S_2)$ of the small parts to decide which of the two is larger. If also this ratio equals to one, one can repeat the process for the small parts of M_iS_i. In principle one can repeat this process so many times that one can decide which of the two primes is larger. Same of course applies to infinite integers and also to infinite rationals build from primes with infinitesimal MS. If NS is not infinitesimal it is not obvious whether this procedure works. If $N_iX_i/M_iS_i = x_i$ is finite for both numbers (this need not be the case in general) then the ratio $\frac{M_1S_1}{M_2S_2}\frac{(1+x_2)}{(1+x_1)}$ provides the needed criterion. In case that this ratio equals one, one can consider use the ratio of the small parts multiplied by $\frac{(1+x_2)}{(1+x_1)}$ of M_iS_i as ordering criterion. Again the procedure can be repeated if needed.

2.7 What is the cardinality of infinite primes at given level?

The basic problem is to decide whether Nature allows also integers S , $R = MN$ represented as infinite product of primes or not. Infinite products correspond to subsystems of infinite size (S) and infinite total occupation number (R) in QFT analogy.

1. One could argue that S should be a finite product of integers since it corresponds to the requirement of finite size for a physically acceptable subsystem. One could apply similar

argument to R. In this case the set of primes at given level has the cardinality of integers ($alef_0$) and the cardinality of all infinite primes is that of integers. If also infinite integers R are assumed to involve only finite products of infinite primes the set of infinite integers is same as that for natural numbers.

2. NMP is well defined in p-adic context also for infinite subsystems and this suggests that one should allow also infinite number of factors for both S and $R = MN$. Super symmetric analogy suggests the same: one can quite well consider the possibility that the total fermion number of the universe is infinite. It seems however natural to assume that the occupation numbers $K(P)$ associated with various primes P in the representations $R = \prod_P P^{K(P)}$ are finite but nonzero for infinite number of primes P. This requirement applied to the modes associated with S would require the integer m to be explicitly expressible in powers of $P_i|S$ ($P_{r_2} = 0$) whereas all values of r_1 are possible. If infinite number of prime factors is allowed in the definition of S, then the application of diagonal argument of Cantor shows that the number of infinite primes is larger than $alef_0$ already at the first level. The cardinality of the first level is $2^{alef_0}2^{alef_0} == 2^{alef_0}$. The first factor is the cardinality of reals and comes from the fact that the sets S form the set of all possible subsets of primes, or equivalently the cardinality of all possible binary valued functions in the set of primes. The second factor comes from the fact that integers $R = NM$ (possibly infinite) correspond to all natural number-valued functions in the set of primes: if only finite powers $k(p)$ are allowed then one can map the space of these functions to the space of binary valued functions bijectively and the cardinality must be 2^{alef_0}. The general formula for the cardinality at given level is obvious: for instance, at the second level the cardinality is the cardinality of all possible subsets of reals. More generally, the cardinality for a given level is the cardinality for the subset of possible subsets of primes at the previous level.

2.8 How to generalize the concepts of infinite integer, rational and real?

The allowance of infinite primes forces to generalize also the concepts concepts of integer, rational and real number. It is not obvious how this could be achieved. The following arguments lead to a possible generalization which seems practical (yes!) and elegant.

2.8.1 Infinite integers form infinite-dimensional vector space with integer coefficients

The first guess is that infinite integers N could be defined as products of the powers of finite and infinite primes.

$$N = \prod_k p_k^{n_k} = nM \ , \ n_k \geq 0 \ , \qquad (10)$$

where n is finite integer and M is infinite integer containing only powers of infinite primes in its product expansion.

It is not however not clear whether the sums of infinite integers really allow similar decomposition. Even in the case that this decomposition exists, there seems to be no way of deriving it. This would suggest that one should regard sums

$$\sum_i n_i M_i$$

of infinite integers as infinite-dimensional linear space spanned by M_i so that the set of infinite integers would be analogous to an infinite-dimensional algebraic extension of say p-adic numbers such that each coordinate axes in the extension corresponds to single infinite integer of form $N = mM$. Thus the most general infinite integer N would have the form

$$N = m_0 + \sum m_i M_i \ . \tag{11}$$

This representation of infinite integers indeed looks promising from the point of view of practical calculations. The representation looks also attractive physically. One can interpret the set of integers N as a linear space with integer coefficients m_0 and m_i:

$$N = m_0|1\rangle + \sum m_i|M_i\rangle \ . \tag{12}$$

$|M_i\rangle$ can be interpreted as a state basis representing many-particle states formed from bosons labelled by infinite primes p_k and $|1\rangle$ represents Fock vacuum. Therefore this representation is analogous to a quantum superposition of bosonic Fock states with integer, rather than complex valued, superposition coefficients. If one interprets M_i as orthogonal state basis and interprets m_i as p-adic integers, one can define inner product as

$$\langle N_a, N_b\rangle = m_0(a)m_0(b) + \sum_i m_i(a)m_i(b) \ . \tag{13}$$

This expression is well defined p-adic number if the sum contains only enumerable number of terms and is always bounded by p-adic ultrametricity. It converges if the p-adic norm of of m_i approaches to zero when M_i increases.

2.8.2 Generalized rationals

Generalized rationals could be defined as ratios $R = M/N$ of the generalized integers. This works nicely when M and N are expressible as products of powers of finite or infinite primes but for more general integers the definition does not look attractive. This suggests that one should restrict the generalized rationals to be numbers having the expansion as a product of positive and negative primes, finite or infinite:

$$N = \prod_k p_k^{n_k} = \frac{n_1 M_1}{nM} \ . \tag{14}$$

2.8.3 Generalized reals form infinite-dimensional real vector space

One could consider the possibility of defining generalized reals as limiting values of the generalized rationals. A more practical definition of the generalized reals is based on the generalization of the pinary expansion of ordinary real number given by

$$\begin{aligned} x &= \sum_{n\geq n_0} x_n p^{-n} \ , \\ x_n &\in \{0,..,p-1\} \ . \end{aligned} \tag{15}$$

It is natural to try to generalize this expansion somehow. The natural requirement is that sums and products of the generalized reals and canonical identification map from the generalized reals to generalized p-adcs are readily calculable. Only in this manner the representation can have practical value.

These requirements suggest the following generalization

$$X = x_0 + \sum_N x_N p^{-N} \ ,$$
$$N = \sum_i m_i M_i \ , \quad (16)$$

where x_0 and x_N are ordinary reals. Note that N runs over infinite integers which has *vanishing finite part*. Note that generalized reals can be regarded as infinite-dimensional linear space such that each infinite integer N corresponds to one coordinate axis of this space. One could interpret generalized real as a superposition of bosonic Fock states formed from single single boson state labelled by prime p such that occupation number is either 0 or infinite integer N with a vanishing finite part:

$$X = x_0|0\rangle + \sum_N x_N|N> \ . \quad (17)$$

The natural inner product is

$$\langle X, Y\rangle = x_0 y_0 + \sum_N x_N y_N \ . \quad (18)$$

The inner product is well defined if the number of N:s in the sum is enumerable and x_N approaches zero sufficiently rapidly when N increases. Perhaps the most natural interpretation of the inner product is as R_p valued inner product.

The sum of two generalized reals can be readily calculated by using only sum for reals:

$$X + Y = x_0 + y_0 + \sum_N (x_N + y_N) p^{-N} \ , \quad (19)$$

The product XY is expressible in the form

$$XY = x_0 y_0 + x_0 Y + X y_0 + \sum_{N_1, N_2} x_{N_1} y_{N_2} p^{-N_1 - N_2} \ , \quad (20)$$

If one assumes that infinite integers form infinite-dimensional vector space in the manner proposed, there are no problems and one can calculate the sums N_1+N_2 by summing component wise manner the coefficients appearing in the sums defining N_1 and N_2 in terms of infinite integers M_i allowing expression as a product of infinite integers.

Canonical identification map from ordinary reals to p-adics

$$x = \sum_k x_k p^{-k} \ \rightarrow \ x_p = \sum_k x_k p^k \ ,$$

generalizes to the form

$$x = x_0 + \sum_N x_N p^{-N} \rightarrow (x_0)_p + \sum_N (x_N)_p p^N \quad , \tag{21}$$

so that all the basic requirements making the concept of generalized real calculationally useful are satisfied.

There are several interesting questions related to generalized reals.

1. Are the extensions of reals defined by various values of p-adic primes mathematically equivalent or not? One can map generalized reals associated with various choices of the base p to each other in one-one manner using the mapping

$$X = x_0 + \sum_N x_N p_1^{-N} \rightarrow x_0 + \sum_N x_N p_2^{-N} \quad . \tag{22}$$

   The ordinary real norms of *finite* (this is important!) generalized reals are identical since the representations associated with different values of base p differ from each other only infinitesimally. This would suggest that the extensions are physically equivalent. It these extensions are not mathematically equivalent then p-adic primes could have a deep role in the definition of the generalized reals.

2. One can generalize previous formulas for the generalized reals by replacing the coefficients x_0 and x_i by complex numbers, quaternions or octonions so as to get generalized complex numbers, quaternions and octonions. Also inner product generalizes in an obvious manner. The 8-dimensionality of the imbedding space provokes the question whether it might be possible to regard the infinite-dimensional configuration space of 3-surfaces, or rather, its tangent space, as a Hilbert space realization of the generalized octonions. This kind of identification could perhaps reduce TGD based physics to generalized number theory.

2.9 Comparison with the approach of Cantor

The main difference between the approach of Cantor and the proposed approach is that Cantor uses only the basic arithmetic concepts such as sum and multiplication and the concept of successor defining ordering of both finite and infinite ordinals. Cantor's approach is also purely set theoretic. The problems of purely set theoretic approach are related to the question what the statement 'Set is Many allowing to regard itself as One' really means and to the fact that there is no obvious connection with physics. The proposed approach is based on the introduction of the concept of prime as a basic concept whereas ordering is based on the use of ratios: using these one can recursively define ordering and get precise quantitative information based on finite reals. Together with canonical identification the concept of infinite primes becomes completely physical in the sense that all probabilities are always finite real numbers. The 'Set is Many allowing to regard itself as One' is defined as quantum physicist would define it: many particle states become single particle states in the second quantization describing the counterpart for the construction of the set of subsets of a given set. One could also say that integer as such corresponds to set as 'One' and its decomposition to a product of primes corresponds to the set as 'Many'. The concept of prime, the ultimate 'One', has as its physical counterpart the concept of elementary particle understood in very general sense. The new element is the physical interpretation: the sum of two numbers whose ratio is zero correspond to completely physical finite-subsystem-infinite complement division and the iterated construction of the set of subsets of a set at given level is basically p-adic evolution understood in the most general possible sense and realized as a repeated second quantization. What is attractive is that this repeated second quantization can be regarded also as a model of abstraction process and actually the process of abstraction itself.

The possibility to interpret the construction of infinite primes either as a repeated bosonic quantization involving subsystem-complement division or as a repeated super-symmetric quantization could have some deep meaning. A possible interpretation consistent with these two pictures is based on the hypothesis that fermions provide a reflective level of consciousness in the sense that the 2^N element Fock basis of many-fermion states formed from N single-fermion states can be regarded as a set of all possible statements about N basic statements. Statements about whether a given element of set X belongs to some subset S of X are certainly the fundamental statements from the point of view of mathematics. Hence one could argue that many-fermion states provide cognitive representation for the subsets of some set. Single fermion states represent the points of the set and many-fermion states represent possible subsets.

3 Generalizing the notion of infinite prime to the non-commutative context

The notion of prime and more generally, that of irreducible, makes sense also in more general number fields and even algebras. The considerations of [E2] suggests that the notion of infinite prime should be generalized to the case of complex numbers, quaternions, and octonions as well as to their hyper counterparts which seem to be physically the most interesting ones [E2]. Also the hierarchy of infinite primes should generalize as well as the representation of infinite primes as polynomials and as space-time surfaces. The proposed number theoretic realization of the dynamics defined by the absolute minimization of Kähler action can be realized if it is possible to assign hyper-octonion analytic functions to infinite hyper-octonionic primes [E2].

3.1 General view about the construction of generalized infinite primes

The consideration of basic objections against quaternionic and octonionic infinite primes allows to identify the basic philosophical ideas serving as guidelines for the construction of infinite primes.

3.1.1 Infinite primes should be commutative and associative

The basic objections against (hyper-)quaternionic and (hyper-)octonionic infinite primes relate to the non-commutativity and non-associativity.

1. In the case of quaternionic infinite primes non-commutativity, and in the case of octonionic infinite primes also non-associativity, might be expected to cause difficulties in the definition of X. Fortunately, the fact that all conjugates of a given finite prime appear in the product defining X, implies that the contribution from each irreducible with a given norm p is real and X is real. Therefore the multiplication and division of X with quaternionic or octonionic primes is a well-defined procedure, and generating infinite primes are well-defined apart from the degeneracy due to non-commutativity and non-associativity of the finite number of lower level primes. Also the products of infinite primes are well defined, since by the reality of X it is possible to tell how the products AB and BA differ. Of course, also infinite primes representing physical states containing infinite numbers of fermions and bosons are possible and infinite primes of this kind must be analogous to generators of a free algebra for which AB and BA are not related in any manner.

2. The sums of products of monomials of generating infinite primes define higher level infinite primes and also here non-commutativity and associativity cause potential difficulties. The

assignment of a monomial to a quaternionic or octonionic infinite prime is not unique since the rational obtained by dividing the finite part mr with the integer n associated with infinite part can be defined either as $(1/n) \times mr$ or $mr \times (1/n)$ and the resulting non-commuting rationals are different.

If the polynomial associated with infinite prime has real-rational coefficients these difficulties do not appear. This would imply universality in the sense that the polynomials as such would not contain information about the number field in question. This number theoretic universality is highly attractive also physically.

The reduction of the roots of polynomials to complex roots encourages the idea about the analogy with quantum measurement theory. Although it is possible to define more general infinite primes, it seems that the primes having representation as space-time surface are reducible to those represented by polynomials with real-rational coefficients. This would mean that the number field field would not be seen at all in the characterization of the polynomial. The roots of the polynomial would be in general complex and effective 2-dimensionality would prevail in this sense. Complex planes of quaternions and octonions space define maximal commutative sub-fields of them. In the case of hyper-quaternions and hyper-octonions hyper-complex planes take the role of maximal sub-algebra which is closed and at the same time commutative. Interestingly, the hyper-octonionic solution ansatz involves a local fixing of a hyper-complex algebra at each point of $HO = M^8$ physically equivalent with the fixing the space of longitudinal polarizations.

At space-time level this should correspond to effective 2-dimensionality in the sense that quantum states and space-time surfaces are coded by the data associated with 2-dimensional partonic surfaces at the intersections of 3-D and 7-D light-like causal determinants. The tangent spaces of these surfaces should be dual to the local hyper-complex longitudinal polarization planes. The induced selection of the transversal polarization plane at each space-time point could be also seen as the number theoretical analog for the selection of a rest frame and of quantization axis for spin.

Commutativity requirement for infinite primes allows real-rationals or possibly algebraic extensions of them as the coefficients of the polynomials formed from hyper-octonionic infinite primes. If only infinite primes with complex rational coefficients are allowed and only the vacuum state $V_{\pm} = X \pm 1$ involving product over all primes of the number field, would reveal the number field. One could thus construct the generating infinite primes using the notion of hyper-octonionic prime for any algebraic extension of rationals.

3.1.2 Do hyper-octonionic infinite primes correspond to space-time surfaces?

The general philosophy behind the construction of infinite primes involves at least the following ideas.

1. Quantum TGD should result as an algebraic continuation of rational number based physics to various number fields. Similar continuation principle should hold true also for infinite primes. This means that the formal expressions for infinite primes should be essentially same as those associated with the infinite primes associated with the field or rational numbers or complex rationals. As far as space-time representation in terms of polynomials is considered, this means that the polynomials involved should have real coefficients. An analogous situation should prevail at the higher levels of the hierarchy.

2. Hyper-octonionic primes are favored physically and if they have representation as polynomials or more general rational functions of hyper-octonion with real-rational coefficients, it is possible to assign to each prime a 4-parameter foliation of $M^4 \times CP_2$ hyper-quaternionic space-time surfaces by the construction of [E2]. Also the dual of the foliation defines a foliation and canonically imbedded M^4 and CP_2 provide a basic example of dual 4-surfaces. The foliations are parameterized by functions $HO = M^8 \to S^6$ fixing the preferred octonionic

imaginary unit. A possible identification is in terms of vacuum degeneracy. The fixing of the imaginary unit means fixing of complex plane of octonions and the physical interpretation is as a local fixing of longitudinal polarization directions having interpretation as gauge degrees of freedom.

3.1.3 The decomposition of rational infinite primes to hyper-octonionic could have a physical meaning

The requirement that hyper-octonionic infinite primes correspond at the highest level to polynomials with rational coefficients would mean an effective reducibility to rational infinite primes.

The reduction to rational infinite primes does not mean trivialization of the theory. One can decompose infinite rational primes to a product of hyper-octonionic primes just as one can decompose them to a product of primes in algebraic extensions of rational numbers and this decomposition might have a physical interpretation as a decomposition of a particle to its composites if one accepts the idea that the hierarchy of algebraic extensions corresponds to a hierarchy of increasing measurement resolutions. The reduction to a rational infinite prime implies that hyper-octonionic primes and their conjugates appear in a pairwise manner in the products involved. Hence the net values of the transversal parts of infinite hyper-octonic 8-momenta vanish and one could speak about the vanishing of transversal M^8 momenta in HO context. In H context this brings in mind the vanishing of transversal M^4 momenta for hadron and vanishing of color quantum numbers.

3.1.4 Commutativity and associativity for infinite primes does not imply commutativity and associativity for corresponding polynomials

The commutativity of infinite primes is not enough to eliminate completely the effects due to non-commutativity and non-associativity in case of corresponding polynomials. For the hyper-octonionic infinite primes at higher levels of hierarchy non-associativity causes delicate effects since the grouping of infinite primes affects the polynomial associated with the infinite prime and thus space-time surface associated with the infinite prime. Only for arguments $h_1, ..h_n$ restricted to a 2-dimensional subspace H_2 of HO the effects due to non-commutativity and non-associativity are completely absent and this conforms nicely with the notion of effective 2-dimensionality meaning that the physical on-associativity and non-commutativity are trivial and correspond to gauge degrees of freedom.

The unique solution to the problems is to assign to infinite hyper-octonionic primes polynomials for which all arguments h_i are identical $h_n = ... = h_1 = h$. A more general solution would be based on the assumption that the arguments of the polynomial are related by hyper-octonion real-analytic rational function. This option also allows to assign to hyper-octonionic infinite primes 4-D surfaces in a natural manner if hyper-octonion real-analyticity gives rise to a foliation of HO by quaternionic 4-surfaces. In this framework the proposed mapping of infinite primes to space-time surfaces could be seen as being natural because hyper-octonionic primes are associated with a maximal algebraic completion.

3.1.5 The interpretation of two vacuum primes in terms of positive and negative energy Fock states

In the rational case the positivity of primes means that $V_{\pm} = X \pm 1$ correspond to two non-equivalent Fock vacua. For hyper-octonionic primes the two vacua correspond to the to different signs of energy related by time reflection since the units with $n_0 < 0$ correspond to time reflection combined with Lorentz boost. The real part of a hyper-octonionic generating prime can be made non-vanishing by an application of a suitable boost represented by unit.

In TGD the time-orientation of the space-time sheet can be also negative and this means that energies can be either positive or negative [D3, D5]. The interpretation of the two vacua is as vacua associated with space-time sheets of negative and positive time orientation. The possibility that the sign of inertial energy is negative has profound implications and defines one of the most important differences between TGD and competing theories.

Physically it would be desirable that also more complex infinite primes having interpretation as representations of bound states could be interpreted as composites of states of unique positive and negative energy generating primes. If the positive and negative energy infinite primes correspond to states with fermion numbers, one must assume that the polynomials of the generating infinite primes are superpositions of products of monomials of degree n_+ and n_- with respect to the generating infinite primes $P_\pm(m,n,s)$ such that $n = n_+ - n_-$ is constant.

The vacua $X \pm 1$ can be interpreted as rational infinite primes, which are however not constructible from rational vacuum $X = \prod_p p$ by a finite number of steps since each rational prime p appears with some power $N(p)$ counting the number of positive primes with norm

$$N(\pi) = h_0^2 - \sum h_i^2 = p \ .$$

Thus one has

$$X = \prod_{\pi>0} \pi = \prod_p p^{N(p)} \ .$$

Numbers with components in real algebraic extensions of rationals would pop-up dynamically, when one factorizes polynomials which are irreducible in the field of rationals.

If algebraic extensions of rationals are allowed as a fundamental number field, $N(\pi)$ must be replaced with

$$N(\pi) = N_K(h_0^2 - \sum_i h_i^2) = p \ .$$

Only one representative of positive primes related by a multiplication with real Dirichlet units representable as fractal scalings can be included (note that the number of Dirichlet units is always infinite for the real extensions of rationals). This gives a finite number of primes for given p. This option is however not attractive physically since it is in conflict with the idea that algebraic extensions pop up dynamically from the representations of the polynomial as space-time surface.

3.2 Quaternionic and octonionic primes and their hyper counterparts

The loss of commutativity and associativity implies that the definitions of (hyper-)quaternionic and (hyper-)octonionic primes are not completely straightforward.

3.2.1 Basic facts about quaternions and octonions

Both quaternions and octonions allow both Euclidian norm and the Minkowskian norm defined as a trace of the linear operator defined by the multiplication with octonion. Minkowskian norm has the metric signature of $H = M^4 \times CP_2$ or $M^4_+ \times CP_2$ so that H can be regarded locally as an octonionic space. Both norms are a multiplicative and the notions of both quaternionic and octonionic prime are well defined despite non-associativity. Quaternionic and octonionic primes have length squared equal to rational prime.

In the case of quaternions different basis of imaginary units I, J, K are related by 3-dimensional rotation group and different quaternionic basis span a 3-dimensional sphere. There is 2-sphere of complex structures since imaginary unit can be any unit vector of imaginary 3-space.

A basis for octonionic imaginary units J, K, L, M, N, O, P can be chosen in many manners and fourteen-dimensional subgroup G_2 of the group $SO(7)$ of rotations of imaginary units is the group labelling the octonionic structures related by octonionic automorphisms to each other. It deserves to be mentioned that G_2 is unique among the simple Lie-groups in that the ratio of the square roots of lengths for long and short roots of G_2 Lie-algebra are in ratio 3 : 1 [33]. For other Lie-groups this ratio is either 2:1 or all roots have same length. The set of equivalence classes of the octonion structures is $SO(7)/G_2 = S^7$. In the case of quaternions there is only one equivalence class.

The group of automorphisms for octonions with a fixed imaginary part is $SU(3)$. The coset space $S^6 = G_2/SU(3)$ labels possible complex structures of the octonion space specified by a selection of a preferred imaginary unit. $SU(3)/U(2) = CP_2$ could be thought of as the space of octonionic structures giving rise to a given quaternionic structure with complex structure fixed. This can be seen as follows. The units $1, I$ are $SU(3)$ singlets whereas J, J_1, J_2 and K, K_1, K_2 form $SU(3)$ triplet and antitriplet. Under $U(2)$ J and K transform like objects having vanishing $SU(3)$ isospin and suffer only a $U(1)$ phase transformation determined by multiplication with complex unit I and are mixed with each other in orthogonal mixture. Thus $1, I, J, K$ is transformed to itself under $U(2)$.

3.2.2 Quaternionic and octonionic primes

Quaternionic primes with $p \ mod \ 4 = 1$ can correspond to (n_1, n_2) with n_1 even and n_2 odd or vice versa. For $p \ mod \ 4 = 3$ (n_1, n_2, n_3) with n_i odd is the minimal option. In this case there is however large number of primes having only two components: in particular, Gaussian primes with $p \ mod \ 4 = 1$ define also quaternionic primes. Purely real Gaussian primes with $p \ mod \ 4 = 3$ with norm $z\overline{z}$ equal to p^2 are not quaternionic primes, and are replaced with 3-component quaternionic primes allowing norm equal to p. Similar conclusions hold true for octonionic primes.

The reality condition for polynomials associated with Gaussian infinite primes requires that the products of generating prime and its conjugate are present so that the outcome is a real polynomial of second order.

3.2.3 Hyper primes

The notion of prime generalizes to hyper-quaternionic and octonionic case. The factorization $n_0^2 - n_3^2 = (n_0 + n_3)(n_0 - n_3)$ implies that any hyper-quaternionic and -octonionic primes can be represented as $(n_0, n_3, 0, ...) = (n_3 + 1, n_3, 0, ...)$, $n_3 = (p-1)/2$ for $p > 2$. $p = 2$ is exceptional: a representation with minimal number of components is given by $(2, 1, 1, 0, ...)$. Notice that the interpretation of hyper-quaternionic primes (or integers) as four-momenta implies that it is not possible to find rest system for them: only a system where energy is minimum is possible.

The notion of "irreducible" (see Appendix of [E1]) is defined as the equivalence class of primes related by a multiplication with a unit and is more fundamental than that of prime. All Lorentz boosts of a hyper prime combine to form an irreducible. Note that the units cannot correspond to real particles in corresponding arithmetic quantum field theory.

If the situation for $p > 2$ is effectively 2-dimensional in the sense that it is always possible to transform the hyper prime to a 2-component form by multiplying it by a suitable unit representing Lorentz boost, the theory for time-like hyper primes effectively reduces to the 2-dimensional hyper-complex case when irreducibles are chosen to belong to H_2. The physical counterpart for the choice of H_2 would be the choice of the plane of longitudinal polarizations, or equivalently, of quantization axis for spin. This hypothesis is physically highly attractive since it would imply number theoretic universality and conform with the effective 2-dimensionality. Of course, the hyper-octonionic primes related by $SO(7, 1)$ boosts need not represent physically equivalent states.

Also the rigorous notion of hyper primeness seems to require effective 2-dimensionality. If effective 2-dimensionality holds true, hyper integers have a decomposition to a product of hyper primes multiplied by a suitable unit. The representation is obtained by Lorentz boosting the hyper integer first to a 2-component form and then decomposing it to a product of hyper-complex primes.

The situation becomes certainly more complex if also space-like primes with negative norm squared $n_0^2 - n_1^2 - ... = -p$ are allowed. Gaussian primes with $p \ mod4 \ = 1$ are representable as space-like primes of form $(0, n_1, n_2, 0)$: $n_1^2 + n_2^2 = p$. Space-like primes with $p \ mod \ 4 = 3$ have at least 3 non-vanishing components which are odd integers.

3.3 Hyper-octonionic infinite primes

The infinite-primes associated with hyper-octonions are the most natural ones physically because of the underlying Lorentz invariance and the possibility to interpret them as 8-momenta with mass squared equal to prime. HO is consistent with the metric signature of the tangent space of H, and the four additional momentum components bring strongly in mind the tangent space counterpart of CP_2 contribution to the mass squared. Also the interpretation of quaternionic part of finite hyper-octonionic primes in terms of electro-weak and color quantum numbers could be considered since the total number of them is $2 + 2 = 4$.

3.3.1 Construction recipe at the lowest level of hierarchy assuming reduction to rational infinite primes

The condition that allowed hyper-octonionic infinite primes correspond to decompositions of rational infinite primes to products of their hyper-octonionic counterparts is the simplest manner to define them and generalizes the decomposition of rational infinite primes to products of primes in algebraic extensions of rationals.

This allows primes in algebraic extensions of rationals containing $\sqrt{-1}$ only if one interprets the commuting unit of hyper-octonionic integers as imaginary unit associated with the algebraic extensions of rationals. Composites of infinite primes in complexification of octonions would be in question. The reality of the coefficients of the polynomials assignable to infinite primes would also mean that the M^8 coordinates of HO stay real.

The physical interpretation for the reduction to rational infinite primes would be in terms of number theoretic analog of color confinement meaning decomposition of particles to their composites becoming visible in an improved algebraic resolution. Also the interpretation in terms of non-commutative geometry in transversal degrees of freedom meaning that only longitudinal momenta corresponding to non-vanishing of only hyper-complex part of hyper-octonionic 8-momentum. Indeed, the commutation relations $xy = qyz$, $q = exp(i\pi/n)$ for quantum plane would allow the vanishing of x and y identified now as components of transversal momentum.

3.3.2 More general construction recipe at the lowest level of hierarchy

The following argument represents the construction recipe for the first level hyper-octonionic primes without the assumption about the reduction to rational infinite primes.

1. Infinite prime property requires that X must be defined by taking one representative from each equivalence class representing irreducible and forming the product of their conjugates. The representative hyper-octonionic primes can be taken to be time-like positive energy primes. The conjugates of each irreducible appear in X so for a given norm p the net result is real for each rational prime p.

   The number of conjugates depends on the number of non-vanishing components of the the prime with norm p in the minimal representation having minimal energy. Several primes with a given norm p not related by a multiplication with unit or by automorphism are in principle possible. The degeneracy is determined by the number of elements of a subgroup of Galois group acting non-trivially on the prime. Galois group is generated by the permutations of 7 imaginary units and 7 conjugations of units consistent with the octonionic product. X is proportional to $p^{N(p)}$ where $N(p)$ in principle depends on p.

2. If the conjectured effective 2-dimensionality holds true, the situation reduces effectively to hyper-complex case and X is product of the squares of all primes multiplied by a power of 2. In the case of ordinary infinite primes there are two different vacuum primes $X \pm 1$. This is the case also now. Since the sign of the time-like component part corresponds to the sign of energy, the sign degeneracy $X \pm 1$ for the vacua could relate to the degeneracy corresponding to positive and negative energy space-time sheets. An alternative interpretation is in terms of fermion-antifermion degeneracy.

3. The product X of all hyper-octonionic irreducibles can be regarded as the counterpart of Dirac vacuum in a rather concrete sense. Moreover, in the hyper-quaternionic and octonionic case the norm of X is analogous to the Dirac determinant of a fermionic field theory with prime valued mass spectrum and integer valued momentum components. The inclusion of only irreducible eliminates from the infinite product defining Dirac determinant product over various Lorentz boosts of $p^k\gamma_k - m$.

4. An interesting question is what happens when the finite part of an infinite prime is multiplied by light like integer k. The obvious guess is that k describes the presence of a massless particle. If the resulting infinite integer is multiplied with conjugates $k_{c,i}$ of k an integer of form $\prod_i k_{c,i}mX/n$ having formally zero norm results. It would thus seem that there is a kind of gauge invariance in the sense that infinite primes for which both finite and infinite part are multiplied with the same light-like primes, are divisors of zero and correspond to gauge degrees of freedom.

5. More complex infinite hyper-octonionic primes can be always decomposed to products of generating infinite primes which correspond to polynomials with zeros in algebraic extensions of rationals so that the resulting polynomial has real-rational coefficients but has no rational zeros. An interpretation as bound states is suggestive and the replacement of the zero of corresponding polynomial with non-rational number is analogous to the change of particle rest mass in bound state formation. The sign of energy is well defined for each factor of this kind.

6. Hyper-octonionic infinite primes correspond to real-rational polynomials if all conjugates of given hyper-octonionic prime occur in the definition of generating infinite primes. The reality requirement satisfied in this manner would exclude the presence of light-like factors in the finite part of the infinite prime. Physically the presence of these factors would seem to be desirable (at least in the finite part of the infinite prime) since they could be interpreted physically as representations of massless particles. The reality condition can be also satisfied for a product of conjugates of infinite primes. In this case the constant part of the resulting infinite primes vanishes.

3.3.3 Zeta function and infinite primes

Fermionic Zeta function is expressible as a product of fermionic partition functions $Z_{F,p} = 1 + p^{-z}$ and could be seen as an image of X under algebraic homomorphism mapping prime p to $Z_{F,p}$ defining an analog of prime in the commutative function algebra of complex numbers. For hyper-octonionic infinite primes the contribution of each p to the norm of X is same finite power of p since only single representative from each Lorentz equivalence class is included, and there is one-one correspondence with ordinary primes so that an appropriate power of ordinary ζ_F might be regarded as a representation of X also in the case of hyper-octonionic primes.

Infinite primes suggest a generalization of the notion of ζ function. Real-rational infinite prime $X \pm 1$ would correspond to $\zeta_F \pm 1$. General infinite prime is mapped to a generalized zeta function by dividing ζ_F with the product of partition functions $Z_{F,p}$ corresponding to fermions kicked out from sea. The same product multiplies '1'. The powers p^n present in either factor correspond to the presence of n bosons in mode p and to a factor $Z^n_{p,B}$ in corresponding summand of the generalized Zeta. In the case of hyper-octonionic infinite primes some power of Z_F multiplied by p-dependent powers $Z^{n(p)}_{F,p}$ of fermionic partition functions with $n(p) \to 0$ for $p \to \infty$ should replace the image of X. If effective 2-dimensionality holds true $n(p) = 2$ holds true for $p > 2$.

For zeros of ζ_F which are same as those of Riemann ζ the image of infinite part of infinite prime vanishes and only the finite part is represented faithfully. Whether this might have some physical meaning is an interesting question.

3.4 Mapping of the hyper-octonionic infinite primes to polynomials

Infinite primes can be mapped to polynomial primes which in turn have geometric representation as algebraic surfaces. This inspires the idea that physics could be reduced to algebraic number theory and algebraic geometry [19, 26, 24] in some general sense. In the following consideration is restricted to hyper-octonionic primes which are the most interesting ones on basis of the considerations of [E2].

3.4.1 Mapping of infinite primes to polynomials at the first level of the hierarchy

The mapping of the generating infinite primes to first order monomials labelled by their rational zeros is extremely simple at the first level of the hierarchy:

$$P_{\pm}(m,n,s) = \frac{mX}{s} \pm ns \to h \pm \frac{m}{sn} \quad .$$

Note that a monomial having zero as its root is not obtained. This mapping induces the mapping of all infinite primes to polynomials.

The simplest infinite primes are constructed using ordinary primes and second quantization of an arithmetic number theory corresponds in one-one manner to rationals. Indeed, the integer $s = \prod_i p_i^{k_i}$ defining the numbers k_i of bosons in modes k_i, where fermion number is one, and the integer r defining the numbers of bosons in modes where fermion number is zero, are co-prime. Moreover, the generating infinite primes can be written as $(n/s)X \pm ms$ corresponding to the two vacua $V = X \pm 1$ and the roots of corresponding monomials are positive *resp.* negative rationals.

More complex infinite primes correspond sums of powers of infinite primes with rational coefficients such that the corresponding polynomial has real coefficients and roots which are not rational but belong to some algebraic extension of rationals. These infinite primes correspond simply to products of infinite primes associated with some algebraic extension of rationals. Obviously the construction of higher infinite primes gives rise to a hierarchy of higher algebraic extensions.

3.4.2 The representation of higher level infinite primes as polynomials

It is possible to continue the process indefinitely by constructing the Dirac vacuum at the n:th level as a product of primes of previous levels and applying the same procedure. At the second level Dirac vacuum $V = X \pm 1$ involves X which is the product of all primes at previous levels and in the polynomial correspondence X thus correspond to a new independent variable. At the n:th level one has polynomials $P(h_1|h_2|...)$ of h_1 with coefficients which are real-rational functions of h_2 with coefficients which are.... The hierarchy of infinite primes is thus mapped to the functional hierarchy in which polynomial coefficients depend on parameters depending on ....

The so called Slaving Hierarchy appearing in Haken's theory of self-organization has similar form: the non-dynamical coupling parameters of the system depend on slowly varying external parameters which in turn depend on... The lowest level of the hierarchy corresponding to the ordinary rationals takes the role of the highest boss in the hierarchy of infinite primes.

For higher level infinite primes the effects of non-commutativity and non-associativity cannot be avoided except when the arguments are restricted to the same hyper-complex sub-space of HO defining the polarization plane. The non-associativity implies that the grouping of the arguments in the polynomial matters and affects the space-time surface. It is not clear whether non-associativity and non-commutative can be really allowed for infinite primes.

A very attractive manner to avoid effects of non-associativity is to assume that all infinite primes are reducible to rational infinite primes and that representations in terms of infinite primes associated with various extensions of rationals (algebraic extensions of rationals and of non-commutative and non-associative completions of rationals) emerge from the decompositions of rational primes to these primes.

4 The representation of hyper-octonionic infinite primes as space-time surfaces

In algebraic geometry numbers, in particular primes, are represented as zeros of polynomials. The obvious idea is that the polynomials associated with infinite primes could represent as their zeros higher-dimensional surfaces. Also infinite integers and rationals could define space-time surfaces as their zeros but infinite primes are in a special position since the polynomials are irreducible in this case and represent connected surfaces.

This starting point turned out to be too naive, and only the hard work trying to take into account the known facts about the dynamics of absolute minimization of Kähler action led to the a more realistic view. The metric of the imbedding space leaves only hyper-octonionic primes under consideration.

The considerations of [E2] demonstrate that the hyper-quaternionic four-surfaces of hyper-octonionic space $HO = M^8$ define 4-surfaces of $M^4 \times CP_2$. Furthermore, hyper-octonion analytic maps of $HO \rightarrow HO$ define foliation of HO and thus $M^4 \times CP_2$ by 4-surfaces.

Since the infinite primes at the first level of hierarchy can be mapped to hyper-octonion analytic maps $OH \rightarrow OH$, they can thus be mapped to four-parameter families of 4-surfaces defining candidates for the solutions of field equations. At the higher levels the crucial input is associativity condition satisfied if one assumes that the hyper-octonionic arguments $h_1, ...h_n$ are not independent but related by hyper-octonion real-analytic map and thus commute. This however means that the resulting hyper-octonioni analytic functions also now define foliations of HO by 4-dimensional surfaces.

The construction generalizes to the higher levels of hierarchy and has an interpretation in terms of the notion of many-sheeted space-time. Although the discussion of these ideas can be found in [E2], the crucial importance of the representation of hyper-octonion analytic functions as space-time surfaces for the representation of infinite primes, motivates the inclusion of the key arguments also in this chapter.

4.1 Hyper-quaternionic 4-surfaces in HO correspond to space-time surfaces in $M^4 \times CP_2$

The observations about the role of $SU(3)$ and CP_2 imply that hyper-quaternionic 4-surface in HO correspond in one-one manner to 4-surfaces in $M^4 \times CP_2$ and to a general ansatz producing this kind of surfaces.

4.1.1 A map $HO \to SU(3)$ defining an integrable distribution of hyper-quaternionic planes defines a foliation of $M^4 \times CP_2$ by 4-surfaces

A distribution of quaternionic planes in HO defines its foliation by 4-surfaces X^4 of HO and therefore also that of $M^4 \times CP_2$ if integrability conditions, which state that hyper-quaternionic planes define tangent planes of X^4 in the foliation, are satisfied. The M^4 coordinates of X^4 are obtained as the projection of HO to a fixed HQ sub-space of HO whereas the selection of the quaternionic plane defines CP_2 coordinates.

If the distribution of the hyper-quaternionic tangent planes in HO defined by a map $g : HO \to SU(3)$ is integrable, a foliation of HO by four-surfaces results, and defines a foliation of $H = M^4 \times CP_2$ so that a 4-parameter family of solutions is obtained. This could perhaps interpreted as stating that the allowed maps $g : HO \to SU(3)$ are consistent with the bundle structure $\pi : SU(3) \to CP_2$ in the sense that g induces a bundle structure $HO \to g^{-1}(CP_2)$. Now however the projection of X^4 to a given fiber need not be a point but can be even four-dimensional surface as in the case of the canonical imbedding of M^4 to H.

4.1.2 A generalization of the solution ansatz to take into account vacuum degeneracy

Vacuum degeneracy is a characteristic feature of Kähler action and implies the presence of infinite number of non-quantum fluctuating zero modes, which do not contribute to the metric of the configuration space. Also the solution ansatz should have analogous degeneracy. This suggests that the solution ansatz assuming that the preferred imaginary unit is same in entire HO is too restricted and should be made local. This means local $S^6 = G_2/SU(3)$ labelling different orientations of the imaginary unit.

Thus the degeneracy due to the possible local choices of the hyper-octonionic complexification corresponding physically to the choice of the plane of non-physical polarizations, becomes a candidate for this degeneracy and would expand the group of local symmetries from $SU(3)$ constrained by the integrability conditions to the entire automorphism group G_2 of octonions. The local fixing of complexification in HO would mean fixing of a map $f : OH \to S^6$. Probably this map could satisfy some constraints forced by the absolute minimization. If the choice of f is completely free, the integrability conditions would be invariant under $G_2 \subset SO(7)$ automorphisms.

The maps $f : HO \to S^6$ and $g : HO \to SU(3)$ can be interpreted as a map $h : HO \to G_2$ in the local trivialization $G_2 = S^6 \times SU(3)$.

4.1.3 Also dual solutions are needed

It seems that also the dual solutions for which the normal space is hyper-quaternionic must be allowed since otherwise it is not possible to understand CP_2 type extremals, which are definitely quaternionic objects. The four parameters labelling the solutions become space-time coordinates for the dual solution whereas the space-time coordinates for the solution parameterize dual solutions.

The surfaces at which the induced metric becomes light-like might allow to glue together solutions corresponding to different functions g and f. The intuitive expectation is that the light-likeness for 3-D surfaces should correspond to the number-theoretic light-likeness of a hyper-quaternionic space-time coordinate. If the hyper-quaternionic functions are rational functions, 3-D light-like causal determinants can appear as a generalization of the poles of a rational function.

4.1.4 Why not octonion analyticity instead of hyper-octonion analyticity?

Mind must be kept open also for the octonionic variant of the solution ansatz might make sense. $HO = M^8$ can be replaced with $O = E^8$, space-time surfaces as hyper-quaternionic sub-manifolds can be replaced with quaternionic sub-manifolds, and the map $M^8 \rightarrow M^4 \times CP_2$ can be replaced with a map $E^8 \rightarrow M^4 \times CP_2$.

The map $O \rightarrow M^4$ is defined as the canonical projection to Q followed by the multiplication of quaternionic imaginary units with $\sqrt{-1}$. Hence the possibility that octonionic ansatz might make sense must be left open.

The differences between the two solution ansätze become obvious when the hypothesis that infinite hyper-octonionic primes are representable in terms of hyper-octonionic polynomials is discussed. As found, these notions of primeness differ in a profound manner, and the fact that hyper-octonionic primes allow an interpretation as Minkowskian 8-momenta encourages to think that they define the correct option.

4.2 Integrability conditions

If the distribution of hyper-quaternionic planes are identifiable as tangent planes of space-time surface $X^4 \subset HO$, commuting tangent vector fields $\partial\alpha$ associated with X^4 coordinate variables x^α exist. Integrability conditions express the commutativity of these vector fields, which be lifted to HO vector fields when space-time surfaces form a foliation of HO. Note that tangent vector fields correspond to H vector fields defined by the gradients $\partial_\alpha h^k$ of imbedding space-coordinates. Note also that $X^4 \subset H = M^4 \times CP_2$ imbedding exists by definition, and it is only the integrability to a 4-surface in HO, which requires additional conditions, hoped to be equivalent with field equations, to be satisfied.

4.2.1 Induction of $SU(3)$ Lie algebra vector fields to HO and tangent plane

$SU(3)$ Lie-algebra generators T^A define vector fields in $SU(3)$. The dual forms ω_A can be induced to HO and either $HO = M^8$ Minkowski metric m_{ij} can be used to lift them to vector fields of HO by the index raising operation

$$\hat{T}^{Ai}\partial_i \quad = \quad m^{ij}\omega_{Ak}\partial_j g^k\partial_i \ . \tag{23}$$

The induced ω_A induced to HO can in turn be induced to forms in the local hyper-quaternionic tangent plane and the metric of the tangent plane allow to transform these forms to vector fields in X^4. The natural tangent plane metric is the metric $g_{\alpha\beta}$ induced either from the metric of HO in HO picture.

The integrability conditions in HO should be equivalent with the field equations defined by Kähler action and in these equations the induced metric and Kähler form of H appear.

4.2.2 Hyper-quaternionicity condition

The hyper-quaternionicity condition states that it is possible to select at each point of HO local $U(2)$ sub-algebra of $SU(3)$. This means that the local algebra is obtained by adjoint action from the standard $U(2)$ algebra at the unit element of $SU(3)$:

$$\begin{aligned} T^m_h \quad &= \quad Ad_g(h)(T^m) = g(h)T^m g^{-1}(h) \ , \\ [T^m_h, T^n_h] \quad &= \quad f^{mn}{}_r T^r_h \end{aligned} \tag{24}$$

4.2.3 The analogy of integrability conditions with those for a flat connection

Integrability implies that X^4 has tangent vector field basis ∂_α in HO. It is possible to express tangent vector fields $\partial_\alpha h^k$ as linear combinations of HO vector fields $\hat{T}^m$ defined the local $U(2)$

Lie-algebra generators T_h^m, where the subscript h tells that the $U(2)$ subalgebra depends on HO coordinate h and is obtained by the adjoint action:

$$\partial_\alpha = A_{\alpha m}\hat{T}_h^m \ .$$

The interpretation of A as an analog of $U(2)$ gauge potentials suggests itself. The difference is that $\hat{T}^m$ does not represent $SU(3)$ vector field but induced HO vector field.

The integrability conditions express the commutativity condition $[\partial_\alpha, \partial_\beta] = 0$. If $\hat{T}_h^m$ would represent $SU(3)$ vector field, the integrability conditions would translate to the flatness of $U(2)$ connection:

$$dF \ = \ dA + [A, A] = 0 \ . \tag{25}$$

In the recent case the conditions have a more complex but analogous form:

$$\left[A_{\alpha m}(\hat{T}_h^m \circ A_{\beta n}) - A_{\beta m}(\hat{T}_h^m \circ A_{\alpha n})\right]\hat{T}_h^n + A_{\alpha m}A_{\beta n}\left[\hat{T}_h^m, \hat{T}_h^m\right] = 0 \ . \tag{26}$$

The generators $\hat{T}^m$ defining $U(2)$ basis differ by a local $SU(3)$ gauge transformation and by the effects caused by induction from the standard basis. An analog of with flat $U(2)$ connection is obvious. This brings in mind the structure of CP_2 as coset space: also in this case $U(2)$ acts as a local gauge group and permutes points inside $U(2)$ cosets. Now these cosets wold be replaced by cosets of local $SU(3)$.

To gain a better understanding of what is involved it is good to clarify what are the basic bundle structures involved.

1. $SU(3) \rightarrow CP_2$ defines a $U(2)$ bundle and is essential for H picture.

2. The tangent bundle $T(G) \rightarrow G = SU(3)$ and corresponding cotangent bundle $T^*(G)$ are essential for HO picture and appear in the integrability conditions. $T(G) \rightarrow G = SU(3)$ bundle structure is induced to give bundle with base space HO by mapping it first to cotangent bundle by assigning to vector fields their duals, inducing the cotangent bundle by standard procedure, and lifting it back to (possibly sub-) vector bundle of the tangent bundle of HO. The induction procedure for vector fields is what brings in dynamics involving metric.

3. $U(2)$ is identified as a sub-manifold of the base G at given point of CP_2, and $U(2)$ tangent space vector fields are induced to vector fields in local hyper-quaternionic spaces and integrability conditions imply that these vector fields define tangent space basis in X^4.

4.3 How to solve the integrability conditions?

In the following some attempts to understand integrability conditions are made. After more or less ad hoc attempts an ansatz based on hyper-octonion analyticity is proposed.

4.3.1 Guesses for the solution of integrability conditions

A trivial vacuum solution with constant CP_2 coordinates results if the local trivialization $SU(3) = U(2) \times CP_2$ is induced to HO by the map g and space-time surfaces correspond to inverse images of $U(2)$. Hyper-quaternionic sub-space is same at each point of X^4 now. Any invertible map g defines trivial vacuum solutions in this manner. Obviously, non-trivial solutions cannot be consistent with the local foliation $SU(3) = U(2) \times CP_2$.

One might wishfully think that the expression for a flat connection generalizes and defines a solution of the integrability conditions also now. This would boil down to the replacement

$$A_\alpha = h^{-1}\partial_\alpha h = (h^{-1}\partial_\alpha h)_m T^m h \rightarrow \quad (h^{-1}\partial_\alpha h)_m \hat{T}^m \ . \tag{27}$$

h should be $U(2)$ valued map $X^4 \rightarrow U(2)$ for each sheaf of the foliation and would define $U(2)$ coordinate for X^4.

A second possibility popping in mind is that the induced vector fields in X^4 define the original $U(2)$ Lie-algebra for the solutions of the integrability conditions apart from the scaling of Lie-algebra generators, and thus of structure constants, by functions of $U(2)$ invariants. This would mean that the effect of the adjoint action and index raising operation with HO metric preserves $U(2)$. In this case A would define a genuine $U(2)$ connection and integrability conditions would state its flatness.

4.3.2 Do hyper-octonion analytic maps $HO \rightarrow HO$ define solutions to the integrability conditions?

The proposed view raises two challenges. First of all, a general solution to the integrability conditions should be found. Second task is to demonstrate that field equations determined by Kähler action are under some additional conditions equivalent with the solution.

1. Hyper-octonion analytic ansatz when the preferred imaginary unit is same everywhere

Hyper-octonion analytic maps with real coefficients from OH to itself suggests themselves as candidates for this kind of maps. The key observation is that it is possible to assign to a map $HO \rightarrow HO$ a map $HO \rightarrow SU(3)$. HO tangent space has $1 \oplus 1 \oplus 3 \oplus \overline{3}$ decomposition so that the tensor product of $3 \otimes \overline{3}$ gives a color octet vector field identifiable as an element of local $SU(3)$ Lie algebra. The exponentiation of this vector field defines an element of local $SU(3)$ defining in HO a distribution of hyper-quaternionic tangent planes.

If hyper-octonion analyticity guarantees integrability conditions, a foliation of HO by 4-surfaces X^4 and hence of $H = M^4 \times CP_2$ results. There is a definite analogy with spontaneous compactification in that TGD in flat 8-D non-compact space HO would be equivalent TGD in $M^4 \times CP_2$.

Hyper-quaternion analytic maps with real Laurent coefficients are of form

$$h_0 + \overline{h} \rightarrow a(h)h_0 + b(h)\overline{h}$$

as is easy to find by looking what happens in the map $h \rightarrow h^2$ and by generalizing using induction.

The solution ansatz involves two $U(2)$ algebras. The first one corresponds to the hyper-quaternionic tangent space and for this representation hyper charge generators is represented by unit matrix. This algebra is very much analogous to electro-weak $U(2)$. Second representation of $U(2)$ algebra results from $2 \times \overline{2}$ tensor product. It would not be too surprising if these algebras could be mapped to each other in the sense that octonionic products for $2 + \overline{2}$ would give the hyper-quaternionic $U(2)$.

2. Hyper-octonionic solution ansatz allowing a local choice of the preferred imaginary unit

Also more general maps defined as composites of a local G_2 rotation $O(h)$ performed for the imaginary part of h and followed by hyper-octonion analytic map are possible and give rise to the result

$$h_0 + \overline{h} \rightarrow h_0 + O(h)(\overline{h}) \rightarrow a(h)h_0 + b(h)O(h)(\overline{h}) \ ,$$

and correspond to more general solution ansatz giving hopes of understanding vacuum degeneracy of the Kähler action.

The generalization of solution ansatz allowing a local G_2 rotation of octonionic basis means that the hyper-octonionic transformation is replaced with

$$h_0 + \overline{h} \rightarrow a(h_0, |h|^2) + b(h_0, |h|^2)O(h) \circ \overline{h} \ , \tag{28}$$

where $O(h) \in G_2$ is HO-local $SO(7)$ rotation. The solution ansatz defines an element of local G_2 in the local trialization $G_2 = S^6 \times SU(3)$. The imaginary part of the hyper-octonion forms a 7-D representation of G_2 and $7 \otimes 7$ tensor product defines an element of the Lie-algebra of G_2 and hence the tensor product defines a map to local G_2. The obvious question is whether the two elements of G_2 defined in this manner are identical.

There is an interesting connection with the conjecture that S^6 does not allow complex structure although it allows almost complex structure which does not allow the representations of imaginary unit in local coordinates. Just before his death Chern published a proposal for a proof of this conjecture [43, 44]. Kähler structure is prerequisite for quantization so that the conjecture would be consistent with the idea that neither Kähler nor complex structure are possible in these degrees degrees of freedom so that local S^6 should indeed represent non-quantum fluctuating zero modes.

A numerologically interesting observation is that the dimensionality of the local parameter space would be $6+4 = 10$: together with the effective 2-dimensionality this means a definite analogy with superstring limits of M(embrane)-theory. If this picture is correct, also the identification of space-time as a 4-surface in $M^4 \times S^6 \times CP_2$ would be also possible. A possible physical interpretation of S^6 degeneracy, possibly constrained by some dynamical conditions, is in terms of vacuum degeneracy of Kähler action.

3. Is reduction to Lie-algebra level possible?

The $SU(3)$ generator given by the tensor product $3 \times \overline{3}$ is of form

$$X = b^2(h) h_3^i h_{\overline{3}}^j C_{ijA} T^A \ , \tag{29}$$

when the preferred imaginary unit is same everywhere. The expression in the general case involves the action of G_2 rotation on the triplets.

The Lie-algebra element at a given point of HO differs only by the scaling factor $b^2(h)$ for different maps when the choice of imaginary unit is kept fixed. Therefore, at a given point of HO the values of $g(h)$ for various hyper-quaternion analytic maps belong to the same one-parameter sub-group $U(1)_h$ determined by $X(h)$.

This reduction and the fact that g is otherwise arbitrary and can be arbitrary near to identity map raises the hope that it is enough to consider the conditions infinitesimally so that $Ad(g) - 1$ reduces infinitesimally to a commutator in Lie algebra. If this is the case, the conditions are satisfied if X is annihilated by the adjoint action of the $U(2)$ generators T^m and would thus define an $U(2)$ invariant vector field in X^4. Taking into account the universal nature of X this not be surprising. Since b^2 is $U(2)$ invariant function so that the remaining universal vector field should be invariant under local $U(2)$ and analogous to the $SU(3)$ invariant vector fields in CP_2.

4.4 About the physical interpretation of the solution ansatz

4.4.1 Hyper-octonionic analyticity and effective 2-dimensionality

The number of local integrability conditions is 6 corresponding to all index pairs for $U(2)$ algebra so that $8 - 6 = 2$ free functions should appear in the map g. The effective 2-dimensionality for the absolute minima is basic ideas of TGD and means that they are determined by the data at partonic 2-surfaces so that also this suggests algebraic two-dimensionality.

The effective 1-dimensionality due to the real analyticity of the hyper-octonionic map, would suggests that the ansatz is too limited. As a matter fact, g is expressible in terms of the conjugate of the hyper-octonionic map and its conjugate so that g depends on both h_T (that is $3 \oplus \overline{3}$ of h) and its conjugate and in this the situation is algebraically 2-dimensional. That the longitudinal degrees of freedom corresponding to $(1, e_1)$ tangent plane do not appear in the expression for g, has physical interpretation in terms of the elimination of longitudinal polarizations. An analogous phenomenon occurs also for the known solutions and in Hamilton Jacobi coordinates $(u, v, w, \overline{w}$ for M^4 it corresponds to the plane spanned by the light-like coordinates u and v.

4.4.2 $HO - H$ duality as color-electro-weak duality?

One can wonder whether the imbedding defines naturally classical electro-weak and color gauge potentials at the space-time surface. One can also wonder how the two dual pictures corresponding to HO and $M^4 \times CP_2$ relate to this.

1. The projections of the duals of $SU(3)$ Lie algebra generators lifted to vector fields at the space-time surface would be natural candidates for classical color gauge fields. If the $U(2)$ algebra is preserved in the induction procedure, the integrability conditions imply the vanishing of a genuine $U(2)$ gauge field. A natural interpretation would be as an electro-weak gauge field. Electro-weak gauge fields would not appear in HO picture.

2. In H picture electro-weak gauge potentials can be induced from the spinor connection of $M^4 \times CP_2$. The projections of Killing vector fields of $SU(3)$ in CP_2 define analogs of gluons but since they do not appear in the modified Dirac equation for induced spinors nor in the Dirac equation for imbedding space, one might argue that genuine gluon fields are not in question.

These observations give some hints about the concrete physical interpretation of $HO - H$ duality. For HO representation of the space-time surface classical color gauge fields are naturally present whereas for H representation this is the case only for electro-weak gauge fields. A vague hunch about this kind of duality has been present in TGD framework from beginning. For instance, induced spinor fields do not carry color as a spin like quantum number whereas color triplet and antitriplet occur naturally in HO representation and could multiply the solutions of the modified Dirac equation in HO.

If this duality makes sense, H picture could correspond to the description of hadron physics using hadrons as basic particles and using the current algebra defined by the electro-weak currents. HO picture would correspond to QCD approach based on the use of color currents. Color confinement might be seen as an impossibility to detect color in the experiments based on $M^4 \times CP_2$ description.

4.5 Mapping of infinite primes to space-time surfaces

At the lowest level of hierarchy the mapping of hyper-octonionic infinite primes to 4-surfaces is a special case of assigning to a hyper-octonion analytic function a foliation of imbedding space by 4-surfaces. At higher levels of hierarchy the mapping of infinite primes to space-time surfaces requires a generalization of this procedure and the constraints from non-commutativity and non-associativity dictate the generalization completely.

4.5.1 Associativity as the basic constraint

On basis of the general vision about how hyper-octonion analytic maps of HO to itself correspond to four-surfaces in $M^4 \times CP_2$ and perhaps also absolute minima of Kähler action, it is clear that the hyper-octonionic polynomials defined by the infinite primes at the first level of hierarchy indeed define a foliations of $M^4 \times CP_2$ by four-dimensional surfaces with an additional degeneracy corresponding to the possibility to choose freely the map $f : HO \rightarrow S^6$ characterizing the choice of preferred imaginary octonionic unit, or equivalently the plane defined by time-like polarizations. There is also a degeneracy related to the choice of the origin of HO coordinates and due to the $SO(7,1)$ invariance acting at the level of $M^8 = HO$.

The basic objection is that the polynomials representing infinite are ill defined at the higher levels of hierarchy due to the problems caused by non-associativity even in case that one restricts the consideration to rational functions with real coefficients. The only resolution of this objection is that the arguments h_i are functionally independent so that one can express h_i, $i > 1$ as hyper-octonion real-analytic function of h_1. Rational functions look especially natural and one can consider also the identification $h_n = h_{n-1} = ... = h_1$.

This assumption reduces the representation to one-dimensional case and if hyper-octonion real-analytic functions define foliations of imbedding space by quaternionic space-time surfaces, one obtains a hierarchy of increasingly complex space-time surfaces. An open question is whether the hierarchy of infinite primes indeed corresponds to a hierarchy of space-time sheets.

The requirement that the theory allows p-adicization is not only a challenge but also a heavy constraint. If everything is rational at the basic level in the proposed sense, there are indeed good hopes for the p-adicization at space-time level. This optimistic view is also encouraged by the recent formulation of quantum TGD as almost topological conformal field theory [C1].

The ordering of the arguments of the polynomials characterizes the thoughts about thoughts hierarchy as a hierarchy in which algebraic complexity increases and, as already noticed, also the Slaving Hierarchy. h_n corresponds to the highest level of the hierarchy and h_1 to its lowest level. Topological condensate indeed gives rise to this kind of hierarchy very naturally. This hierarchy is not lost even in the reduction of variables to single hyper-octonionic variable.

The identification allows a generalization of the basic philosophy of algebraic geometry. The rational functions associated with infinite primes have natural ordering with respect to their degree and dimension of algebraic extension of rationals associated with the roots of these polynomials. This makes sense for both functions of n complex arguments and single hyper-octonionic argument. Hence the space-time surfaces can be ordered in a natural manner with respect to their algebraic complexity. One could hope that this kind of ordering might be of decisive help in the physical interpretation of the predictions of the theory.

The most elegant theory results if all infinite primes are assumed to reduce to rational infinite primes and that the decomposition to primes in algebraic completions of rationals and to quaternionic, octonionic, hyper-octonionic infinite primes and their variants in the complexification of quaternions and octonions reflects to or is at least analogous to the possibility to decompose a particle into its more elementary constituents. One might hope that number theoretic analog of color confinement translates to a deep physical principle.

4.5.2 Interaction between infinite primes fixes the scaling of the polynomials associated with infinite primes

The assignment of a polynomial with an infinite prime is unique only up to an over-all scaling and the following argument suggests that the only physically acceptable scaling corresponds to the normalization of the constant term, call it c, of the polynomial to $c = 1$.

In algebraic geometry the zeros of polynomials as their representations has the property that the product of polynomials corresponds to a union of disjoint surfaces and there is no interaction between the surfaces. For infinite integers represented in terms of hyper-quaternionic surfaces this is not the case. This raises the question whether this state of affairs makes possible a realistic number theoretical description of interactions. This description could the counterpart for the description based on the absolute minima of Kähler action which are not simply disjoint unions of absolute minima associated with two 3-surfaces. It would also be analog for the description of the interaction between different space-time sheets in terms of polynomials defined by higher level infinite primes.

This interaction should be consistent with the idea that the interaction of the systems described by infinite primes is weak in some space-time regions. This is certainly the case if the polynomial approaches constant equal to one. To see what happens consider the product of polynomials associated with two infinite primes. The expectation is that in the regions where second hyper-octonion analytic polynomials P_1 approaches to a constant value, which must be real by real-analyticity, the product of infinite primes defines a 4-surface which resembles the surfaces associated with P_2.

The product of hyper-octonion analytic functions $g_1 = a_1 + b_1\overline{h}$ and $g_2 = a_2 + b_2\overline{h}$ is $a_1a_2 + b_1b_2\overline{h}\cdot\overline{h} + (a_1b_2 - a_2b_1)\overline{h}$. If b_1 approaches to zero, the product behaves as $a_1a_2 + a_1b_2\overline{h}$, so that a_1 should approach to $a_1 = 1$ in order that interaction would be negligible.

The observation would suggest that the mapping of infinite primes to polynomials must involve a scaling taking care that the constant term appearing in the polynomial equals to one. This kind of scaling is of course possible. It would however mean that infinite primes with polynomials for which constant term vanishes are not allowed. This would mean that products of conjugates of infinite primes for which finite part is proportional to a light-like integer are not allowed since in this case the constant term vanishes. This is true if one assumes that hyper-octonionic infinite primes reduce to rational infinite primes.

5 How to interpret the infinite hierarchy of infinite primes?

From the foregoing it should be clear that infinite primes might play key role in quantum physics. One can even consider the possibility that physics reduces to a generalized number theory, and that infinite primes are crucial for understanding mathematically consciousness and cognition. Of course, one must leave open the question whether infinite primes really provide really the mathematics of consciousness or whether they are only a beautiful but esoteric mathematical construct. In this spirit the following subsections give only different points of view to the problem with no attempt to a coherent overall view.

5.1 Infinite primes and hierarchy of super-symmetric arithmetic quantum field theories

Infinite primes are a generalization of the notion of prime. They turn out to provide number theoretic correlates of both free, interacting and bound states of a super-symmetric arithmetic quantum field theory. The mapping of infinite primes to polynomials in turn allows to assign to infinite prime space-time surface as a geometric correlate. Hence infinite primes serve as a bridge between classical and quantum and realize quantum classical correspondence stating that quantum states have classical counterparts, and has served as a basic heuristic guideline of TGD. More precisely, the natural hypothesis is that infinite primes code for the ground states of super-canonical representations (for instance, ordinary particles correspond to states of this kind).

5.1.1 Generating infinite primes as counterparts of Fock states of a super-symmetric arithmetic quantum field theory

The basic construction recipe for infinite primes is simple and generalizes to the quaternionic case.

1. Form the product of all primes and call it X:

$$X = \prod_p p \quad .$$

2. Form the vacuum states

$$V_{\pm} = X \pm 1 \quad .$$

3. From these vacua construct all *generating* infinite primes by the following process. Kick out from the Dirac sea some negative energy fermions: they correspond to a product s of first powers of primes: $V \rightarrow X/s \pm s$ (s is thus square-free integer). This state represents a state with some fermions represented as holes in Dirac sea but no bosons. Add bosons by multiplying by integer r, which decomposes into parts as $r = mn$: m corresponding to bosons in X/s is product of powers of primes dividing X/s and n corresponds to bosons in s and is product of powers of primes dividing s. This step can be described as $X/s \pm s \rightarrow mX/s \pm ns$.

Generating infinite primes are thus in one-one correspondence with the Fock states of a super-symmetric arithmetic quantum field theory and can be written as

$$P_{\pm}(m, n, s) = \frac{mX}{s} \pm ns \quad ,$$

where X is product of all primes at previous level. s is square free integer. m and n have no common factors, and neither m and s nor n and X/s have common factors.

The physical analog of the process is the creation of Fock states of a super-symmetric arithmetic quantum field theory. The factorization of s to a product of first powers of primes corresponds to many-fermion state and the decomposition of m and n to products of powers of prime correspond to bosonic Fock states since p^k corresponds to k-particle state in arithmetic quantum field theory.

5.1.2 More complex infinite primes as counterparts of bound states

Generating infinite primes are not all that are possible. One can construct also polynomials of the generating primes and under certain conditions these polynomials are non-divisible by both finite primes and infinite primes already constructed.

The physical counterpart of n:th order irreducible polynomial is as a bound state of n particles whereas infinite integers constructed as products of infinite primes correspond to non-bound but interacting states. This process can be repeated at the higher levels by defining the vacuum state to be the product of all primes at previous levels and repeating the process. A repeated second quantization of a super-symmetric arithmetic quantum field theory is in question.

The fact that more general infinite primes can be constructed as polynomials of the generating infinite primes, suggest strongly that infinite primes can be mapped to ordinary polynomials by replacing the argument X in $V_{\pm} = X \pm 1$ with variable h. This indeed turns out to be the case. This correspondence allows to deduce that more general infinite primes correspond to irreducible polynomials of generating infinite primes not allowing decomposition to a product of generating infinite primes.

The infinite primes represented by irreducible polynomials correspond to quantum states obtained by mapping the superposition of the products of the generating infinite primes to a superposition of the corresponding Fock states. If complex rationals are the coefficient field for infinite integers, this gives rise to states in a complex Hilbert space and irreducibility corresponds to a superposition of states with varying particle number and the presence of entanglement. For instance, the superpositions of several products of type $\prod_{i=1,..,n} P_i$ of n generating infinite primes are possible and in general give rise to irreducible infinite primes decomposing into a product of infinite primes in algebraic extension of rationals.

5.2 Prime Hilbert spaces and infinite primes

There is a result of quantum information science providing an additional reason why for p-adic physics. Suppose that one has N-dimensional Hilbert space which allows $N+1$ unbiased basis. This means that the moduli squared for the inner product of any two states belonging to different basis equals to $1/N$. If one knows all transition amplitudes from a given state to all states of all $N+1$ mutually unbiased basis, one can fully reconstruct the state. For $N=p^n$ dimensional $N+1$ unbiased basis can be found and the article of Durt[57] gives an explicit construction of these basis by applying the properties of finite fields. Thus state spaces with p^n elements - which indeed emerge naturally in p-adic framework - would be optimal for quantum tomography. For instance, the discretization of one-dimensional line with length of p^n units would give rise to p^n-dimensional Hilbert space of wave functions.

The observation motivates the introduction of prime Hilbert space as as a Hilbert space possessing dimension which is prime and it would seem that this kind of number theoretical structure for the category of Hilbert spaces is natural from the point of view of quantum information theory. One might ask whether the tensor product of mutually unbiased bases in the general case could be constructed as a tensor product for the bases for prime power factors. This can be done but since the bases cannot have common elements the number of unbiased basis obtained in this manner is equal to $M+1$, where M is the smallest prime power factor of N. It is not known whether additional unbiased bases exists.

5.2.1 Hierarchy of prime Hilbert spaces characterized by infinite primes

The notion of prime Hilbert space provides also a new interpretation for infinite primes, which are in 1-1 correspondence with the states of a supersymmetric arithmetic QFT. The earlier interpretation was that the hierarchy of infinite primes corresponds to a hierarchy of quantum states. Infinite primes could also label a hierarchy of infinite-D prime Hilbert spaces with product and sum for infinite primes representing unfaithfully tensor product and direct sum.

1. At the lowest level of hierarchy one could interpret infinite primes as homomorphisms of Hilbert spaces to generalized integers (tensor product and direct sum mapped to product and sum) obtained as direct sum of infinite-D Hilbert space and finite-D Hilbert space. (In)finite-D Hilbert space is (in)finite tensor product of prime power factors. The map of N-dimensional Hilbert space to the set of N-orthogonal states resulting in state function reduction maps it to N-element set and integer N. Hence one can interpret the homomorphism as giving rise to a kind of shadow on the wall of Plato's cave projecting (shadow quite literally!) the Hilbert space to generalized integer representing the shadow. In category theoretical setting one could perhaps see generalize integers as shadows of the hierarchy of Hilbert spaces.

2. The interpretation as a decomposition of the universe to a subsystem plus environment does not seem to work since in this case one would have tensor product. Perhaps the decomposition could be to degrees of freedom to those which are above and below measurement resolution. One could of course consider decomposition to a tensor product of bosonic and fermionic state spaces.

3. The construction of the Hilbert spaces would reduce to that of infinite primes. The analog of the fermionic sea would be infinite-D Hilbert space which is tensor product of all prime Hilbert spaces H_p with given prime factor appearing only once in the tensor product. One can "add n bosons" to this state by replacing of any tensor factor H_p with its n+1:th tensor power. One can "add fermions" to this state by deleting some prime factors H_p from the tensor product and adding their tensor product as a finite-direct summand. One can also "add n bosons" to this factor.

4. At the next level of hierarchy one would form infinite tensor product of all infinite-dimensional prime Hilbert spaces obtained in this manner and repeat the construction. This can be continued ad infinitum and the construction corresponds to abstraction hierarchy or a hierarchy of statements about statements or a hierarchy of n:th order logics. Or a hierarchy of space-time sheets of many-sheeted space-time. Or a hierarchy of particles in which certain many-particle states at the previous level of hierarchy become particles at the new level (bosons and fermions). There are many interpretations.

5. Note that at the lowest level this construction can be applies also to Riemann Zeta function. ζ would represent fermionic vacuum and the addition of fermions would correspond to a removal of a product of corresponding factors ζ_p from ζ and addition of them to the resulting truncated ζ function. The addition of bosons would correspond to multiplication by a power of appropriate ζ_p. The analog of ζ function at the next level of hierarchy would be product of all these modified ζ functions and might well fail to exist since the product might typically converge to either zero or infinity.

5.2.2 Hilbert spaces assignable to infinite integers and rationals make also sense

1. Also infinite integers make sense since one can form tensor products and direct sums of infinite primes and of corresponding Hilbert spaces. Also infinite rationals exist and this raises the question what kind of state spaces inverses of infinite integers mean.

2. Zero energy ontology suggests that infinite integers correspond to positive energy states and their inverses to negative energy states. Zero energy states would be always infinite rationals with real norm which equals to real unit.

3. The existence of these units would give for a given real number an infinite rich number theoretic anatomy so that single space-time point might be able to represent quantum states of the entire universe in its anatomy (number theoretical Brahman=Atman). Also the world of classical worlds (light-like 3-surfaces of the imbedding space) might be imbeddable to this anatomy so that basically one would have just space-time surfaces in 8-D space and configuration space would have representation in terms of space-time based on generalized notion of number. Note that infinitesimals around a given number would be replaced with infinite number of number-theoretically non-equivalent real units multiplying it.

5.2.3 Should one generalize the notion of von Neumann algebra?

Especially interesting are the implications of the notion of prime Hilbert space concerning the notion of von Neumann algebra -in particular the notion of hyper-finite factors of type II_1 playing a key role in TGD framework. Does the prime decomposition bring in additional structure? Hyper-finite factors of type II_1 are canonically represented as infinite tensor power of 2×2 matrix algebra having a representation as infinite-dimensional fermionic Fock oscillator algebra and allowing a natural interpretation in terms of spinors for the world of classical worlds having a representation as infinite-dimensional fermionic Fock space.

Infinite primes would correspond to something different: a tensor product of all $p\times p$ matrix algebras from which some factors are deleted and added back as direct summands. Besides this some factors are replaced with their tensor powers. Should one refine the notion of von Neumann algebra so that one can distinguish between these algebras as physically non-equivalent? Is the full algebra tensor product of this kind of generalized hyper-finite factor and hyper-finite factor of type II_1 corresponding to the vibrational degrees of freedom of 3-surface and fermionic degrees of freedom? Could p-adic length scale hypothesis - stating that the physically favored primes are near powers of 2 - relate somehow to the naturality of the inclusions of generalized von Neumann algebras to HFF of type II_1?

5.3 Do infinite hyper-octonionic primes represent quantum numbers associated with Fock states?

Hyper-octonionic primes involve so much structure that one can seriously consider the possibility that they could code quantum numbers of elementary particles which in accordance with quantum-classical correspondence would be coded to the shape of space-time surfaces.

5.3.1 Hyper-octonionic infinite primes as representations for quantum numbers of Fock states?

Configuration space spinor fields assign infinite number of quantum states to a given 3-surface as components of configuration space spinor. This suggests that there cannot be one-to-one correspondence between Fock states and space-time surfaces except in the approximation that one replaces configuration space spinor field with single 'quantum average space-time'. This forces to consider critically the identification of the hyper-octonionic primes as quantum numbers.

Perhaps a more realistic identification of infinite primes is as coding for the quantum numbers for the ground states of the representations of super-canonical and Kac-Moody algebras. This identification would be in an agreement with the view that space-time surfaces represent only the classical aspects of physics but not quantum fluctuations. Arithmetic quantum field theory should represent only the sector of ground states of quantum TGD.

It is interesting to check whether hyper-octonionic infinite primes could allow a realistic coding for the quantum numbers of ground states of super Kac-Moody representations.

1. If it is assumed that each prime in the finite part of X corresponds to a fermion, the requirement that the Fock state possesses a well-defined fermion number poses constraints on the structure of the polynomial associated with the infinite prime. A product of generating infinite primes in algebraic extension of real-rationals interpreted as representing states for which rest mass is changed by bound state interactions, would however resolve these constraints. Also super-positions of products are allowed but in this case net fermion numbers associated with various monomials must be same.

2. Hyper-octonionic infinite prime could be interpreted as coding for the relationship between particle four-momentum represented by the hyper-quaternionic part of infinite prime and the quantum numbers associated with CP_2 degrees of freedom represented by the quaternionic part of the infinite prime. Electro-weak isospin and hyper charge and corresponding color quantum numbers indeed give rise to four quantum numbers.

   Mass squared formula for infinite primes, and more generally, infinite integers would be the basic string mass formula. For bound states the mass squared values would be primes in algebraic extension of rationals.

3. Space-like hyper-octonionic primes do not seem to be natural in the case of hyper-octonionic option. Octonionic option would allow them but in this case the interpretation in terms of momenta is lost. This not so plausible option would allow as a special case Gaussian and Eisenstein primes discussed in [E8]. Eisenstein primes correspond to algebraic extension involving $\sqrt{3}$. These primes correspond to time-like primes obtained by multiplying the prime with a suitable unit. The degeneracies of these primes due to units defined by complex phases are 4 and 8. One can ask whether these degeneracies might relate to the spin states of imbedding space spinors.

4. If the proposed interpretation is taken at face value, the question about distinction between quarks and leptons at the level of infinite primes, arises. Somehow the two different chiralities for induced imbedding space spinor fields should have space-time correlates. If the primes $p \ mod \ 4 = 1$ and $p \ mod \ 4 = 3$ correspond to leptons and quarks or vice versa it would be possible to assign to each generating infinite prime lepton or quark number. Bosons could be regarded as fermion-antifermion bound states and bosonic surfaces would correspond to the composites of two infinite primes with either $p \ mod \ 4 = 1$ or $p \ mod \ 4 = 3$ or superposition of this kind of monomials.

5. Since only polynomials with real coefficients are possible, kind of number theoretic analog of color confinement occurs, and requires that at least two generating infinite primes with the hyper-octonionic zero of the corresponding monomial with components belonging to an algebraic extension of real rationals appears in the state. This confinement has counterpart at the level of super-canonical conformal weights which are complex and expressible in terms of zeros of Riemann Zeta: only states with real net conformal weight are possible.

6. One can imagine several interpretations for the two vacua $V_{\pm} = X \pm 1$.
i) The most plausible interpretation for these vacua is in terms of matter and antimatter and thus as representations for states having opposite fermion number. In number theoretic bound states represented by higher degree polynomials both matter and antimatter particles can occur.
ii) A less plausible interpretation is as positive and negative energy vacua associated with the space-time sheets of opposite time orientation predicted by TGD. The fact that negative energy particles do not seem to appear in elementary particle reactions inspires the hypothesis that negative energies are associated with higher level infinite primes and correspond to the infinite primes defining the denominators of the rational functions appearing in the definitions of higher level infinite primes. Phase conjugate photons would be a basic example of negative energy particles.
iii) Also the interpretation in terms of the vacua of associated Ramond and NS type super canonical algebras can be considered.

There are also other degrees of freedom besides Super Kac Moody degrees fo freedom.

1. Zero modes are an essential part of TGD and would correspond to the degrees of freedom associated with the maps $HO \rightarrow S^6$ and their generalization to the higher levels of the hierarchy. Physical interpretation would be as a imbedding space dependent selection of longitudinal degrees of freedom in turn fixing at space-time level the spin quantization axis and the transversal degrees of freedom associated with polarizations of massless particles.

2. There is no obvious relation between super-canonical conformal weights and infinite primes. Perhaps the reason is that these quantum numbers are associated with configuration space spinor fields.

5.3.2 Family replication phenomenon and commutative sub-manifolds of space-time surface

The idea that complex Abelian sub-manifolds of space-time sheets are in preferred role by their commutativity in hyper-octonionic sense, is consistent with the topological explanation of family replication phenomenon [F1] by interpreting different particle families as particles with corresponding 3-surface having boundary with genus $g = 0, 1, 2, ...$

The representations $p = f(q)$ of the algebraic surfaces with real-analytic f, when restricted to complex numbers, define 2-dimensional Riemann surfaces in 4-dimensional complex space. These surfaces are characterized by genus so that genus emerges in very natural manner from the theory.

If the boundary component has same genus as the genus defined by hyper-quaternionicity, then the notion of elementary particle vacuum functional makes sense, and p-adic mass calculations [F2, F3, F4, F5] which rely crucially on the notion of genus, remain unchanged. natural possibility is that the 2-surface where hyper-quaternions are commutative in fact corresponds to a boundary component of 3-surface. The 2-dimensional intersections of 3-D light-like causal determinants X_l^3 and 7-D light-like causal determinants defined by boundaries of future and past light-cones of M^4 are natural candidates for partonic 2-surfaces. If this picture is correct, one can also answer the troublesome question 'What is the two-dimensional sub-manifold of 3-dimensional boundary of space-time surface to which one assigns elementary particle vacuum functional?'. This question is of high relevance since the conformal equivalence class of boundary component depends on how the boundary component is identified.

5.4 The physical interpretation of infinite integers at the first level of the hierarchy

The idea that primes are for the number theory what elementary particles are for physics, suggests that the decomposition of an infinite integer to a product of infinite primes corresponds to the decomposition of a physical system to elementary systems allowing no further decomposition.

5.4.1 Higher degree polynomial primes as bound states

The sums for the products of infinite primes defining irreducible polynomials define infinite primes describing many particle states and the interpretation as composites of space-time surfaces associated with simpler 'effective' generating infinite primes belonging to the extension of quaternions is natural and leads to a dynamical generation of algebraic symmetries. A natural interpretation is as topological composites formed from space-time surfaces describing bound states. Each root of the polynomial equation defining a branch of the space-time surface would correspond to a particle present in the composite. Indeed, n:th order irreducible polynomial factors to product of monomials $x-l$, $l \notin K$. If the polynomial differs only slightly from a product of prime polynomials, it is natural to interpret the slight change of the roots as a slight change of the composite states induced by the mutual interaction.

5.4.2 Infinite integers as interacting many particle states

The space-time surfaces representing infinite integers could represent many-particle states. The space-time surface associated with the integer is in general not a union of the space-time surfaces associated with the primes composing the integer. This means that classical description of interactions emerges automatically. The description of classical states in terms of infinite integers is completely analogous to the description of many particle states as finite integers in arithmetic quantum field theory.

The finite primes which correspond to particles of an arithmetic quantum field theory present in Fock state, correspond to the space-time sheets of finite size serving as the building blocks of the space-time sheet characterized by infinite prime. Real topology is the space-time topology in the regions, where matter resides whereas 'mind stuff' corresponds to the regions obeying p-adic topology. This is in accordance with the fact that the physics based on real numbers is so successful. The success of p-adic physics could be understood as resulting from the fact that it describes the physics of the mind like regions mimicking the physics of the real matter-like regions.

5.5 What is the interpretation of the higher level infinite primes?

Interesting questions are related to the higher level infinite primes obtained by taking X to be a product of all lower level primes and repeating the construction.

5.5.1 Infinite hierarchy of infinite primes

Infinite hierarchy of infinite primes codes for a hierarchy of Fock states such that many-particle Fock states of a given level serve as elementary particles at next level. The unavoidable conclusion is that higher levels represent totally new physics not described by the standard quantization procedures. In particular, the assignment of fermion/boson property to arbitrarily large system would be in some sense exact. Topologically these higher level particles could correspond to space-time sheets containing many-particle states and behaving as higher level elementary particles.

This view suggests that the generating quantum numbers are present already at the lowest level and somehow coded by the hyper-octonionic primes taking the role of momentum quantum number they have in arithmetic quantum field theories. The task is to understand whether and how hyper-octonionic primes can code for quantum numbers predicted by quantum TGD.

The quantum numbers coding higher level states are collections of quantum numbers of lower level states. At geometric level the replacement of the coefficients of polynomials with rational functions is the equivalent of replacing single particle states with new single particle states consisting of many-particle states.

5.5.2 Rationals of the previous level appear at given level

What is remarkable is that the rationals formed from the integers of $n-1$:th level label the simplest primes of n:th level. The numerator and denominator of the rational number correspond to a pair of integers representing physical states at previous level, which suggests that the new states are higher level physical states representing information about pairs of physical states at the previous level. The most natural guess is that the states of the pair correspond to the initial and final states of a quantum jump. In this manner the infinite hierarchy give rise to physical states representing increasingly abstract information about dynamics. The fact that I am a physical system ponder physics problems could be seen as a direct evidence for the existence for these higher levels of physical existence.

At the next level physical states represent information about pairs of quantum jumps which in TGD inspired theory of consciousness correspond to memories about primary conscious experiences determined by quantum jumps. They clearly represent experiences about experiences. At n:th level quantum jump represent n-fold abstraction giving conscious information about experiences about.....about experiences.

TGD allows space-time sheets with both positive and negative time orientation and the sign of classical energy correlates with the orientation of the space-time sheet. This leads to a radical revision of the energy concept and clarifies the relationship between gravitational and inertial energy. The interpretation of the numerator and denominator of the infinite rational in terms of positive and negative energy space-time sheets looks natural. Of course, one must be ready to consider the possibility that "energy" might be replaced by some other conserved quantity. This interpretation would also explain why negative energy particles appear only at higher organization level of matter and are not detected in accelerators. Indeed, the basic TGD applications relate to quantum biology, consciousness [K1], and free energy [G2].

The interpretation of particle reactions as quantum jumps between zero energy states is implied by this vision, and this interpretation is consistent with crossing symmetry. Zero energy states can be seen also as representations of quantum jumps with positive and negative energy components of the state identifiable as counterparts of initial and final states. One could say that all states of the entire Universe, even at classical space-time level, represent reflective level of existence, being always about something. Only in the approximation that positive and negative energy components of the state do not interact the western view about objective reality with conserved energy makes sense.

5.6 Infinite primes and the structure of many-sheeted space-time

The mapping of infinite primes to space-time surfaces codes the structure of infinite prime to the structure of space-time surface in rather non-implicit manner, and the question arises about the concrete correspondence between the structure of infinite prime and topological structure of the space-time surface.

5.6.1 A possible interpretation for the lowest level infinite primes

The concrete prediction of the general vision is that the hierarchy of infinite primes should correspond to the hierarchy of space-time sheets. The challenge is to find space-time counterparts for infinite primes at the lowest level of hierarchy.

One could hope that the Fock space structure of infinite prime would have a more concrete correspondence with the structure of the many-sheeted space-time. One might that the space-time sheets labelled by primes p would directly correspond to the primes appearing in the definition of infinite prime. This expectation seems to be too simplistic.

1. What seems to be a safe guess is that the simplest infinite primes at the lowest level of the hierarchy should correspond to elementary particles. If inverses of infinite primes correspond to negative energy space-time sheets, this would explain why negative energy particles are not encountered in elementary particle physics.

2. More complex infinite primes at the lowest level of the hierarchy could be interpreted in terms of structures formed by connecting these structures by join along boundaries bonds to get space-time correlates of bound states. Even simplest infinite primes must correspond to bound state structures if the condition that the corresponding polynomial has real-rational coefficients is taken seriously.

Infinite primes at lowest level of hierarchy correspond to several finite primes rather than single finite prime. The number of finite primes is however finite. This conforms with the idea that this level indeed corresponds to space-time sheets associated with elementary particles.

1. A possible interpretation for multi-p property is in terms of multi-p p-adic fractality prevailing in the interior of space-time surface. The effective p-adic topology of these space-time sheets would depend on length scale. In the longest scale the topology would correspond to p_n, in some shorter length scale there would be smaller structures with $p_{n-1} < p_n$-adic topology, and so on... . A good metaphor would be a wave containing ripples, which in turn would contain still smaller ripples. The multi-p p-adic fractality would be assigned with the 4-D space-time sheets associated with elementary particles.

2. Effective 2-dimensionality would suggest that p-adic topologies could be assigned with the 2-dimensional partonic surfaces or corresponding 3-D light-like causal determinants. Thus infinite prime would characterize at the lowest level space-time sheet and corresponding partonic 2-surfaces. This interpretation is consistent with the fact that modified Dirac operator assigns to its generalized eigen modes p-adic prime p characterizing the p-adic topology of corresponding p-adic parton obeying same algebraic equations.

5.6.2 How to interpret higher level infinite primes?

A possible interpretation for higher level infinite primes is in terms of q-adicity assignable to the function spaces defined by the rational functions assignable to them. The role of finite prime p would be taken by the rational function defined by the infinite prime. This interpretation makes sense both when one assigns to infinite primes functions of rational arguments $q_1, ...q_n$ or when one identifies these arguments. This function space is q-adic for some rational number q. At the lowest level the infinite prime indeed defines naturally an ordinary rational number.

At higher levels of the hierarchy one can assign to infinite prime an infinite rational number of previous level. By continuing the assignments of lower level rationals to the infinite primes appearing in this infinite rational one ends up with an assignment of a unique rational number with a given infinite prime. This rational serves as a good candidate for a rational defining the q-adicity. The question is whether this q-adicity can be assign with space-time topology or some function space topology.

1. The modified Dirac operator associated with a partonic 2-surface assignable to the largest space-time sheet of topological condensation hierarchy would naturally assign q to its eigen modes. It is however not clear whether one can assign to partonic 2-surface characterized by algebraic equations unique q-adic space-time sheet. The problem is that q-adic numbers do not form number field so that the algebraic equations defining the partonic 2-surface need not make sense.

2. The q-adic function spaces might have a natural interpretation in terms of the fields assignable to the space-time sheet by replacing complex argument with quaternionic one. One possible interpretation is that primes appearing in the lowest level infinite prime correspond to partonic 2-surfaces and infinite prime itself defines q-adic topology for a functions space assignable to the space-time sheet. The q-adic topology associated with the function space associated with a space-time sheet containing topologically condensed space-time sheets would be characterized by the infinite prime and corresponding polynomial determined by the infinite primes associated with the topologically condensed space-time sheets that it contains. Note that the modified Dirac operator would assign to partonic 2-surfaces at all levels of hierarchy a p-adic prime.

3. Quantum criticality suggests strongly that configuration space of 3-surfaces effectively reduces to discrete spin glass energy landscape corresponding to the maxima of Kähler function. Spin glass property suggests strongly that this space obeys ultrametric topology. Therefore a natural conjecture is that the q-adic topology can be assigned with this space.

5.7 How infinite integers could correspond to p-adic effective topologies?

Besides the hierarchy of space-time sheets, TGD predicts, or at least suggests, several hierarchies such as the hierarchy of infinite primes, hierarchy of Jones inclusions [C6], dark matter hierarchy characterized by increasing values of $\hbar$ [F9, J6], the hierarchy of extensions of given p-adic number field, and the hierarchy of selves and quantum jumps with increasing duration with respect to geometric time. There are good reasons to expect that these hierarchies are closely related. Number theoretical considerations allow to develop more quantitative vision about the relationship between the hierarchy of infinite primes and p-adic length scale hierarchy.

5.7.1 How to define the notion of elementary particle?

p-Adic length scale hierarchy forces to reconsider carefully also the notion of elementary particle. p-Adic mass calculations led to the idea that particle can be characterized uniquely by single p-adic prime characterizing its mass squared [F3, F4, F5]. It however turned out that the situation is probably not so simple.

The work with modelling dark matter suggests that particle could be characterized by a collection of p-adic primes to which one can assign weak, color, em, gravitational interactions, and possibly also other interactions [F6, F8, F9]. It would also seem that only the space-time sheets containing common primes in this collection can interact. This leads to the notions of relative and partial darkness. An entire hierarchy of weak and color physics such that weak bosons and gluons of given physics are characterized by a given p-adic prime p and also the fermions of this physics contain space-time sheet characterized by same p-adic prime, say M_{89} as in case of weak interactions. In this picture the decay widths of weak bosons do not pose limitations on the number of light particles if weak interactions for them are characterized by p-adic prime $p \neq M_{89}$. Same applies to color interactions.

The p-adic prime characterizing the mass of the particle would perhaps correspond to the largest p-adic prime associated with the particle. Graviton which corresponds to infinitely long ranged interactions, could correspond to the same p-adic prime or collection of them common to all particles. This might apply also to photons. Infinite range might mean that the join along boundaries bonds mediating these interactions can be arbitrarily long but their transversal sizes are characterized by the p-adic length scale in question.

The natural question is what this collection of p-adic primes characterizing particle means? The hint about the correct answer comes from the number theoretical vision, which suggests that at fundamental level the branching of boundary components to two or more components, completely analogous to the branching of line in Feynman diagram, defines vertices [C2, C5].

1. If space-time sheets correspond holographically to multi-p p-adic topology such that largest p determines the mass scale, the description of particle reactions in terms of branchings indeed makes sense. This picture allows also to understand the existence of different scaled up copies of QCD and weak physics. Multi-p p-adicity could number theoretically correspond to q-adic topology for $q = m/n$ a rational number consistent with p-adic topologies associated with prime factors of m and n ($1/p$-adic topology is homeomorphic with p-adic topology).

2. One could also imagine that different p-adic primes in the collection correspond to different space-time sheets condensed at a larger space-time sheet or boundary components of a given space-time sheet. If the boundary topologies for gauge bosons are completely mixed, as the model of hadrons forces to conclude, this picture is consistent with the topological explanation of the family replication phenomenon and the fact that only charged weak currents involve mixing of quark families. The problem is how to understand the existence of different copies of say QCD. The second difficult question is why the branching leads always to an emission of gauge boson characterized by a particular p-adic prime, say M_{89}, if this p-adic prime does not somehow characterize also the particle itself.

5.7.2 What effective p-adic topology really means?

The need to characterize elementary particle p-adically leads to the question what p-adic effective topology really means. p-Adic mass calculations leave actually a lot of room concerning the answer to this question.

1. At the fundamental level this problem seems to be well understood now. By the almost topological QFT property of quantum real and p-adic variants of light-like partonic 3-surfaces can satisfy same algebraic equations. Modified Dirac operator assigns well-defined p-adic prime p to its eigenmodes with non-vanishing eigenvalues. Zero modes are an exception.

2. The naivest option would be that each space-time sheet corresponds to single p-adic prime. This view is not favored by the view that each particle corresponds to a collection of p-adic primes each characterizing one particular interaction that the particle in question participates. A more natural possibility is that the boundary components of space-time sheet, and more generally, light-like 3-surfaces serving as causal determinants, correspond to different p-adic primes.

3. This implies that a given space-time sheet to several p-adic primes. Indeed, a power series in powers of given integer n gives rise to a well-defined power series with respect to all prime factors of n and effective multi-p-adicity could emerge at the level of field equations in this manner in the interior of space-time sheets. One could say that space-time sheet corresponds to several p-adic primes through its effective p-adic topology in a hologram like manner. This option is the most natural as far as physical interpretation is considered. It is also supported by the number theoretical considerations predicting the value of gravitational coupling constant.

An attractive hypothesis is that only space-time sheets characterized by integers n_i having common prime factors can be connected by join along boundaries bonds and can interact by particle exchanges and that each prime p in the decomposition corresponds to a particular interaction mediated by an elementary boson characterized by this prime.

5.7.3 Do infinite primes code for effective q-adic space-time topologies?

As found, one can assign to a given infinite prime a rational number. The most obvious question concerns the possible space-time interpretation of this rational number. Also the question arises about the possible relation with the integers characterizing space-time sheets having interpretation in terms of multi-p-adicity. On can assign to any rational number $q = m/n$ so called q-adic topology. This topology is not consistent with number field property like p-adic topologies. Hence the rational number q assignable to infinite prime could correspond to an effective q-adic topology.

If this interpretation is correct, arithmetic fermion and boson numbers could be coded into effective q-adic topology of the space-time sheets characterizing the non-determinism of Kähler action in the relevant length scale range. For instance, the power series of $q > 1$ in positive powers with integer coefficients in the range $[0, q)$ define q-adically converging series, which also converges with respect to the prime factors of m and can be regarded as a p-adic power series. The power series of q in negative powers define in similar converging series with respect to the prime factors of n.

I have proposed earlier that the integers defining infinite rationals and thus also the integers m and n characterizing finite rational could correspond at space-time level to particles with positive *resp.* negative time orientation with positive *resp.* negative energies. Phase conjugate laser beams would represent one example of negative energy states. With this interpretation super-symmetry exchanging the roles of m and n and thus the role of fermionic and bosonic lower level primes would correspond to a time reversal.

1. The first interpretation is that there is single q-adic space-time sheet and that positive and negative energy states correspond to primes associated with m and n respectively. Positive (negative) energy space-time sheets would thus correspond to p-adicity ($1/p$-adicity) for the field modes describing the states.

2. Second interpretation is that particle (in extremely general sense that entire universe can be regarded as a particle) corresponds to a pair of positive and negative energy space-time sheets labelled by m and n characterizing the p-adic topologies consistent with $m-$ and n-adicities. This looks natural since Universe has necessary vanishing net quantum numbers.

Unless one allows the non-uniqueness due to $m/n = mr/nr$, positive and negative energy space-time sheets can be connected only by # contacts so that positive and negative energy space-time sheets cannot interact via the formation of $\#_B$ contacts and would be therefore dark matter with respect to each other. Antiparticles would also have different mass scales. If the rate for the creation of # contacts and their CP conjugates are slightly different, say due to the presence of electric components of gauge fields, matter antimatter asymmetry could be generated primordially.

These interpretations generalize to higher levels of the hierarchy. There is a homomorphism from infinite rationals to finite rationals. One can assign to a product of infinite primes the product of the corresponding rationals at the lower level and to a sum of products of infinite primes the sum of the corresponding rationals at the lower level and continue the process until one ends up with a finite rational. Same applies to infinite rationals. The resulting rational $q = m/n$ is finite and defines q-adic effective topology, which is consistent with all the effective p-adic topologies corresponding to the primes appearing in factorizations of m and n. This homomorphism is of course not 1-1.

q would associate with the particle q-adic topology consistent with a collection of p-adic topologies corresponding to the prime factors of m and n and characterizing the interactions that the particle can participate directly. In a very precise sense particles would represent both infinite and finite numbers.

5.7.4 Under what conditions boundary components can be connected by $\#_B$ contact?

Assume that particles are characterized by a p-adic prime determining it mass scale plus p-adic primes characterizing the gauge bosons to which they couple and assume that $\#_B$ contacts mediate gauge interactions. Assume that these primes label the boundary components of the space-time sheet representing the particle or more general light-like 3-surfaces. The question is what kind of space-time sheets can be connected by $\#_B$ contacts.

The first working hypothesis that comes in mind is that the p-adic primes associated with the two boundary components connected by $\#_B$ contact must be identical. If the notion of multi-p p-adicity is accepted, space-time sheets are characterized by integers and the largest prime dividing the integer might characterize the mass of the particle. This makes sense if the p-adic temperature $T = 1/n$ associated with small primes is small enough. In this case a common prime factor p for the integers characterizing the two space-time sheets could be enough for the possibility of $\#_B$ contact and this contact would be characterized by this prime. If no common prime factors exist, only # contacts could connect the space-time sheets. This option conforms with the number theoretical vision. This option would predict that the transition to large $\hbar$ phase occurs simultaneously for all interactions.

5.7.5 What about the integer characterizing graviton?

If one accepts the hypothesis that graviton couples to both visible and dark matter, graviton should be characterized by an integer dividing the integers characterizing all particles. This leaves two options.

Option I: gravitational constant characterizes graviton number theoretically

The argument leading to an expression for gravitational constant in terms of CP_2 length scale led to the proposal that the product of primes $p \leq 23$ are common to all particles and one interpretation was in terms of multi-fractality. If so, graviton would be characterized by a product of some or all primes $p \leq 23$ and would thus correspond to a very small p-adic length scale. This might be also the case for photon although it would seem that photon cannot couple to dark matter always. $p = 23$ might characterize the transversal size of the massless extremal associated with the space-time sheet of graviton.

Option II: gravitons are characterized by Mersenne prime M_{127}

The arguments related to the model of coupling constant evolution [C4] lead to the proposal that graviton coupling strength behaves as L_p^2 as a function of the p-adic length scale and that effective renormalization group invariance of tge gravitational coupling strength is due to the fact that gravitational interactions are carried by $\#_B$ contacts which correspond to Mersenne prime M_{127}. This would mean that each elementary particle contains partonic 2-surface labelled by M_{127}. This is possible if the p-adic temperature associated with M_{127} is $T = 1/n$, $n > 1$, for all particles lighter than electron so that p-adic thermodynamics does not contribute appreciably to the mass squared of the particle.

Option III: graviton behaves as a unit with respect to multiplication

One can also argue that if the largest prime assignable to a particle characterizes the size of the particle space-time sheet it does not make sense to assign any finite prime to a massless particle like graviton. Perhaps graviton corresponds to simplest possible infinite prime $P = X \pm 1$, X the product of all primes.

As found, one can assign to any infinite prime, integer, and rational a rational number $q = m/n$ to which one can assign a q-adic topology as effective space-time topology and as a special case effective p-adic topologies corresponding to prime factors of m and n.

In the case of $P = X \pm 1$ the rational number would be equal to ± 1. Graviton could thus correspond to $p = 1$-adic effective topology. The "prime" $p = 1$ indeed appears as a factor of any integer so that graviton would couple to any particle. Formally the 1-adic norm of any number would be 1 or 0 which would suggest that a discrete topology is in question.

The following observations help in attempts to interpret this.

1. CP_2 type extremals having interpretation as gravitational instantons are non-deterministic in the sense that M^4 projection is random light-like curve. This condition implies Virasoro conditions which suggests interpretation in terms topological quantum theory limit of gravitation involving vanishing four-momenta but non-vanishing color charges. This theory would represent gravitation at the ultimate CP_2 length scale limit without the effects of topological condensation. In longer length scales a hierarchy of effective theories of gravitation corresponds to the coupling of space-time sheets by join along boundaries bonds would emerge and could give rise to "strong gravities" with strong gravitational constant proportional to L_p^2. It is quite possible that the M-theory based vision about duality between gravitation and gauge interactions applies to electro-weak interactions and in these "strong gravities".

2. p-Adic length scale hypothesis $p \simeq 2^k$, k integer, implies that $L_k \propto \sqrt{k}$ corresponds to the size scale of causal horizon associated with # contact. For $p = 1$ k would be zero and the causal horizon would contract to a point which would leave only generalized Feynman diagrams consisting of CP_2 type vacuum extremals moving along random light-like orbits and obeying Virasoro conditions so that interpretation as a kind of topological gravity suggests itself.

3. $p = 1$ effective topology could make marginally sense for vacuum extremals with vanishing Kähler form and carrying only gravitational charges. The induced Kähler form vanishes identically by the mere assumption that X^4, be it continuous or discontinuous, belongs to $M^4 \times Y^2$, Y^2 a Lagrange sub-manifold of CP_2.

Why topological graviton, or whatever the particle represented by CP_2 type vacuum extremals should be called, should correspond to the weakest possible notion of continuity? The most plausible answer is that discrete topology is *consistent* with any other topology, in particular with any p-adic topology. This would express the fact that CP_2 type extremals can couple to any p-adic prime. The vacuum property of CP_2 type extremals implies that the splitting off of CP_2 type extremal leaves the physical state invariant and means effectively multiplying integer by $p = 1$.

It seems that Option I suggested by the deduction of the value of gravitational constant looks more plausible as far as the interpretation of gravitation is considered. This does not however mean that CP_2 type vacuum extremals carrying color quantum numbers could not describe gravitational interactions in CP_2 length scale.

5.8 An alternative interpretation for the hierarchy of functions defined by infinite primes

Suppose that infinite primes code for the ground states of super-conformal representations. Supersymmetry suggests that the corresponding polynomials or their zeros could code for the moduli space associated with these states. At the limit of algebraic closure of rationals the vanishing of the polynomial would code for a complex codimension one surface of C^n at n:th level of hierarchy.

The recent progress in the understanding of S-matrix [C2] relies on the idea that the data needed to construct S-matrix is provided by the intersection of real and p-adic parton 2-surface obeying same algebraic equations. Quantum TGD is almost topological QFT since only the light-likeness of orbits of partonic 2-surfaces brings in the notion of metric. This leads to the idea that the braiding S-matrices of topological quantum field theories generalize to give a realistic S-matrix in TGD framework. The number theoretical braids at partonic 2-surface for which the strands of the braid project to the same point of the geodesic sphere S^2 of CP_2 play a key role in this approach. Braids are thus characterized by complex numbers labelling the points of S^2.

In this framework the natural idea would be that that the n, in general complex, algebraic numbers, code for the positions of braids and that vanishing of the polynomial gives correlation between the positions of braids so that the position of n^{th} level braid is fixed almost uniquely once the positions of lower level braids are known. One must however admit that this kind of correlation does not look too convincing and that the interpretation involves ad hoc elements such as the selection of the geodesic sphere. It must be however added that infinite primes could allow several mutually consistent interpretations and that this interpretation or some interpretation analogous to it might make sense.

6 Does the notion of infinite-P p-adicity make sense?

In this section speculations related to infinite-P p-adicity are represented in the form of shy questions in order to not irritate too much the possible reader. The basic open question causing tension is whether infinite primes relate only to the physics of cognition or whether they might allow to say something non-trivial about the physics of matter too.

The obvious question is whether the notion of p-adic number field makes sense makes sense for infinite primes and whether it might have some physical relevance. One can certainly introduce power series in powers of any infinite prime P and the coefficients can be taken to belong to any ordinary number field. In the representation by polynomials P-Adic power series correspond to Laurent series in powers of corresponding polynomial and are completely finite.

For straightforward generalization of the norm all powers of infinite-P prime have vanishing norm. The infinite-p p-adic norm of infinite-p p-adic integer would be given by its finite part so that in this sense positive powers of P would represent infinitesimals. For Laurent series this would mean that the lowest term would give the whole approximation in the real topology. For finite-primes one could however replace the norm as a power of p by a power of some other number. This would allow to have a finite norm also for P-adic primes. Since the simplest P-adic primes at the lowest level of hierarchy define naturally a rational one might consider the possibility of defining the norm of P as the inverse of this rational.

6.1 Does infinite-P p-adicity reduce to q-adicity?

Any non-vanishing p-adic number is expressible as a product of power of p multiplied by a p-adic unit which can be infinite as a normal integer and has pinary expansion in powers of p:

$$x = p^n(x_0 + \sum_{k>0} x_k p^k) \ , \ x_k \in \{0,..,p-1\} \ , \ x_0 > 0 \ . \tag{30}$$

The p-adic norm of x is given by $N_p(x) = p^{-n}$. Each unit has p-adic inverse which for finite integers is always infinite as an ordinary integer.

To define infinite-P p-adic numbers one must generalize the pinary expansion to a infinite-P p-adic expansion of an infinite rational. In particular, one must identify what the statement 'infinite integer modulo P' means when P is infinite prime, and what are the infinite integers N satisfying the condition $N < P$. Also one must be able to construct the p-adic inverse of any infinite prime. The correspondence of infinite primes with polynomials allows to construct infinite-P p-adics in a straightforward manner.

Consider first the infinite integers at the lowest level.

1. Infinite-P p-adics at the first level of hierarhcy correspond to Laurent series like expansions using an irreducible polynomial P of degree n representing infinite prime. The coefficients of the series are numbers in the coefficient fields. Modulo p operation is replaced with modulo polynomial P operation giving a unique result and one can calculate the coefficients of the expansion in powers of P by the same algorithm as in the case of the ordinary p-adic numbers. In the case of n-variables the coefficients of Taylor series are naturally rational functions of at most $n-1$ variables. For infinite primes this means rationals formed from lower level infinite-primes.

2. Infinite-P p-adic units correspond to expansions of this type having non-vanishing zeroth order term. Polynomials take the role of finite integers. The inverse of a infinite integer in P-adic number field is obtained by developing the polynomial counterpart of $1/N$ in the following manner. Express N in the form $N = N_0(1+x_1P+..)$, where N_0 is polynomial with degree at most equal to $n-1$. The factor $1/(1+x_1P+...)$ can be developed in geometric series so that only the calculation of $1/N_0$ remains. Calculate first the inverse $\hat{N}_0^{-1}$ of N_0 as an element of the 'finite field' defined by the polynomials modulo P: a polynomial having degree at most equal to $n-1$ results. Express $1/N_0$ as

$$\frac{1}{N_0} = \hat{N}_0^{-1}(1 + y_1P + ...)$$

and calculate the coefficients in the expansion iteratively using the condition $N \times (1/N) = 1$ by applying polynomial modulo arithmetics. Generalizing this, one can develop any rational function to power series with respect to polynomial prime P. The expansion with respect to a polynomial prime can in turn be translated to an expansion with respect to infinite prime and also mapped to a superposition of Fock states.

3. What about the norm of infinite-P p-adic integers? Ultra-metricity suggest a straightforward generalization of the usual p-adic norm. The direct generalization of the finite-p p-adic norm would mean the identification of infinite-P p-adic norm as P^{-n}, where n corresponds to the

lowest order term in the polymomial expansion. Thus the norm would be infinite for $n < 0$, equal to one for $n = 0$ and vanish for $n > 0$. Any polynomial integer N would have vanishing norm with respect to those infinite-P p-adics for which P divides N. Essentially discrete topology would result.

This seems too trivial to be interesting. One can however replace P^{-n} with a^{-n}, where a is any finite number a without losing the multiplicativity and ultra-metricity properties of the norm. The function space associated with the polynomial defined by P serves as a guideline also now. This space is naturally q-adic for some rational number q. At the lowest level the infinite prime defines naturally an ordinary rational number as the zero of the polynomial as is clear from the definition of the polynomial. At higher levels of the hierarchy the rational number is rational function of lower level infinite primes and by continuing the assignments of lower level rational functions to the infinite primes one ends up with an assignment of a unique rational number with a given infinite prime serving as an excellent candidate for a rational defining the q-adicity.

6.2 q-Adic topology determined by infinite prime as a local topology of the configuration space

Since infinite primes correspond to polynomials, infinite-P p-adic topology, which by previous considerations would be actually q-adic topology, is a natural candidate for a topology in function spaces, in particular in the configuration space of 3-surfaces.

This view conforms also with the idea of algebraic holography. The sub-spaces of configuration space can be modelled in terms of function spaces of rational functions, their algebraic extensions, and their P-adic completions. The mapping of the elements of these spaces to infinite rationals would make possible the correspondence between configuration space and number theoretic anatomy of point of the imbedding space.

The q-adic norm for these function spaces is in turn consistent with the ultra-metricity for the space of maxima of Kähler functions conjectured to be all that is needed to construct S-matrix. Ultra-metricity conforms nicely with the expected four-dimensional spin glass degeneracy due to the enormous vacuum degeneracy meaning that maxima of Kähler function define the analog of spin glass free energy landscape. That only maxima of Kähler function would be needed would mean that radiative corrections to the configuration space integral would vanish as quantum criticality indeed requires. This TGD can be regarded as an analog of for an integrable quantum theory. Quantum criticality is absolutely essential for guaranteing that S-matrix and U-matric elements are algebraic numbers which in turn guarantees number theoretic universality of quantum TGD.

6.3 The interpretation of the discrete topology determined by infinite prime

Also $p = 1$-adic topology makes formally sense and corresponds to a discrete topology in which all rationals have unit norm. It results also results if one naively generalizes p-adic topology to infinite-p p-adic topology by defining the norm of infinite prime at the lowest level of hierarchy as $|P|_P = 1/P = 0$. In this topology the distance between two points is either 1 or 0 and this topology is the roughest possible topology one can imagine.

It must be however noticed that if one maps infinite-P p-adics to real by the formal generalization of the canonical identification then one obtains real topology naturally if coefficients of powers of P are taken to be reals. This would mean that infinite-P p-adic topology would be equivalent with real topology.

Consider now the possible interpretations.

1. At the level of function spaces infinite-p p-adic topology in the naive sense has a completely natural interpretation and states that the replacement of the Taylor series with its lowest term.

2. The formal possibility of $p = 1$-adic topology at space-time level suggests a possible interpretation for the mysterious infinite degeneracy caused by the presence of the absolute minima of the Kähler function: one can add to any absolute minimum a vacuum extremal, which behaves completely randomly except for the constraints forcing the surface to be a vacuum extremal. This non-determinism is much more general than the non-determinism involving a discrete sequence of bifurcations (I have used the term association sequence about this kind of sequences). This suggests that one must replace the concept of 3-surface with a more general one, allowing also continuous association sequences consisting of a continuous family of space-like 3-surfaces with infinitesimally small time like separations. These continuous association sequences would be analogous to vacuum bubbles of the quantum field theories.

One can even consider the possibility that vacuum extremals are non-differentiable and even discontinuous obeying only effective $p = 1$-adic topology. Also modified Dirac operator vanishes identically in this case. Since vacuum surfaces are in question, $p = 1$ regions cannot correspond to material sheets carrying energy and also the identification as cognitive space-time sheets is questionable. Since $p = 1$, the smallest possible prime in generalized sense, it must represent the lowest possible level of evolution, primordial chaos. Quantum classical correspondence suggests that $p = 1$ level is indeed present at the space-time level and might realized by the mysterious vacuum extremals.

7 Infinite primes and mathematical consciousness

The mathematics of infinity relates naturally with the mystery of consciousness and religious and mystic experience. In particular, mathematical cognition might have as a space-time correlate the infinitely structured space-time points implied by the introduction of infinite-dimensional space of real units defined by infinite (hyper-)octonionic rationals having unit norm in the real sense. I hope that the reader takes this section as a noble attempt to get a glimpse about unknown rather than final conclusions.

7.1 Infinite primes, cognition and intentionality

Somehow it is obvious that infinite primes must have some very deep role to play in quantum TGD and TGD inspired theory of consciousness. What this role precisely is has remained an enigma although I have considered several detailed interpretations, one of them above.

In the following an interpretation allowing to unify the views about fermionic Fock states as a representation of Boolean cognition and p-adic space-time sheets as correlates of cognition is discussed. Very briefly, real and p-adic partonic 3-surfaces serve as space-time correlates for the bosonic super algebra generators, and pairs of real partonic 3-surfaces and their algebraically continued p-adic variants as space-time correlates for the fermionic super generators. Intentions/actions are represented by p-adic/real bosonic partons and cognitions by pairs of real partons and their p-adic variants and the geometric form of Fermi statistics guarantees the stability of cognitions against intentional action. It must be emphasized that this interpretation is not identical with the one discussed above since it introduces different identification of the space-time correlates of infinite primes.

7.1.1 Infinite primes very briefly

Infinite primes have a decomposition to infinite and finite parts allowing an interpretation as a many-particle state of a super-symmetric arithmetic quantum field theory for which fermions and bosons are labelled by primes. There is actually an infinite hierarchy for which infinite primes of a given level define the building blocks of the infinite primes of the next level. One can map infinite primes to polynomials and these polynomials in turn could define space-time surfaces or at least light-like partonic 3-surfaces appearing as solutions of Chern-Simons action so that the classical dynamics would not pose too strong constraints.

The simplest infinite primes at the lowest level are of form $m_B X/s_F + n_B s_F$, $X = \prod_i p_i$ (product of all finite primes). The simplest interpretation is that X represents Dirac sea with all states filled and $X/s_F + s_F$ represents a state obtained by creating holes in the Dirac sea. m_B, n_B, and s_F are defined as $m_B = \prod_i p_i^{m_i}$, $n_B = \prod_i q_i^{n_i}$, and $s_F = \prod_i q_i$, m_B and n_B have no common prime factors. The integers m_B and n_B characterize the occupation numbers of bosons in modes labelled by p_i and q_i and $s_F = \prod_i q_i$ characterizes the non-vanishing occupation numbers of fermions.

The simplest infinite primes at all levels of the hierarchy have this form. The notion of infinite prime generalizes to hyper-quaternionic and even hyper-octonionic context and one can consider the possibility that the quaternionic components represent some quantum numbers at least in the sense that one can map these quantum numbers to the quaternionic primes.

The obvious question is whether configuration space degrees of freedom and configuration space spinor (Fock state) of the quantum state could somehow correspond to the bosonic and fermionic parts of the hyper-quaternionic generalization of the infinite prime. That hyper-quaternionic (or possibly hyper-octonionic) primes would define as such the quantum numbers of fermionic super generators does not make sense. It is however possible to have a map from the quantum numbers labelling super-generators to the finite primes. One must also remember that the infinite primes considered are only the simplest ones at the given level of the hierarchy and that the number of levels is infinite.

7.1.2 Precise space-time correlates of cognition and intention

The best manner to end up with the proposal about how p-adic cognitive representations relate bosonic representations of intentions and actions and to fermionic cognitive representations is through the following arguments.

1. In TGD inspired theory of consciousness Boolean cognition is assigned with fermionic states. Cognition is also assigned with p-adic space-time sheets. Hence quantum classical correspondence suggets that the decomposition of the space-time into p-adic and real space-time sheets should relate to the decomposition of the infinite prime to bosonic and fermionic parts in turn relating to the above mention decomposition of physical states to bosonic and fermionic parts.

   If infinite prime defines an association of real and p-adic space-time sheets this association could serve as a space-time correlate for the Fock state defined by configuration space spinor for given 3-surface. Also spinor field as a map from real partonic 3-surface would have as a space-time correlate a cognitive representation mapping real partonic 3-surfaces to p-adic 3-surfaces obtained by algebraic continuation.

2. Consider first the concrete interpretation of integers m_B and n_B. The most natural guess is that the primes dividing $m_B = \prod_i p^{m_i}$ characterize the effective p-adicities possible for the real 3-surface. m_i could define the numbers of disjoint partonic 3-surfaces with effective p_i-adic topology and associated with with the same real space-time sheet. These boundary

conditions would force the corresponding real 4-surface to have all these effective p-adicities implying multi-p-adic fractality so that particle and wave pictures about multi-p-adic fractality would be mutually consistent. It seems natural to assume that also the integer n_i appearing in $m_B = \prod_i q_i^{n_i}$ code for the number of real partonic 3-surfaces with effective q_i-adic topology.

3. Fermionic statistics allows only single genuinely q_i-adic 3-surface possibly forming a pair with its real counterpart from which it is obtained by algebraic continuation. Pairing would conform with the fact that n_F appears both in the finite and infinite parts of the infinite prime (something absolutely essential concerning the consistency of interpretation!).

   The interpretation could be as follows.

   i) Cognitive representations must be stable against intentional action and fermionic statistics guarantees this. At space-time level this means that fermionic generators correspond to pairs of real effectively q_i-adic 3-surface and its algebraically continued q_i-adic counterpart. The quantum jump in which q_i-adic 3-surface is transformed to a real 3-surface is impossible since one would obtain two identical real 3-surfaces lying on top of each other, something very singular and not allowed by geometric exclusion principle for surfaces. The pairs of boson and fermion surfaces would thus form cognitive representations stable against intentional action.

   ii) Physical states are created by products of super algebra generators Bosonic generators can have both real or p-adic partonic 3-surfaces as space-time correlates depending on whether they correspond to intention or action. More precisely, m_B and n_B code for collections of real and p-adic partonic 3-surfaces. What remains to be interpreted is why m_B and n_B cannot have common prime factors (this is possible if one allows also infinite integers obtained as products of finite integer and infinite primes).

   iii) Fermionic generators to the pairs of a real partonic 3-surface and its p-adic counterpart obtained by algebraic continuation and the pictorial interpretation is as fermion hole pair. Unrestricted quantum super-position of Boolean statements requires that many-fermion state is accompanied by a corresponding many-antifermion state. This is achieved very naturally if real and corresponding p-adic fermion have opposite fermion numbers so that the kicking of negative energy fermion from Dirac sea could be interpreted as creation of real-p-adic fermion pairs from vacuum.

   If p-adic space-time sheets obey same algebraic expressions as real sheets (rational functions with algebraic coefficients), the Chern-Simons Noether charges associated with real partons defined as integrals can be assigned also with the corresponding p-adic partons if they are rational or algebraic numbers. This would allow to circumvent the problems related to the p-adic integration. Therefore one can consider also the possibility that p-adic partons carry Noether charges opposite to those of corresponding real partons sheet and that pairs of real and p-adic fermions can be created from vacuum. This makes sense also for the classical charges associated with Kähler action in space-time interior if the real space-time sheet obeying multi-p p-adic effective topology has algebraic representation allowing interpretation also as p-adic surface for all primes involved.

   iv) This picture makes sense if the partonic 3-surfaces containing a state created by a product of super algebra generators are unstable against decay to this kind of 3-surfaces so that one could regard partonic 3-surfaces as a space-time representations for a configuration space spinor field.

4. Are alternative interpretations possible? For instance, could $q = m_B/m_B$ code for the effective q-adic topology assignable to the space-time sheet. That q-adic numbers form a

ring but not a number field casts however doubts on this interpretation as does also the general physical picture.

7.1.3 Number theoretical universality of S-matrix

The discreteness of the intersection of the real space-time sheet and its p-adic variant obtained by algebraic continuation would be a completely universal phenomenon associated with all fermionic states. This suggests that also real-to-real S-matrix elements involve instead of an integral a sum with the arguments of an n-point function running over all possible combinations of the points in the intersection. S-matrix elements would have a universal form which does not depend on the number field at all and the algebraic continuation of the real S-matrix to its p-adic counterpart would trivialize. Note that also fermionic statistics favors strongly discretization unless one allows Dirac delta functions.

7.2 Algebraic Brahman=Atman identity

The proposed view about cognition and intentionality emerges from the notion of infinite primes [E10], which was actually the first genuinely new mathematical idea inspired by TGD inspired consciousness theorizing. Infinite primes, integers, and rationals have a precise number theoretic anatomy. For instance, the simplest infinite primes correspond to the numbers $P_\pm = X \pm 1$, where $X = \prod_k p_k$ is the product of all finite primes. Indeed, $P_\pm \ mod \ p = 1$ holds true for all finite primes. The construction of infinite primes at the first level of the hierarchy is structurally analogous to the quantization of super-symmetric arithmetic quantum field theory with finite primes playing the role of momenta associated with fermions and bosons. Also the counterparts of bound states emerge. This process can be iterated: at the second level the product of infinite primes constructed at the first level replaces X and so on.

The structural similarity with repeatedly second quantized quantum field theory strongly suggests that physics might in some sense reduce to a number theory for infinite rationals M/N and that second quantization could be followed by further quantizations. As a matter fact, the hierarchy of space-time sheets could realize this endless second quantization geometrically and have also a direct connection with the hierarchy of logics labelled by their order. This could have rather breathtaking implications.

1. One is forced to ask whether this hierarchy corresponds to a hierarchy of realities for which level below corresponds in a literal sense infinitesimals and the level next above to infinity.

2. Second implication is that there is an infinite number of infinite rationals behaving like real units ($M/N \equiv 1$ in real sense) so that space-time points could have infinitely rich number theoretical anatomy not detectable at the level of real physics. Infinite integers would correspond to positive energy many particle states and their inverses (infinitesimals with number theoretic structure) to negative energy many particle states and $M/N \equiv 1$ would be a counterpart for zero energy ontology to which oneness and emptiness are assigned in mysticism.

3. Single space-time point, which is usually regarded as the most primitive and completely irreducible structure of mathematics, would take the role of Platonia of mathematical ideas being able to represent in its number theoretical structure even the quantum state of entire Universe. Algebraic Brahman=Atman identity and algebraic holography would be realized in a rather literal sense [E10].

This number theoretical anatomy should relate to mathematical consciousness in some manner. For instance, one can ask whether it makes sense to speak about quantum jumps changing the number theoretical anatomy of space-time points and whether these quantum jumps give rise to mathematical ideas. In fact, the identifications of Platonia as spinor fields in WCW on one hand and as the set number theoretical anatomies of point of imbedding space force the conclusion that configuration space spinor fields (recall also the identification as correlates for logical mind) can be realized in terms of the space for number theoretic anatomies of imbedding space points. Therefore quantum jumps would be correspond to changes in the anatomy of the space-time points. Or more precisely, to the changes of the configuration space spinor fields regarded as wave functions in the set of imbedding space points which are equivalent in real sense. Imbedding space would be experiencing genuine number theoretical evolution. The whole physics would reduce to the anatomy of numbers. All mathematical notions which are more than mere human inventions would be imbeddable to the Platonia realized as the number theoretical anatomies of single imbedding space point.

To realize this picture would require that both configuration space and configuration space spinor fields are mappable to the number theoretic anatomies of space-time point. The possibility to map infinite primes to polynomials and vice versa gives support for the possibility to map configuration space or at least the space of maxima of Kähler function defining the counterpart of spin glass energy landscape to the number theoretic anatomy of imbedding space point.

Function spaces provide a natural model for the subspaces of the world of classical worlds. The spaces of rational functions, their extensions, and q-adic completions, provide natural candidates for these function spaces, so that a mapping to real units defined by infinite rationals, their extensions, and q-adic completions emerge naturally. In the same manner Fock states can be mapped to infinite primes and one can see the polynomial-infinite prime correspondence also as an articulation of fermion-boson super-symmetry.

The commutativity requirement for infinite primes implies that infinite primes at n:th level can define rational functions of n complex variables. This relates naturally to the effective 2-dimensionality of TGD in the sense that configuration space geometry involves only data about 2-dimensional partonic surfaces at boundaries of $\delta M^4_\pm \times CP_2$. Allowing non-commutativity one would also obtain 4-D surfaces but algebraic continuation would mean that 2-D data is enough.

7.3 The generalization of the notion of ordinary number field

The notion of infinite rationals leads also to the generalization of the notion of a finite number. The obvious generalization would be based on the allowance of infinitesimals. Much more interesting approach is however based on the observation that one obtains infinite number of real units by taking two infinite primes with a finite rational valued ratio q and by dividing this ratio by ordinary rational number q. As a real number the resulting number differs in no manner from ordinary unit of real numbers but in p-adic sense the points are not equivalent. This construction generalizes also to quaternionic and octonionic case.

Space-time points would become structured since infinite rationals normed to unity define naturally a gigantically infinite-dimensional free algebra generated by the units serving in well-define sense as Mother of All Algebras. The units of the algebra multiplying ordinary rational numbers (and also other elements) of various number fields are invisible at the level of real physics so that the interpretation as the space-time correlate of mathematical cognition realizing the idea of monad is natural. Universe would be an algebraic hologram with single point being able to represent the state of the Universe in its structure. Infinite rationals would allow the realization of the Platonia of all imaginable mathematical constructs at the level of space-time.

7.3.1 The generalized units for quaternions and octonions

In the case of real and complex rationals the group of generalized units generated by primes *resp.* infinite Gaussian primes is commutative. In the case of unit quaternions and hyper-quaternions group becomes non-commutative and in case of unit hyper-octonions the group is replaced by a kind non-associative generalization of group.

For infinite primes for which only finite number of bosonic and fermionic modes are excited it is possible to tell how the products AB and BA of two infinite primes explicitly since the finite hyper-octonionic primes can be assumed to multiply the infinite integer X from say left.

Situation changes if infinite number of bosonic excitations are present since one would be forced to move finite H- or O-primes past a infinite number of primes in the product AB. Hence one must simply assume that the group G generated by infinite units with infinitely many bosonic excitations is a free group. Free group interpretation means that non-associativity is safely localized inside infinite primes and reduced to the non-associativity of ordinary hyper-octonions. Needless to say free group is the best one can hope of achieving since free group allows maximal number of factor groups.

The free group G can be extended into a free algebra A by simply allowing superpositions of units with coefficients which are real-rationals or possibly complex rationals. Again free algebra fulfils the dreams as system with a maximal representative power. The analogy with quantum states defined as functions in the group is highly intriguing and unit normalization would correspond to the ordinary normalization of Schrödinger amplitudes. Obviously this would mean that single point is able to mimic quantum physics in its structure. Could state function reduction and preparation be represented at the level of space-time surfaces so that initial and final 3-surfaces would represent pure states containing only bound state entanglement represented algebraically, and could the infinite rationals generating the group of quaternionic units (no sums over them) represent pure states?

The free algebra structure of A together with the absolutely gigantic infinite-dimensionality of the endless hierarchy of infinite rational units suggests that the resulting free algebra structure is universal in the sense that any algebra defined with coefficients in the field of rationals can be imbedded to the resulting algebra or represented as a factor algebra obtained by the sequence $A \rightarrow 1_1 = A/I_1 \rightarrow A_1/I_2...$ where the ideal I_k is defined by $k : th$ relation in A_{k-1}.

Physically the embedding would mean that some field quantities defined in the algebra are restricted to the subalgebra. The representation of algebra B as an iterated factor algebra would mean that some field quantities defined in the algebra are constant inside the ideals I_k of A defined by the relations. For instance, the induced spinor field at space-time surface would have same value for all points of A which differ by an element of the ideal. At the configuration space level, the configuration space spinor field would be constant inside an ideal associated with the algebra of A-valued functions at space-time surfaces.

The units can be interpreted as defining an extension of rationals in C, H, or O. Galois group is defined as automorphisms of the extension mapping the original number field to itself and obviously the transformations $x \rightarrow gxg^{-1}$, where g belongs to the extended number field act as automorphisms. One can regard also the extension by real units as the extended number field and in this case the automorphisms contain also the automorphisms induced by the multiplication of each infinite prime Π_i by a real unit U_i: $\Pi_i \rightarrow \hat{\Pi}_i = U_i\Pi_i$.

7.3.2 The free algebra generated by generalized units and mathematical cognition

One of the deepest questions in theory of consciousness concerns about the space-time correlates of mathematical cognition. Mathematician can imagine endlessly different mathematical structures. Platonist would say that in some sense these structures exist. The claim classical physical worlds correspond to certain 4-surfaces in $M^4_+ \times CP_2$ would leave out all these beautiful mathematical structures unless they have some other realization than the physical one.

The free algebra A generated by the generalized multiplicative units of rationals allows to understand how Platonia is realized at the space-time level. A has no correlate at the level of real physics since the generalized units correspond to real numbers equal to one. This holds true also in quaternionic and octonionic cases since one can require that the units have net quaternionic and octonionic phases equal to one. By its gigantic size A and free algebra character might be able represent all possible algebras in the proposed manner. Also non-associative algebras can be represented.

Algebraic equations are the basic structural building blocks of mathematical thinking. Consider as a simple example the equation $AB = C$. The equations are much more than tautologies since they contain the information at the left hand side about the variables of the algebraic operation giving the outcome on the right hand side. For instance, in the case of multiplication $AB = C$ the information about the factors is present although it is completely lost when the product is evaluated. These equations pop up into our consciousness in some mysterious manner and the question is what are the space-time correlates of these experiences suggested to exist by quantum-classical correspondence.

The algebra of units is an excellent candidate for the sought for correlate of mathematical cognition. I must admit that that it did not occur to me that Leibniz might have been right about his monads! The idealization is however in complete accordance with the idea about the Universe as an algebraic hologram taken to its extreme. One can say that each point represents an equation. The left hand side of the equation corresponds to the element of the free algebra defined by octonionic units. Consider as an example product of powers of $X/\Pi(Q_q)$ representing infinite quaternionic rationals. Equality sign corresponds to the evaluation of this expression by interpreting it as a real quaternionic rational number: real physics does the evaluation automatically. The information about the primes appearing as factors of the result is not however lost at cognitive level. Note that the analogs of quantum states represented by superpositions of the unit elements of the algebra A can be interpreted as equations defining them.

7.3.3 When two points are cobordant?

Topological quantum field theories have led to a dramatic success in the understanding of 3- and 4-dimensional topologies and cobordisms of these manifolds (two n-manifolds are cobordant if there exists an $n+1$-manifold having them as boundaries). In his thought-provoking and highly inspiring article Pierre Cartier [45] poses a question which at first sounds absurd. What might be the the counterpart of cobordism for points? The question is indeed absurd unless the points have some structure.

If one takes seriously the idea that each point of space-time sheet corresponds to a unit defined by an infinite rational, the obvious question is under what conditions there is a continuous line connecting these points with continuity being defined in some generalized sense. In real sense the line is continuous always but in p-adic sense only if all p-adic norms of the two units are identical. Since the p-adic norm of the unit of $Y(n/m) = X/\Pi(n/m)$ is that of $q = n/m$, the norm of two infinite rational numbers is same only if they correspond to the same ordinary rational number.

Suppose that one has

$$
\begin{aligned}
Y_I &= \frac{\prod_i Y(q^I_{1i})}{\prod_i Y(q^I_{2i})} \ , \quad Y_F = \frac{\prod_i Y(q^F_{1i})}{\prod_i Y(q^F_{2i})} \ , \\
q^I_{ki} &= \frac{n^I_{k_i}}{m^I_{k_i}} \ , \qquad q^F_{ki} = \frac{n^F_{k_i}}{m^F_{k_i}} \ ,
\end{aligned}
\tag{31}
$$

Here $m^{\cdot}_{\cdot}$ representing arithmetic many-fermion state is a square free integer and $n^{\cdot}_{\cdot}$ representing arithmetic many-boson state is an integer having no common factors with $m^{\cdot}_{\cdot}$.

The two units have same p-adic norm in all p-adic number fields if the rational numbers associated with Y_I and Y_F are same:

$$\frac{\prod_i q^I_{1i}}{\prod_i q^I_{2i}} = \frac{\prod_i q^F_{1i}}{\prod_i q^F_{2i}} \quad . \tag{32}$$

The logarithm of this condition gives a conservation law of energy encountered in arithmetic quantum field theories, where the energy of state labelled by the prime p is $E_p = log(p)$:

$$\begin{aligned} E^I &= \sum_i log(n^I_{1i}) - \sum_i log(n^I_{2i}) - \sum_i log(m^I_{1i}) + \sum_i log(m^I_{2i}) = \\ &= \sum_i log(n^F_{1i}) - \sum_i log(n^F_{2i}) - \sum_i log(m^F_{1i}) + \sum_i log(m^F_{2i}) = E^F \quad . \end{aligned} \tag{33}$$

There are both positive and negative energy particles present in the system. The possibility of negative energies is indeed one of the basic predictions of quantum TGD distinguishing it from standard physics. As one might have expected, Y^I and Y^F represent the initial and final states of a particle reaction and the line connecting the two points represents time evolution giving rise to the particle reaction. In principle one can even localize various steps of the reaction along the line and different lines give different sequences of reaction steps but same overall reaction. This symmetry is highly analogous to the conformal invariance implying that integral in complex plane depends only on the end points of the curve.

Whether the entire four-surface should correspond to the same value of topological energy or whether E can be discontinuous at elementary particle horizons separating space-time sheets and represented by light-like 3-surfaces around wormhole contacts remains an open question. Discontinuity through elementary particle horizons would make possible the arithmetic analogs of poles and cuts of analytic functions since the limiting values of Y from different sides of the horizon are different. Note that the construction generalizes to the quaternionic and octonionic case.

7.4 Leaving the world of finite reals and ending up to the ancient Greece

If strong number theoretic vision is accepted, all physical predictions of quantum TGD would be numbers in finite algebraic extensions of rationals. Just the numbers which ancient Greeks were able to construct by the technical means at use! This seems rather paradoxical but conforms also with the hypothesis that the dicrete algebraic intersections of real and p-adic 2-surfaces provide the fundamental cognitive representations.

The proposed construction for infinite primes gives a precise division of infinite primes to classes: the ratios of primes in given class span a *subset of rational numbers*. These classes give much more refined classification of infinities than infinite ordinals or alephs. They would correspond to separate phases in the evolution of consciousness identified as a sequence of quantum jumps defining sequence of primes $\rightarrow p_1 \rightarrow p_2$...... Infinite primes could mean a transition from space-time level to the level of function spaces. Configuration space is example of a space which can be parameterized by a space of functions locally.

The minimal assumption is that infinite primes reflect their presence only in the possibility to multiply the coordinates of imbedding space points by real units formed as ratios of infinite integers. The correspondence between polynomials and infinite primes gives hopes of mapping at least the reduced configuration space consisting of the the maxima of Kähler function to the anatomy of space-time point. Also configuration space spinors and perhaps also the the modes of configuration space spinor fields would allow this kind of map.

One can consider also the possibility that infinite integers and rationals give rise to a hierarchy of imbedding spaces such that given level represents infinitesimals from the point of view of higher levels in hierarchy. Even 'simultaneous' time evolutions of conscious experiences at different aleph levels with completely different time scales (to put it mildly) are possible since the time values around which the contents of conscious experience are possibly located, are determined by the quantum jump: also multi-snapshots containing snapshots also from different aleph levels are possible. Un-integrated conscious experiences with all values of p could be contained in given quantum jump: this would give rise to a hierarchy of conscious beings: the habitants above given level could be called Gods with full reason: those above us would probably call us just 'epsilons' if ready to admit that we exist at all except in non-rigorous formulations of elementary calculus!

Quantum entanglement between subsystems belonging to different aleph levels of infinity would make possible experiences containing information about this finite world and about the higher level worlds, too. Perhaps our brightest mathematical thoughts (at least) could correspond to cognitive space-time sheets of infinite duration glued to cognitive space-time sheets with even more infinite duration whereas the contents of sensory experiences would be located around finite values of geometric time.

7.5 Infinite primes and mystic world view

The proposed interpretation deserves some additional comments from the point of consciousness theory.

1. An open problem is whether the finite integer S appearing in the infinite prime is product of only finite or possibly even infinite number of lower level primes at a given level of hierarchy. The proposed physical identification of S indeed allows S to be a product of infinitely many primes. One can allow also M and N appearing in the infinite and infinite part to be contain infinite number of factors. In this manner one obtains a hierarchy of infinite primes expressible in the form

$$\begin{aligned} &P = nY^{r_1} + mS \ , \quad r = 1,2,... \\ &m = m_0 + P_{r_2}(Y) \ , \\ &Y = \frac{X}{S} \ , \\ &S = \prod_i P_i \ . \end{aligned}$$

   Note that this ansatz is in principle of the same general form as the original ansatz $P = nY + mS$. These primes correspond in physical analogy to states containing infinite number of particles.

   If one poses no restrictions on S this implies that that the cardinality for the set of infinite primes at first level would be $c = 2^{alef_0}$ ($alef_0$ is the cardinality of natural numbers). This is the cardinality for *all* subsets of natural numbers equal to the cardinality of reals. At the next level one obtains the cardinality 2^c for *all* subsets of reals, etc....

   If S were always a product of *finite number of primes* and $k(p)$ would differ from zero for finite number of primes only, the cardinality of infinite primes would be $alef_0$ at each level. One could pose the condition that mS is infinitesimal as compared to nX/S. This would guarantee that the ratio of two infinite primes at the same level would be well defined and equal to n_1S_2/n_2S_1. On the other hand, the requirement that all rationals are obtained as ratios of infinite primes requires that no restrictions are posed on $k(p)$: in this case the cardinality coming from possible choices of $r = ms$ is the cardinality of reals at first level.

   The possibility of primes for which also S is finite would mean that the algebra determined by the infinite primes must be generalized. For the primes representing states containing infinite number of bosons and/or fermions it would be be possible to tell how P_1P_2 and P_2P_1 differ and these primes would behave like elements of free algebra. As already found, this kind of free algebra would provide single space-time point with enormous algebraic representative power and analog of Brahman=Atman identity would result.

2. There is no physical subsystem-complement decomposition for the infinite primes of form $X \pm 1$ since fermionic degrees of freedom are not excited at all. Mystic could interpret it as a state of consciousness in which all separations vanish and there is no observer-observed distinction anymore. A state of pure awareness would be in question if bosonic and fermionic excitations represent the contents of consciousness! Since fermionic many particle states identifiable as Boolean statements about basic statements are identified as representation for reflective level of consciousness, $S = 1$ means that the reflective level of consciousness is absent: enlightment as the end of thoughts according to mystics.

   The mystic experiences of oneness ($S = 1$!), of emptiness (the subset of primes defined by S is empty!) and of the absence of all separations (there is no subsystem-complement separation and hence no division between observer and observed) could be related to quantum jumps to this kind of sectors of the configuration space. In super-symmetric interpretation $S = 1$ means that state contains no fermions.

3. There is entire hierarchy of selves corresponding to the hierarchy of infinite primes and the relationship between selves at different levels of the hierarchy is like the relationship between God and human being. Infinite primes at the lowest level would presumably represent elementary particles. This implies a hierarchy for moments of consciousness and it would be un-natural to exclude the existence of higher level 'beings' (one might call them Angels, Gods, etc...).

7.6 Infinite primes and evolution

The original argument leading to the notion of infinite primes was simple. Generalized unitarity implies evolution as a gradual increase of the p-adic prime labelling the configuration space sector D_p to which the localization associated with quantum jump occurs. Infinite p-adic primes are forced by the requirement that p-adic prime increases in a statistical sense and that the number of quantum jumps already occurred is infinite (assuming finite number of these quantum jumps and therefore the first quantum jump, one encounters the problem of deciding what was the first configuration space spinor field).

Quantum classical correspondence requires that p-adic evolution of the space-time surface with respect to geometric time repeats in some sense the p-adic evolution by quantum jumps implied by the generalized unitarity [E6]. Infinite p-adic primes are in a well defined sense composites of the primes belonging to lower level of infinity and at the bottom of this de-compositional hierarchy are finite primes. This decomposition corresponds to the decomposition of the space-time surface into p-adic regions which in TGD inspired theory of consciousness correspond to selves. Therefore the increase of the composite primes at lower level of infinity induces the increase of the infinite p-adic prime. p-Adic prime can increase in two manners.

1. One can introduce the concept of the p-adic sub-evolution: the evolution of infinite prime P is induced by the sub-evolution of infinite primes belonging to a lower level of infinity being induced by .... being induced by the evolution at the level of finite primes. For instance, the increase of the cell size means increase of the p-adic prime characterizing it: neurons are indeed very large and complicated cells whereas bacteria are small. Sub-evolution occurs both in subjective and geometric sense.

   i) For a given value of geometric time the p-adic prime of a given space-time sheet gradually increases in the evolution by quantum jumps: our geometric past evolves also!

ii) The p-adic prime characterizing space-time sheet also increases as the geometric time associated with the space-time sheet increases (say during morphogenesis).

The notion of sub-evolution is in accordance with the "Ontogeny recapitulates phylogeny" principle (ORP): the evolution of organism, now the entire Universe, contains the evolutions of the more primitive organisms as sub-evolutions.

2. Infinite prime increases also when entirely new finite primes emerge in the decomposition of an infinite prime to finite primes. This means that entirely new space-time sheets representing new structures emerge in quantum jumps. The creation of space-time sheets in quantum jumps could correspond to this process. By quantum classical correspondence this process corresponds at the space-time level to phase transitions giving rise to new material space-time sheets with more and more refined effective p-adic effective topology.

8 Local zeta functions, Galois groups, and infinite primes

The recent view about TGD leads to some conjectures about Riemann Zeta.

1. Non-trivial zeros should be algebraic numbers.

2. The building blocks in the product decomposition of ζ should be algebraic numbers for non-trivial zeros of zeta.

3. The values of zeta for their combinations with positive imaginary part with positive integer coefficients should be algebraic numbers.

These conjectures are motivated by the findings that Riemann Zeta seems to be associated with critical systems and by the fact that non-trivial zeros of zeta are analogous to complex conformal weights. The necessity to make such a strong conjectures, in particular conjecture c), is an unsatisfactory feature of the theory and one could ask how to modify this picture. Also a clear physical interpretation of Riemann zeta is lacking.

8.1 Local zeta functions and Weil conjectures

Riemann Zeta is not the only zeta [47, 46]. There is entire zoo of zeta functions and the natural question is whether some other zeta sharing the basic properties of Riemann zeta having zeros at critical line could be more appropriate in TGD framework.

The so called local zeta functions analogous to the factors $\zeta_p(s) = 1/(1-p^{-s})$ of Riemann Zeta can be used to code algebraic data about say numbers about solutions of algebraic equations reduced to finite fields. The local zeta functions appearing in Weil's conjectures [48] associated with finite fields $G(p,k)$ and thus to single prime. The extensions $G(p,nk)$ of this finite field are considered. These local zeta functions code the number for the points of algebraic variety for given value of n. Weil's conjectures also state that if X is a mod p reduction of non-singular complex projective variety then the degree for the polynomial multiplying the product $\zeta(s)\times\zeta(s-1)$ equals to Betti number. Betti number is 2 times genus in 2-D case.

It has been proven that the zetas of Weil are associated with single prime p, they satisfy functional equation, their zeros are at critical lines, and rather remarkably, they are rational functions of p^{-s}. For instance, for elliptic curves zeros are at critical line [48].

The general form for the local zeta is $\zeta(s) = exp(G(s))$, where $G = \sum g_n p^{-ns}$, $g_n = N_n/n$, codes for the numbers N_n of points of algebraic variety for n^{th} extension of finite field F with nk elements assuming that F has $k = p^r$ elements. This transformation resembles the relationship $Z = exp(F)$ between partition function and free energy $Z = exp(F)$ in thermodynamics.

The exponential form is motivated by the possibility to factorize the zeta function into a product of zeta functions. Note also that in the situation when N_n approaches constant N_∞, the division of N_n by n gives essentially $1/(1-N_\infty p^{-s})$ and one obtains the factor of Riemann Zeta at a shifted argument $s - log_p(N_\infty)$. The local zeta associated with Riemann Zeta corresponds to $N_n = 1$.

8.2 Local zeta functions and TGD

The local zetas are associated with single prime p, they satisfy functional equation, their zeros lie at the critical lines, and they are rational functions of p^{-s}. These features are highly desirable from the TGD point of view.

8.2.1 Why local zeta functions are natural in TGD framework?

In TGD framework modified Dirac equation assigns to a partonic 2-surface a p-adic prime p and inverse of the zeta defines local conformal weight. The intersection of the real and corresponding p-adic parton 2-surface is the set containing the points that one is interested in. Hence local zeta sharing the basic properties of Riemann zeta is highly desirable and natural. In particular, if the local zeta is a rational function then the inverse images of rational points of the geodesic sphere are algebraic numbers. Of course, one might consider a stronger constraint that the inverse image is rational. Note that one must still require that p^{-s} as well as s are algebraic numbers for the zeros of the local zeta (conditions 1) and 2) listed in the beginning) if one wants the number theoretical universality.

Since the modified Dirac operator assigns to a given partonic 2-surface a p-adic prime p, one can ask whether the inverse $\zeta_p^{-1}(z)$ of some kind of local zeta directly coding data about partonic 2-surface could define the generalized eigenvalues of the modified Dirac operator and radial super-canonical conformal weights so that the conjectures about Riemann Zeta would not be needed at all.

The eigenvalues of the modified Dirac operator would in a holographic manner code for information about partonic 2-surface. This kind of algebraic geometric data are absolutely relevant for TGD since U-matrix and probably also S-matrix must be formulated in terms of the data related to the intersection of real and partonic 2-surfaces (number theoretic braids) obeying same algebraic equations and consisting of algebraic points in the appropriate algebraic extension of p-adic numbers. Note that the hierarchy of algebraic extensions of p-adic number fields would give rise to a hierarchy of zetas so that the algebraic extension used would directly reflect itself in the eigenvalue spectrum of the modified Dirac operator and super-canonical conformal weights. This is highly desirable but not achieved if one uses Riemann Zeta.

One must of course leave open the possibility that for real-real transitions the inverse of the zeta defined as a product of the local zetas (very much analogous to Riemann Zeta) defines the conformal weights. This kind of picture would conform with the idea about real physics as a kind of adele formed from p-adic physics.

8.2.2 Finite field hierarchy is not natural in TGD context

That local zeta functions are assigned with a hierarchy of finite field extensions do not look natural in TGD context. The reason is that these extensions are regarded as abstract extensions of $G(p, k)$ as opposed to a large number of algebraic extensions isomorphic with finite fields as abstract number fields and induced from the extensions of p-adic number fields. Sub-field property is clearly highly relevant in TGD framework just as the sub-manifold property is crucial for geometrizing also other interactions than gravitation in TGD framework.

The $O(p^n)$ hierarchy for the p-adic cutoffs would naturally replace the hierarchy of finite fields. This hierarchy is quite different from the hierarchy of finite fields since one expects that the number of solutions becomes constant at the limit of large n and also at the limit of large p so that powers in the function G coding for the numbers of solutions of algebraic equations as function of n should not increase but approach constant N_∞. The possibility to factorize $exp(G)$ to a product $exp(G_0)exp(G_\infty)$ would mean a reduction to a product of a rational function and factor(s) $\zeta_p(s) = 1/(1-p^{-s_1})$ associated with Riemann Zeta with argument s shifted to $s_1 = s - log_p(N_\infty)$.

8.2.3 What data local zetas could code?

The next question is what data the local zeta functions could code.

1. It is not at clear whether it is useful to code global data such as the numbers of points of partonic 2-surface modulo p^n. The notion of number theoretic braid occurring in the proposed approach to S-matrix suggests that the zeta at an algebraic point z of the geodesic sphere S^2 of CP_2 or of light-cone boundary should code purely local data such as the numbers N_n of points which project to z as function of p-adic cutoff p^n. In the generic case this number would be finite for non-vacuum extremals with 2-D S^2 projection. The n^{th} coefficient $g_n = N_n/n$ of the function G_p would code the number N_n of these points in the approximation $O(p^{n+1}) = 0$ for the algebraic equations defining the p-adic counterpart of the partonic 2-surface.

2. In a region of partonic 2-surface where the numbers N_n of these points remain constant, $\zeta(s)$ would have constant functional form and therefore the information in this discrete set of algebraic points would allow to deduce deduce information about the numbers N_n. Both the algebraic points and generalized eigenvalues would carry the algebraic information.

3. A rather fascinating self referentiality would result: the generalized eigen values of the modified Dirac operator expressible in terms of inverse of zeta would code data for a sequence of approximations for the p-adic variant of the partonic 2-surface. This would be natural since second quantized induced spinor fields are correlates for logical thought in TGD inspired theory of consciousness. Even more, the data would be given at points $\zeta(s)$, s a rational value of a super-canonical conformal weight or a value of generalized eigenvalue of modified Dirac operator (which is essentially function $s = \zeta_p^{-1}(z)$ at geodesic sphere of CP_2 or of light-cone boundary).

8.3 Galois groups, Jones inclusions, and infinite primes

Langlands program [50, 51] is an attempt to unify mathematics using the idea that all zeta functions and corresponding theta functions could emerge as automorphic functions giving rise to finite-dimensional representations for Galois groups (Galois group is defined as a group of automorphisms of the extension of field F leaving invariant the elements of F). The basic example corresponds to rationals and their extensions. Finite fields $G(p,k)$ and their extensions $G(p,nk)$ represents another example. The largest extension of rationals corresponds to algebraic numbers (algebraically closed set). Although this non-Abelian group is huge and does not exist in the usual sense of the word its finite-dimensional representations in groups $GL(n,Z)$ make sense.

For instance, Edward Witten is working with the idea that geometric variant of Langlands duality could correspond to the dualities discovered in string model framework and be understood in terms of topological version of four-dimensional $N = 4$ super-symmetric YM theory [52]. In particular, Witten assigns surface operators to the 2-D surfaces of 4-D space-time. This brings unavoidably in mind partonic 2-surfaces and TGD as $N = 4$ super-conformal almost topological QFT.

This observation stimulates some ideas about the role of zeta functions in TGD if one takes the vision about physics as a generalized number theory seriously.

8.3.1 Galois groups, Jones inclusions, and quantum measurement theory

The Galois representations appearing in Langlands program could have a concrete physical/cognitive meaning.

1. The Galois groups associated with the extensions of rationals have a natural action on partonic 2-surfaces represented by algebraic equations. Their action would reduce to permutations of roots of the polynomial equations defining the points with a fixed projection to the above mentioned geodesic sphere S^2 of CP_2 or δM^4_+. This makes possible to define modes of induced spinor fields transforming under representations of Galois groups. Galois groups would also have a natural action on configuration space-spinor fields. One can also speak about configuration space spinors invariant under Galois group.

2. Galois groups could be assigned to Jones inclusions having an interpretation in terms of a finite measurement resolution in the sense that the discrete group defining the inclusion leaves invariant the operators generating excitations which are not detectable.

3. The physical interpretation of the finite resolution represented by Galois group would be based on the analogy with particle physics. The field extension K/F implies that the primes (more precisely, prime ideals) of F decompose into products of primes (prime ideals) of K. Physically this corresponds to the decomposition of particle into more elementary constituents, say hadrons into quarks in the improved resolution implied by the extension $F \rightarrow K$. The interpretation in terms of cognitive resolution would be that the primes associated with the higher extensions of rationals are not cognizable: in other words, the observed states are singlets under corresponding Galois groups: one has algebraic/cognitive counterpart of color confinement.

4. For instance, the system labelled by an ordinary p-adic prime could decompose to a system which is a composite of Gaussian primes. Interestingly, the biologically highly interesting p-adic length scale range 10 nm-5 μm contains as many as four Gaussian Mersennes ($M_k = (1+i)^k - 1$, $k = 151, 157, 163, 167$), which suggests that the emergence of living matter means an improved cognitive resolution.

8.3.2 Galois groups and infinite primes

In particular, the notion of infinite prime suggests a manner to realize the modular functions as representations of Galois groups. Infinite primes might also provide a new perspective to the concrete realization of Langlands program.

1. The discrete Galois groups associated with various extensions of rationals and involved with modular functions which are in one-one correspondence with zeta functions via Mellin transform defined as $\sum x_n n^{-s} \rightarrow \sum x_n z^n$ [49]. Various Galois groups would have a natural action in the space of infinite primes having interpretation as Fock states and more general bound states of an arithmetic quantum field theory.

2. The number theoretic anatomy of space-time points due to the possibility to define infinite number of number theoretically non-equivalent real units using infinite rationals [17] allows the imbedding space points themselves to code holographically various things. Galois groups would have a natural action in the space of real units and thus on the number theoretical anatomy of a point of imbedding space.

3. Since the repeated second quantization of the super-symmetric arithmetic quantum field theory defined by infinite primes gives rise to a huge space of quantum states, the conjecture

that the number theoretic anatomy of imbedding space point allows to represent configuration space (the world of classical worlds associated with the light-cone of a given point of H) and configuration space spinor fields emerges naturally [17].

4. Since Galois groups G are associated with inclusions of number fields to their extensions, this inclusion could correspond at quantum level to a generalized Jones inclusion $\mathcal{N} \subset \mathcal{M}$ such that G acts as automorphisms of $\mathcal{M}$ and leaves invariant the elements of $\mathcal{N}$. This might be possible if one allows the replacement of complex numbers as coefficient fields of hyper-finite factors of type II_1 with various algebraic extensions of rationals. Quantum measurement theory with a finite measurement resolution defined by Jones inclusion $\mathcal{N} \subset \mathcal{M}$ [16] could thus have also a purely number theoretic meaning provided it is possible to define a non-trivial action of various Galois groups on configuration space spinor fields via the imbedding of the configuration space spinors to the space of infinite integers and rationals (analogous to the imbedding of space-time surface to imbedding space).

This picture allows to develop rather fascinating ideas about mathematical structures and their relationship to physical world. For instance, the functional form of a map between two sets the points of the domain and target rather than only its value could be coded in a holographic manner by using the number theoretic anatomy of the points. Modular functions giving rise to generalized zeta functions would emerge in especially natural manner in this framework. Configuration space spinor fields would allow a physical realization of the holographic representations of various maps as quantum states.

9 Remarks about correspondence between infinite primes, space-time surfaces, and configuration space spinor fields

The correspondence of CH points with infinite primes and thus with real units can be understood if one assume that the points of CH correspond to infinite rationals via their mapping to hyper-octonion real-analytic rational functions conjectured to define foliations of HO to hyper-quaternionic 4-surfaces inducing corresponding foliations of H. The correspondence of CH spinors with the real units identified as infinite rationals with varying number theoretical anatomies is not so obvious. It is good to approach the problem by making questions.

1. How the points of CH and CH spinors at given point of CH correspond to various real units? Configuration space Hamiltonians and their super-counterparts characterize modes of configuration space spinor fields rather than only spinors. Does this mean that only ground states of super-conformal representations, which are expected to correspond elementary particles, correspond to configuration space spinors and are coded by infinite primes?

2. How do CH spinor fields (as opposed to CH spinors) correspond to infinite rationals? Configuration space spinor fields are generated by elements of super-conformal algebra from ground states. Should one code the matrix elements of the operators between ground states and creating zero energy states in terms of time-like entanglement between ground states represented by real units and assigned to the preferred points of H characterizing the tips of future and past light-cones and having also interpretation as arguments of n-point functions?

The argument to be represented is in a nutshell following.

1. CH itself and CH spinors are by super-symmetry characterized by ground states of super-conformal representations and can be mapped to infinite rationals defining real units U_k multiplying the eight preferred H coordinates h^k whereas configuration space spinor fields

correspond to discrete analogs of Schrödinger amplitudes in the space whose points have U_k as coordinates. The 8-units correspond to ground states for an 8-fold tensor power of a fundamental super-conformal representation or to a product of representations of this kind.

2. General states are coded by quantum entangled states defined as entangled states of positive and negative energy ground states with entanglement coefficients defined by the product of operators creating positive and negative energy states represented by the units. Normal ordering prescription makes the mapping unique.

3. The condition that various symmetries have number theoretical correlates leads to rather detailed view about the map of ground states to real units.

4. It seems that quantal generalization of the fundamental associativity and commutativity conditions might be needed.

Before continuing it is perhaps good to represent the most obvious objection against the idea. The correspondence between CH and CH spinors with infinite rationals and their discreteness means that also CH (world of classical worlds) and space of CH spinors should be discrete. First this looks non-sensible but is indeed what one obtains if space-time surfaces correspond to light-like 3-surfaces expressible in terms of algebraic equations involving rational functions with rational coefficients.

9.1 How CH and CH spinor fields correspond to infinite rationals?

The basic question is how CH and CH spinor fields on quantum fluctuating degrees of freedom (degrees of freedom for which configuration space metric is non-vanishing) correspond to infinite rationals.

9.1.1 Associativity and commutativity or only their quantum variants?

Associativity and commutativity conditions are absolutely essential notions in quantum TGD and also in the mapping of infinite primes to the space-time sheets. Associativity, guaranteed by hyper-octonion real-analyticity and implying rational infinite primes, seems to be necessary in order to obtain well-defined representations but might be too strong a condition.

Associativity implies hyper-quaternionicity and commutativity requirement in turn leads to rational infinite primes. Since one can decompose rational primes to hyper-quaternionic and even hyper-octonionic primes, one might hope that this could allow to represent states which consist of colored constituents. This representations has however the flavor of a formal trick and the considerations related to concrete representations of infinite primes suggest that the rationality of infinite primes might be a too restrictive condition.

A more radical possibility is that physical states are only quantum associative. This means that they are obtained as quantum superpositions in the space of real units over all possible associations performed for a given product of hyper-octonion primes (for instance, $|A(BC)\rangle + |(AB)C\rangle$). These states would be associative in quantum sense but would not reduce to hyper-quaternionic primes. Also the notion of quantum commutativity makes sense. The fact that mesons are quantum superpositions of quark-antiquark pairs which each corresponds to different pair of hyper-quaternionic primes and are thus not representable classically, suggests that one can require only quantum associativity and quantum commutativity.

How this idea relates to the representation of space-time surfaces in terms of rational functions of hyper-octonionic variable obtained as an image of rational infinite prime? If one replaces the coefficients of the polynomial which complex or more complex rational, hyper-octonion real analyticity is lost and one must consider some manner to map associative quantum state defined as superposition of various associations to single hyper-quaternionic prime.

1. The first approach is based on the assumption that only infinite integers reduce to infinite rational integers in the sense that the corresponding rational function has rational coefficients. This would allow partons as colored partons represented as non-associative constituents of infinite integers and there would be no problems with space-time correlates. It is however not clear whether this kind infinite integers are possible.

2. In the case of non-commutative group one can speak about commutator group and define Abelian group as coset group of these. Could it be that one can speak about associator algebra and define associative algebra by identifying additive associators $A(BC) - (AB)C$ with zero or multiplicative associators $(A(BC))((AB)C)^{-1}$ with unit. Hyper-octonionic primes would be mapped to something represented by matrices. A good guess for the representation is in terms of 8-D analog of Pauli spin matrices.

9.1.2 Basic assumptions

The following assumptions serve as constraints when one tries to guess the map of quantum states to infinite primes.

1. Free many-particle states correspond to infinite integers and bound states to infinite primes mappable to irreducible polynomials. The numerator/denominator of the infinite rational should correspond to positive/negative energy states of which zero energy states consist of. At higher levels the mapping should be induced from that for the lowest level. Bosonic (fermionic) elementary particles in ground states should correspond to bosonic (fermionic primes). Phase conjugation as a generalization of that for laser beams) would correspond to the replacement of infinite integer with its inverse.

2. Concerning charge conjugation one can imagine several options but the detailed study of the realization of color symmetry leaves only one option. For this option the two singlets $1 \pm ie_7$ and triplet and antitriplet correspond to leptons and quarks with spin and electro-weak spin represented by the moduli space associated with the hyper-octonionic structures. One must leave open the interpretation of the change of the sign of the small part of the infinite prime, which looks excellent candidate for some discrete symmetry (parity perhaps?).

3. Discrete super-canonical and Super Kac-Moody algebras with bosonic and fermionic generators label the states. One should map the ground states of these representations to infinite primes and thus to real units in a natural manner. The requirement that standard model symmetries reduce to number theory serves as a powerful constraint and will be analyzed in detail later.

4. The excited states of various super-conformal representations can be mapped to quantum superpositions of many particle states formed from infinite primes. The operators creating the positive and negative energy parts are unique combinations of the operators of algebra if normal ordering prescription is applied. The matrix elements of these operators between ground states can be calculated. The entangled state formed from ground states with entanglement coefficients represented by these matrix elements gives the representation of the general state. Note that the real units would be associated with different points of H identifiable as arguments of n-point function in S-matrix elements.

9.1.3 How to map ground states of super-conformal representations to infinite primes?

Under the assumptions just stated the problem reduces to that of guessing the detailed form of the map of the ground states of super-conformal representations to primes at the first level of the hierarchy. The mapping of infinite primes to rational functions could provide a clue about how to achieve a natural one-to-one correspondence.

1. The decomposition of the irreducible polynomials in the algebraic extension of rationals gives interpretation in terms of many-particle states labelled by primes in the extension. This brings in Galois groups and their representations. This seems to be something new to present day physics. Note that color group plays the role of Galois group for octonions regarded as extension of reals.

2. Partonic two-surfaces should correspond to infinite primes but in such a manner that an infinite number of infinite primes are mapped to the same partonic 2-surface since given 3-surface should be able to to carry an arbitrary state of super-canonical and super Kac-Moody representation. This is the case since each light-like 3-surface traversing a given partonic 2-surface corresponds to an infinite prime in turn assumed to code for a foliation of hyper-quaternionic or co-hyper-quaternionic surfaces via corresponding rational function of hyper-octonionic variable. Light-like 3-surfaces and corresponding 4-D space-time sheets would thus code for the ground states of super-conformal representations. Quantum classical correspondence would apply to ground states but not to the excited states of super-conformal representations.

3. One should also understand how light-like partonic 3-surfaces are mapped to the number theoretic anatomies of a point of imbedding space. The natural choice for this point would be the preferred point of H defining the tip of the light-cone and the origin of complex coordinates of CP_2 transforming linearly under $U(2) \subset SU(3)$. This choice should be coded as a zero/pole of infinite rational with unit real norm coding for the zero energy states. Zeros would correspond to the positive energy state and poles to the negative energy state.

9.1.4 The treatment of zero modes

There are also zero modes which are absolutely crucial for quantum measurement theory. They entangle with quantum fluctuating degrees of freedom in quantum measurement situation and thus map quantum numbers to positions of pointers. The interior degrees of freedom of space-time interior must correspond to zero modes and they represent space-time correlates for quantum states realized at light-like partonic 3-surfaces.

As long as states associated with zero modes are represented by operators (such as CH Hamiltonians), the same description applies to them as to the representation of excited states of super-conformal representations. The absence of metric in zero modes means that there is no integration measure. The problems are avoided if one assumes that wave functions in zero modes have a discrete locus as suggested already earlier.

According to the argument represented in [C1], the quantum fluctuating configuration space degrees of freedom are by definition super-symmetrizable since configuration space gamma matrices correspond to the super counterparts of Hamiltonians in the case of super-canonical algebra. Super-symmetrizability condition means that the Poisson brackets of bosonic Hamiltonians reduce to 1-dimensional integrals over "stringy" curves of partonic 2-surface [C1]. This happens for the sub-algebra of super-canonical algebra having vanishing S^2 spin and color charges.

This would mean that zero modes include also the charged Hamiltonians of the super-canonical algebra. This brings in mind induced representations for which one has coset space structure with entire super-canonical group divided by the group generated by neutral super-canonical algebra. The necessary discretization zero modes of freedom suggests a reduction of the representations of isometry groups of H and CH to those for discrete subgroups of isometry groups which indeed appear naturally in Jones inclusions.

One must take this suggestion with some grain of salt. The coset construction for Kac-Moody representations allows to consider the possibility of extending the representations to charged Hamiltonians in such a manner that "stringy" commutators are preserved. The generation of Virasoro and Kac-Moody central extension parameters might be seen as the price paid for the stringy commutation relations.

9.1.5 Configuration space spinor fields as discrete Schrödinger amplitudes in the space of number theoretic anatomies?

It would seem that the analog of a complex Schrödinger amplitude in the space of number-theoretic anatomies of a given imbedding space point represented by single point of H and represented as 8-tuples of real units could naturally represent the dependence of CH spinors understood as ground states of super-conformal representations obtained as an 8-fold tensor power of a fundamental representation or product of representations perhaps differing somehow. The open question is why eight of them are needed. The excited states of super-conformal representations would be represented as time entangled states with entanglement between real units associated with the preferred points characterizing the tips future and past directed light-cones.

This picture conforms with the simple idea that infinite primes label the points in the fibers of the spinor field bundle having CH_h, h a preferred point of H characterizing the preferred origin of hyper-octonion structure, as a base space and that physical states correspond to discrete analogs of Schrödinger amplitude in this kind of bundles and product bundles formed from them. These 8-tuples define a number theoretical analog of $U(1)^8$ group in terms of which all number theoretical symmetries are represented.

9.2 Can one understand fundamental symmetries number theoretically?

One should understand symmetries number theoretically.

1. The basic idea is that color $SU(3) \subset G_2$ acts as automorphisms of hyper-octonion structure with a preferred imaginary unit and preferred point with respect to which hyper-octonionic power series are developed. $SO(7,1)$ would act as symmetries in the moduli space of hyper-octonion structures. Associativity implies symmetry breaking so that only hyper-quaternionic structures are considered and $SO(3,1) \times SO(4)$ acts as symmetries of the moduli space for these structures.

2. Color group is the analog of Galois group for the extension of reals to octonions and has a natural action on the decompositions of rational infinite primes to hyper-octonionic infinite primes. Color confinement is implied by hyper-quaternionicity of primes implied by associativity necessary to assign space-time surfaces to the infinite rationals. If one assumes only quantum associativity, one should have a generalization of the condition guaranteing color confinement. A possible more general condition is that infinite integers give rise to rational polynomials whereas infinite primes can be non-associative and non-commutative if they appear as constituents of N-particle state. This would predict that free quarks are not possible.

3. Electro-weak symmetries and Lorentz group act in the moduli space of hyper-octonionic structures and their actions deform space-time in H picture. CP_2 parameterizes the moduli space of hyper-quaternionic structures induced from a given hyper-octonionic structure with preferred imaginary unit.

4. Four-momenta correspond to translational degrees of freedom associated with the preferred points of M^4 coded by the infinite rational (tip of the light-cone). Color quantum numbers

in cm degrees of freedom can be assigned to the CP_2 projection of the preferred point of H. As a matter fact, the definition of hyper-octonionic structure involves the choice of origin of HO giving rise to the preferred point of H.

9.2.1 Automorphisms and the symmetries of moduli space of hyper structures as basic symmetries

Consider now in more detail various symmetries.

1. G_2 acts as automorphisms on octonionic imaginary units and $SU(3)$ respects the choice of preferred imaginary unit. Associativity requires a reduction to hyper-quaternionic primes and implies color confinement. For hyper-quaternionic primes the automorphisms restrict to $SO(3)$ which has right/left action of fermionic hyper-quaternionic primes and adjoint action on bosonic hyper-quaternionc primes. The choice of hyper-quaternionic structure is global as opposed to the local choice of hyper-quaternionic tangent space of space-time surface assigning to a point of $HQ \subset HO$ a point of CP_2. $U(2) \subset SU(3)$ leaves invariant given hyper-quaternionic structure which are thus parameterized by CP_2. Color partial waves can be interpreted as partial waves in this moduli space.

2. The choice of global hyper-octonionic coordinate is dictated only modulo a transformation of $SO(1,7)$ acting as isometries of hyper-octonionic norm and as transformations in moduli space of hyper-octonion structures $SO(7)$ acting leaves invariant the choice of real unit. $SO(1,3) \times SO(4)$ acts in the moduli space of global hyper-quaternionic structures identified as sub-structures of hyper-octonionic structure. The choice of global HO structures involves also choice of origin implying preferred point of H. The M^4 projection of this point corresponds to the tip of light-cone. Since the integers representing physical states must be hyper-quaternionic by associativity conditions, the symmetry breaking ("number theoretic compactification") to $SO(1,3) \times SO(4)$ occurs very naturally. This group acts as spinor rotations in H picture and as isometries in HO picture.

3. $SO(1,7)$ allows 3 different 8-dimensional representations (8_v, 8_s, and $\overline{8}_s$). All these representations must decompose under $SU(3)$ as $1+1+3+\overline{3}$ as little exercise with $SO(8)$ triality demonstrates. Under $SO(6) \cong SU(4)$ the decompositions are $1+1+6$ and $4+\overline{4}$ for 8_v and 8_s and its conjugate. Both hyper-octonion spinors and gamma matrices are identified as hyper-octonion units rather than as matrices. It would be natural to assign to bosonic HO primes 8_v and to fermionic HO primes 8_s and $\overline{8}_s$. One can distinguish between $8_v, 8_s$ and $\overline{8}_s$ for hyper-octonionic units only if one considers the full $SO(1,3) \times SO(4)$ action in the moduli space of hyper-octonionic structures.

9.2.2 Physical interpretation of the decomposition of rational primes to various hyper-primes

Consider now the physical interpretation for the decomposition of rational primes to hyper-complex, hyper-quaternionic, and hyper-octonionic primes. Here one must keep doors open by allowing also the notion of quantum commutativity and quantum associativity so that infinite hyper-octonionic primes would not in general have these properties whereas their images to gamma matrices would define primes of an associative algebra so that a unique space-time representation in terms of hyper-octonionic polynomial would result. Abelianization would produce a generalization of hyper-complex algebra with 7 commuting imaginary units satisfying $e_i^2 = 1$. I have considered earlier also the possibility that hyper-analytic functions of this kind of variable could define space-time surfaces. At this stage one cannot distinguish between this and hyper-octonion real-analytic option.

1. The net quantum numbers of physical states must vanish in zero energy ontology. This is implied by the reduction of infinite rationals to infinite rationals associated with rationals but one must consider also more general options. The vanishing of net quantum numbers could be achieved in many manners. In the most general case the quantum numbers of positive and negative energy state represented by integers in the numerator and denominator of the infinite rational would compensate. If one requires only associativity for infinite primes (or integers) then positive (negative) energy state can correspond to hyper-quaternionic integer and one ends up with H picture and breaking of HO symmetries to those of H.

2. Commutativity condition implies a restriction to hyper-complex numbers. The only restriction would be due to fermion number conservation. Bosonic rational primes could be decomposed to fermionic and antifermionic hyper-quaternionic/octonionic primes such that the net fermion number vanishes. Fermionic primes could correspond to neutrinos and antineutrinos.

3. Giving up commutativity condition but requiring that the primes are associative gives hyper-quaternionic primes and color confinement. One obtains two states which possess non-vanishing and opposite color hypercharges equal to $\pm 2/3$. Thus only the interpretation as lepton, antilepton, quark and antiquark with no color isospin is possible. Spin, weak spin, and color would not be manifest since it would correspond to degree of freedom in the moduli space of hyper-quaternionic structures.

4. Hyper-quaternionic primes can be decomposed to hyper-octonionic primes. In the fermionic sector the three quark states consisting of hyper-octonion units would give color singlets as linear combination of hyper-octonion real unit and the preferred imaginary unit. A state analogous to baryon would result. Is this representation just a formal trick or does it have a real physical content must be left open. In TGD framework, color quantum numbers correspond to color partial waves in CP_2 labelling the moduli space of hyper-quaternionic structures associated with a given hyper-octonionic structure. One might hope that the decomposition provides a formal representation of information about these partial waves.

5. Giving up also associativity for single hyper-octonionic prime and requiring only quantum associativity and requiring that only infinite integers reduces to rational infinite integers leads to the most general framework allowing to describe entangled many particle states formed from elementary particles with quantum numbers of quark and lepton and basic gauge bosons. Gauge bosons and would correspond to locally entangled fermion antifermion pairs (as predicted by TGD) represented as locally entangled real units.

9.2.3 Electro-weak and color symmetries

The crucial test for this picture is whether color and electro-weak symmetries can be understood number theoretically.

Electro-weak group acts as transformations in the hyper-quaternionic moduli space inducing left or right actions of fermions which cannot interpreted as $U(2) \subset SU(3)$ automorphisms realized via adjoint action. For bosons one adjoint action results. Therefore color singlet states can possess non-vanishing electro-weak quantum numbers as also spin. For bosonic hyper-quaternionic primes one obtains singlet and triplet and for fermionic primes two doublets. The interpretation in terms of electro-weak gauge bosons and electro-weak doublets seems natural. Spin degrees of freedom are not manifestly visible but correspond to the moduli space resulting by $SL(2,C)$ action on hyper-quaternionic units.

Some more detailed comments about color symmetries are in order.

1. Color group SU(3) corresponds to subgroup of G_2 which acts as a Galois group for the extension of reals to octonions. $SU(3)$ leaves invariant real unit and a preferred octonionic

imaginary unit. As noticed 8_v, 8_s and $\overline{8}_s$ decompose in a similar manner under $SU(3)$ and only the action of $SL(2,C) \times SO(4)$ modifying hyper-octonionic structure can distinguish between them.

2. Color group would act as a symmetry group on the composites of hyper-octonionic primes and color confinement in spinorial degrees of freedom would follow automatically from (complex) rationality (and even hyper-quaternionicity) of infinite integers necessitated by associativity. This does not however imply color singlet property in color rotational degrees of freedom in imbedding space. The value of color hypercharge (em charge) assignable to the spinors is the only signature of whether lepton or quark is in question.

9.2.4 Relationship to HO-H duality and two paradoxes

$HO - H$ duality states that the descriptions based on the use of HO and H as imbedding space are equivalent. This can be the case only if one assumes that the breaking of $SO(1,7)$ symmetry to $SO(1,3) \times SO(4)$ symmetry implied by mere associativity is present in both cases as indeed assumed in previous considerations. This forces to reconsider what one really means with HO and H pictures.

What looks to be the basic difference is that the notion of spinor is different in the two cases besides different identification of the imbedding space.

1. Hyper-octonionic spinors identifiable as hyper-octonionic units are used in HO picture. HO spinors reduce to HQ spinors by associativity and $SO(1,3) \times SO(4)$ symmetries act in the moduli space of hyper quaternion structures equivalent with the space of complex 8-spinors. Spin and electroweak spin quantum numbers are thus only implicitly present. For hyper-quaternionic structures induced from a fixed hyper-octonionic structure CP_2 is the moduli space. Color can be represented if one allows decomposition of hyper-quaternionic primes to products of hyper-octonionic primes.

2. In H picture one uses H spinors with CP_2 identified as the space of hyper-quaternionic tangent planes at a given point of HO. Spin and electroweak spin are explicitly present but not color.

3. One might perhaps say that in HO picture the roles of spinor rotations and isometries are changed. Color group takes the role of $SO(3,1) \times SO(4)$ and acts as automorphisms of hyper-octonion structure. In H picture one uses 8-D complex spinors on which $SO(3,1) \times SO(4)$ acts naturally and color groups acts as isometries. Instead of color group $SO(3) \times SO(4)$ would characterize HO Hamiltonians.

Both the spin puzzle of proton implied by the observation that quarks do not seem to contribute to the spin of proton and the statistics paradox implied by the non-visibility of color can be understood in this framework.

1. For HO picture color confinement implies the vanishing of the net spin if attention is restricted to single hyper-octonion structure neglecting thus the zero modes defined by hyper-octonionic moduli parametrized by $SL(2,C)$. Also electroweak quantum numbers vanish under analogous conditions. If the experimental findings correspond to what one observes by using HO picture with a fixed space-time surface, then the spin puzzle of proton can be understood as a neglect of the moduli degrees of freedom characterized by $SO(3,1)$.

2. At low energy limit of hadron physics color is not visible and H picture is natural. This would mean that there is no manifest color and one ends up with spin-statistics paradox if one does not take into account the moduli characterizing hyper-quaternionic structure associated with given hyper-octonionic structure.

10 A little crazy speculation about knots and infinite primes

D-dimensional knots correspond to the isotopy equivalence classes of the imbeddings of spheres S^d to S^{d+2}. One can consider also the isotopy equivalence classes of more general manifolds $M^d \subset M^{d+2}$. Knots [54] are very algebraic objects. The product (or sum, I prefer to talk about product) of knots is defined in terms of connected sum. Connected sum quite generally defines a commutative and associative product, and one can decompose any knot into prime knots.

Knots can be mapped to Jones polynomials $J(K)$ (for instance - there are many other polynomials and there are very general mathematical results about them [54]) and the product of knots is mapped to a product of corresponding polynomials. The polynomials assignable to prime knots should be prime in a well-defined sense, and one can indeed define the notion of primeness for polynomials $J(K)$: prime polynomial does not factor to a product of polynomials of lower degree in the extension of rationals considered.

This raises the idea that one could define the notion of zeta function for knots. It would be simply the product of factors $1/(1-J(K)^{-s})$ where K runs over prime knots. The new (to me) but very natural element in the definition would be that ordinary prime is replaced with a polynomial prime. This observation led to the idea that the hierarchy of infinite primes could correspond to the hierarchy of knots in various dimensions and this in turn stimulated quite fascinating speculations.

10.1 Do knots correspond to the hierarchy of infinite primes?

A very natural question is whether one could define the counterpart of zeta function for infinite primes. The idea of replacing primes with prime polynomials would resolve the problem since infinite primes can be mapped to polynomials. For some reason this idea however had not occurred to me earlier.

The correspondence of both knots and infinite primes with polynomials inspires the question whether $d = 1$-dimensional prime knots might be in correspondence (not necessarily 1-1) with infinite primes. Rational or Gaussian rational infinite primes would be naturally selected: these are also selected by physical considerations as representatives of physical states although quaternionic and octonionic variants of infinite primes can be considered.

If so, knots could correspond to the subset of states of a super-symmetric arithmetic quantum field theory with bosonic single particle states and fermionic states labelled by quaternionic primes.

1. The free Fock states of this QFT are mapped to first order polynomials and irreducible polynomials of higher degree have interpretation as bound states so that the non-decomposability to a product in a given extension of rationals would correspond physically to the non-decomposability into many-particle state. What is fascinating that apparently free arithmetic QFT allows huge number of bound states.

2. Infinite primes form an infinite hierarchy, which corresponds to an infinite hierarchy of second quantizations for infinite primes meaning that n-particle states of the previous level define single particle states of the next level. At space-time level this hierarchy corresponds to a hierarchy defined by space-time sheets of the topological condensate: space-time sheet containing a galaxy can behave like an elementary particle at the next level of hierarchy.

3. Could this hierarchy have some counterpart for knots?In one realization as polynomials, the polynomials corresponding to infinite prime hierarchy have increasing number of variables. Hence the first thing that comes into my uneducated mind is as the hierarchy defined by the increasing dimension d of knot. All knots of dimension d would in some sense serve as building bricks for prime knots of dimension $d+1$ or possibly $d+2$ (the latter option turns out to be the more plausible one). A canonical construction recipe for knots of higher dimensions should exist.

4. One could also wonder whether the replacement of spherical topologies for d-dimensional knot and $d+2$-dimensional imbedding space with more general topologies could correspond to algebraic extensions at various levels of the hierarchy bringing into the game more general infinite primes. The units of these extensions would correspond to knots which involve in an essential manner the global topology (say knotted non-contractible circles in 3-torus). Since the knots defining the product would in general have topology different from spherical topology the product of knots should be replaced with its category theoretical generalization making higher-dimensional knots a groupoid in which spherical knots would act diagonally leaving the topology of knot invariant. The assignment of d-knots with the notion of n-category, n-groupoid, etc.. by putting d=n is a highly suggestive idea. This is indeed natural since are an outcome of a repeated abstraction process: statements about statements about .....

5. The lowest ($d = 1, D = 3$) level would be the fundamental one and the rest would be somewhat boring repeated second quantization;-). This is why the dimension $D = 3$ (number theoretic braids at light-like 3-surfaces!) would be fundamental for physics.

10.2 Further speculations

Some further speculations about the proposed structure of all structures are natural.

1. The possibility that algebraic extensions of infinite primes could allow to describe the refinements related to the varying topologies of knot and imbedding space would mean a deep connection between number theory, manifold topology, sub-manifold topology, and n-category theory.

2. Category theory appears already now in fundamental role in the construction of the generalization of M-matrix unifying the notions of density matrix and S-matrix. Generalization of category to n-category theory and various n-structures would have very direct correspondence with the physics of TGD Universe if one assumes that repeated second quantization makes sense and corresponds to the hierarchical structure of many-sheeted space-time where even galaxy corresponds to elementary fermion or boson at some level of hierarchy.

   This however requires that the unions of light-like 3-surfaces and of their sub-manifolds at different levels of topological condensate are able to represent higher-dimensional manifolds physically albeit not in the standard geometric sense since imbedding space dimension is just 8. This might be possible.

3. As far as physics is considered, the disjoint union of sub-manifolds of dimensions d_1 and d_2 behaves like a $d_1 + d_2$-dimensional Cartesian product of the corresponding manifolds. This is of course used in standard manner in wave mechanics (the configuration space of n-particle system is identified as E^{3n}/S_n with division coming from statistics).

4. If the surfaces have intersection points, one has a union of Cartesian product with punctures (intersection points) and of lower-dimensional manifold corresponding to the intersection points.

5. Note also that by posing symmetries on classical fields one can effectively obtain from a given n-manifold manifolds (and orbifolds) with quotient topologies.

The megalomanic conjecture is that this kind of physical representation of d-knots and their imbedding spaces is possible using many-sheeted space-time. Perhaps even the entire magnificent mathematics of n-manifolds and their sub-manifolds might have a physical representation in terms of sub-manifolds of 8-D $M^4 \times CP_2$ with dimension not higher than space-time dimension $d = 4$.

10.3 The idea survives the most obvious killer test

All this looks nice and the question is how to give a death blow to all this reckless speculation. Torus knots are an excellent candidate for performing this unpleasant task but the hypothesis survives!

1. Torus knots [56] are labelled by a pair integers (m, n), which are relatively prime. These are prime knots. Torus knots for which one has $m/n = r/s$ are isotopic so that any torus knot is isotopic with a knot for which m and n have no common prime power factors.

2. The simplest infinite primes correspond to free Fock states of the supersymmetric arithmetic QFT and are labelled by pairs (m, n) of integers such that m and n do not have any common prime factors. Thus torus knots would correspond to free Fock states! Note that the prime power $p^{k(p)}$ appearing in m corresponds to $k(p)$-boson state with boson "momentum" p and the corresponding power in n corresponds to fermion state plus $k(p) - 1$ bosons.

3. A further property of torus knots is that (m, n) and (n, m) are isotopic: this would correspond at the level of infinite primes to the symmetry $mX + n \rightarrow nX + m$, X product of all finite primes. Thus infinite primes are in $2 \rightarrow 1$ correspondence with torus knots and the hypothesis survives also this murder attempt. Probably the assignment of orientation to the knot makes the correspondence 1-1 correspondence.

10.4 How to realize the representation of the braid hierarchy in many-sheeted space-time?

One can consider a concrete construction of higher-dimensional knots and braids in terms of the many-sheeted space-time concept.

1. The basic observation is that ordinary knots can be constructed as closed braids so that everything reduces to the construction of braids. In particular, any torus knot labelled by (m,n) can be made from a braid with m strands: the braid word in question is $(\sigma_1....\sigma_{m-1})^n$ or by $(m, n) = (n, m)$ equivalence from n strands. The construction of infinite primes suggests that also the notion of d-braid makes sense as a collection of d-braids in $d+2$-space, which move and and define $d+1$-braid in $d+3$ space (the additional dimension being defined by time coordinate).

2. The notion of topological condensate should allow a concrete construction of the pairs of d- and $d+2$-dimensional manifolds. The 2-D character of the fundamental objects (partons) might indeed make this possible. Also the notion of length scale cutoff fundamental for the notion of topological condensate is a crucial element of the proposed construction.

3. Infinite primes have also interpretation as physical states and the representation in terms of knots would mean a realization of quantum classical correspondence.

The concrete construction would proceed as follows.

1. Consider first the lowest non-trivial level in the hierarchy. One has a collection of 3-D light-like 3-surfaces X_i^3 representing ordinary braids. The challenge is to assign to them a 5-D imbedding space in a natural manner. Where do the additional two dimensions come from? The obvious answer is that the new dimensions correspond to the partonic 2-surface X^2 assignable to the $3-D$ lightlike surface X^3 at which these surfaces have suffered topological condensation. The geometric picture is that X_i^3 grow like plants from ground defined by X^2 at 7-dimensional $\delta M_+^4 \times CP_2$.

2. The degrees of freedom of X^2 should be combined with the degrees of freedom of X_i^3 to form a 5-dimensional space X^5. The natural idea is that one first forms the Cartesian products $X_i^5 = X_i^3 \times X^2$ and then the desired 5-manifold X^5 as their union by posing suitable additional conditions. Braiding means a translational motion of X_i^3 inside X^2 defining braid as the orbit in X^5. It can happen that X_i^3 and X_j^3 intersect in this process. At these points of the union one must obviously pose some additional conditions. Same applies to intersection of more than two X_i^3.

   Finite (p-adic) length scale resolution suggests that all points of the union at which an intersection between two or more light-like 3-surfaces occurs must be regarded as identical. In general the intersections would occur in a 2-d region of X^2 so that the gluing would take place along 5-D regions of X_i^5 and there are therefore good hopes that the resulting 5-D space is indeed a manifold. The imbedding of the surfaces X_i^3 to X^5 would define the braiding.

3. At the next level one would consider the 5-d structures obtained in this manner and allow them to topologically condense at larger 2-D partonic surfaces in the similar manner. The outcome would be a hierarchy consisting of $2n+1$-knots in $2n+3$ spaces. A similar construction applied to partonic surfaces gives a hierarchy of $2n$-knots in $2n+2$-spaces.

4. The notion of length scale cutoff is an essential element of the many-sheeted space-time concept. In the recent context it suggests that d-knots represented as space-time sheets topologically condensed at the larger space-time sheet representing $d+2$-dimensional imbedding space could be also regarded effectively point-like objects (0-knots) and that their d-knottiness and internal topology could be characterized in terms of additional quantum numbers. If so then d-knots could be also regarded as ordinary colored braids and the construction at higher levels would indeed be very much analogous to that for infinite primes.

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