# Identification of the Configuration Space Kähler Function

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## Contents

**Abstract**

The general topological structure of the configuration space is described and the difficulties associated with the geometrization attempt relying on the local coset space structure are described. Also the physical and mathematical motivations for $Diff^4$ invariance and -degeneracy and Kähler property are explained in detail. The idea is that configuration space decomposes into union $\cup_i G/H_i$ of coset spaces $G/H_i$ such that $G$ is a subgroup of $Diff(\delta M^4_+ \times CP_2)$ and $H_i$ contains the subgroup of $G$ whose action reduces to diffeomorphisms for given 3-surface $X^3$. Configuration space metric has also zero modes; part of them correspond to the generators of isometries invariant under complexification and part of them corresponds to isometry invariants.

The basic motivation for the construction of configuration space geometry is the vision that physics reduces to the geometry of classical spinor fields in the infinite-dimensional configuration space of 3-surfaces of $M^4_+ \times CP_2$. Hermitian conjugation is the basic operation in quantum theory and its geometrization requires that configuration space possesses Kähler geometry. One of the basic features of the Kähler geometry is that geometry is solely determined by the so called Kähler function.

The task of finding Kähler geometry for the configuration space reduces to that of finding Kähler function. The main constraints on the Kähler function result from the requirement of Diff$^4$ symmetry and degeneracy. General coordinate invariance requires that the definition of the Kähler function assigns to a given 3-surface $X^3$ a unique space-time surface $X^4(X^3)$, the generalized Bohr orbit defining the classical physics associated with $X^3$. The natural guess is that Kähler function is defined by what might be called Kähler action, which is essentially Maxwell action with Maxwell field expressible in terms of $CP_2$ coordinates. Absolute minimization is the most natural manner to fix $X^4(X^3)$ uniquely.

The basic question is what distinguishes physically between $X^3$ and other Diff$^4$ 3-surfaces at $X^4$ so that $X^3$ defines $X^4$ as absolute minimum. For $H = M^4_+ \times CP_2$ the problem can be settled. If Kähler action would define a strictly deterministic variational principle, Diff$^4$ degeneracy and invariance would be achieved by restricting the consideration to 3-surfaces $Y^3$ at the boundary of $M^4_+$ and by defining Kähler function for 3-surfaces $X^3$ at $X^4(Y^3)$ and diffeo-related to $Y^3$ as $K(X^3) = K(Y^3)$. This reduction might be called quantum gravitational holography. The classical non-determinism of Kähler action introduces complications, which might be overcome by generalizing the notion of quantum gravitational holography.

It has however become clear that the gigantic symmetries associated with $\delta M^4_+ \times CP_2$ are also symmetries at laboratory scale and that $M^4$ is favored over $M^4_+$. In this case $X^3$ must be selected uniquely by the internal geometry of $X^4$. The possibility of negative Poincare energies inspires the hypothesis that the total quantum numbers and classical conserved quantities of the Universe vanish. This view is consistent with experimental facts if gravitational energy is defined as a difference of Poincare energies of positive and negative energy matter. Space-time surface consists of pairs of positive and negative energy space-time sheets created at some moment from vacuum and branching at that moment. This allows to select $X^3$ uniquely and define $X^4(X^3)$ as absolute minimum of Kähler action. This view is also consistent with the non-determinism of Kähler action.

A complementary approach to the problem of constructing configuration space geometry is based on symmetries. The work of Dan Freed has demonstrated that the Kähler geometry of loop spaces is unique from the existence of Riemann connection and fixed completely by the Kac Moody symmetries of the space. In 3-dimensional context one has even better reasons to expect uniqueness. The guess is that configuration space is a union symmetric spaces labelled by zero modes not appearing in the line element as differentials. The generalized conformal invariance of metrically 2-dimensional light like 3-surfaces acting as causal determinants is the corner stone of the construction. The construction works only for 4-dimensional space-time and imbedding space which is a product of future light cone of four-dimensional Minkowski space and $CP_2$.

In this chapter a definition of the Kähler function is proposed and various physical and mathematical motivations behind the proposed definition are discussed. The key feature of

the Kähler action is classical non-determinism and various implications of the classical non-determinism are discussed.

# 1 Introduction

The motivation the construction of configuration space geometry is that physics reduces to the geometry of classical spinor fields in the the "world of the classical worlds" identified as the infinite-dimensional configuration space of 3-surfaces of $M^4_+ \times CP_2$ or $M^4 \times CP_2$, where $M^4$ and $M^4_+$ denote Minkowski space and its light cone respectively.

Hermitian conjugation is the basic operation in quantum theory and its geometrization requires that configuration space possesses Kähler geometry. One of the basic features of the Kähler geometry is that it is solely determined by the so called Kähler function, which defines both the Kähler form $J$ and the components of the Kähler metric $g$ in complex coordinates via the formulas [16]

$$\begin{aligned} J &= i\partial_k\partial_{\bar{l}}K dz^k \wedge d\bar{z}^l \ , \\ ds^2 &= 2\partial_k\partial_{\bar{l}}K dz^k d\bar{z}^l \ . \end{aligned} \quad (1)$$

Kähler form is covariantly constant two-form and can be regarded as a representation of imaginary unit in the tangent space of the configuration space

$$J_{mr}J^{rn} = -g_m^{\ n} \ . \quad (2)$$

As a consequence Kähler form defines also symplectic structure in configuration space.

## 1.1 Definition of Kähler function

The task of finding Kähler geometry for the configuration space reduces to that of finding Kähler function. The main constraints on the Kähler function result from the requirement of Diff$^4$ symmetry (general coordinate invariance) and degeneracy. General coordinate invariance requires that the definition of the Kähler function assigns to a given 3-surface $X^3$ a unique space-time surface $X^4(X^3)$, the generalized Bohr orbit defining the classical physics associated with $X^3$. The natural guess is that Kähler function is defined by what might be called Kähler action, which is essentially Maxwell action with Maxwell field expressible in terms of $CP_2$ coordinates. Absolute minimization is the first guess for how to fix $X^4(X^3)$ uniquely.

It has however become clear that this option might well imply that Kähler is negative and infinite for the entire Universe so that the vacuum functional would be identically vanishing. I ended up with a more attractive less global option from number theoretical vision [E2]. According to this option the absolute value of the contribution to the Kähler aaction coming from each region where the action density has definite sign is minimized separately. [E2]. As a consequence the preferred extremals are as near to vacuum extremals as possible. For the dual of this principle maximization of absolute value occurs instead of minimization.

If Kähler action would define a strictly deterministic variational principle, Diff$^4$ degeneracy and invariance would be achieved by restricting the consideration to 3-surfaces $Y^3$ at the boundary of $M^4_+$ and by defining Kähler function for 3-surfaces $X^3$ at $X^4(Y^3)$ and diffeo-related to $Y^3$ as $K(X^3) = K(Y^3)$. This reduction might be called quantum gravitational holography. The classical non-determinism of the Kähler action however introduces complications which might be however overcome by generalizing the notion of quantum gravitational holography.

## 1.2 Minkowski space or its light cone?

For a long time I believed that the question "$M^4_+$ or $M^4$?" had been settled in favor of $M^4_+$. The work with the conceptual problems related to energy and time, and with the symmetries of quantum TGD, however led gradually to the realization that there are strong reasons for considering $M^4$ instead of $M^4_+$.

1. It has become clear that the gigantic symmetries associated with the $\delta M^4_+ \times CP_2$ and more general surfaces $X^3_l \times CP_2$, $X^3_l$ light like 3-surface of $M^4$ are also laboratory symmetries visible directly at the level of propagators and vertices [C5]. $X^3_l$ could be restricted to be a union of future and past directed light cone boundaries since arbitrary light like surfaces $X^3_l$ contain singularities such as self intersections. Poincare invariance fits very elegantly with the two types of super-conformal symmetries of TGD. The first conformal symmetry corresponds to the light-like surfaces $X^3_l \times CP_2$ of the imbedding space. More general light like 7-surfaces are not favored because they do not possess the huge conformal symmetries of $X^3_l \times CP_2$. Second conformal symmetry corresponds to light like boundaries of $X^4$ and light-like surfaces separating space-time regions with different signatures of the induced metric and is identifiable as the counterpart of the Kac Moody symmetry of string models. A rather plausible conclusion is that configuration space is a union of configuration spaces associated with $X^3_l \times CP_2$, with $X^3_l$ identified as unions of future and past directed light cones. Thus the construction reduces to a high degree to a study of a simple special case $\delta M^4_+ \times CP_2$.

2. The replacement of the energy momentum tensor by a collection of conserved currents means that the sign of the energy depends on the time orientation of the space-time sheet. The simplest theory results if one assumes that the net quantum numbers of physical states vanish. Crossing symmetry guarantees consistency with elementary particle physics. A consistency with the macroscopic physics results if gravitational energy is the difference of positive and negative inertial (Poincare) energies of matter. This option allows also $M^4$ and leads to a fractal cosmology in which light like 7-surfaces of $X^3_l \times CP_2$ serve as causal determinants at the level of the imbedding space.

3. There is however a conceptual hurdle involved. Suppose that $X^4$ is the absolute minimum associated with $X^3$, and let $Y^3$ be some other $Diff^4$ related 3-surface at $X^4$. One cannot require that the absolute minima $X^4(X^3)$ and $X^4(Y^3)$ are same and one certainly cannot assign an absolute minimum or more general preferred extremal to every 3-surface separately since general coordinate invariance for 3-surfaces would imply that Kähler function is infinitely many-valued. $X^4$ can thus be an absolute minimum only for some preferred 3-surface $X^3(X^4)$ at $X^4$ and the question is what makes this 3-surface preferred.

4. It seems that the internal geometry of $X^4(X^3)$ must be such that it defines uniquely $X^3(X^4)$, or perhaps even more generally, a light like causal determinant $X^3_l \times CP_2$ of $H$ to which $X^3$ belongs. If one requires that the net values of the conserved classical quantities are zero, one could regard $X^4$ as consisting of space-time sheets with opposite time orientation which are created at some moment from vacuum and possibly also disappear to vacuum. If this is the case then the 3-surface at which positive and negative energy space-time sheets are created and begin to evolve as separate branches presents a natural candidate for $X^3$. Also this view is also consistent with quantum holography and supported strongly by number theoretical considerations ($M^4$ has an interpretation as quaternion space with Minkowski metric defined as the real part of $q^2$).

## 1.3 Configuration space metric from symmetries

A complementary approach to the problem of constructing configuration space geometry is based on symmetries. The work of Dan Freed has demonstrated that the Kähler geometry of loop spaces is unique from the existence of Riemann connection and fixed completely by the Kac Moody symmetries of the space. In 3-dimensional context one has even better reasons to expect uniqueness. The guess is that configuration space is a union symmetric spaces labelled by zero modes not appearing in the line element as differentials and having interpretations as classical degrees providing a rigorous formulation of quantum measurement theory. The generalized conformal invariance of metrically 2-dimensional light like 3-surfaces acting as causal determinants is the corner stone of the construction. The construction works only for 4-dimensional space-time and imbedding space which is a product of four-dimensional Minkowski space or its future light cone with $CP_2$.

## 1.4 Is absolute minimization the correct variational principle?

One can criticize the assumption that extremals correspond to the absolute minima of Kähler action. Any other principle allowing to assign to a given 3-surface a unique space-time surface in principle must in principle be considered as a viable alternative. The number theoretical vision discussed in [E2] indeed favors the separate minimization of magnitudes of positive and negative contributions to the Kähler action.

For this option Universe would do its best to save energy, being as near as possible to vacuum. Also vacuum extremals would become physically relevant: note that they would be only inertial vacua and carry non-vanishing density gravitational energy. The non-determinism of the vacuum extremals would have an interpretation in terms of the ability of Universe to engineer itself.

The 3-surfaces for which $CP_2$ projection is at least 2-dimensional and not a Lagrange manifold would correspond to non-vacua since conservation laws do not leave any other option. The variational principle would favor equally magnetic and electric configurations whereas absolute minimization of action based on $S_K$ would favor electric configurations. The positive and negative contributions would be minimized for 4-surfaces in relative homology class since the boundary of $X^4$ defined by the intersections with 7-D light-like causal determinants would be fixed. Without this constraint only vacuum bubbles would result.

The attractiveness of the number theoretical variational principle from the point of calculability of TGD would be that the initial values for the time derivatives of the imbedding space coordinates at $X^3$ at light-like 7-D causal determinant could be computed by requiring that the energy of the solution is minimized. This could mean a computerizable solution to the construction of Kähler function. The number theoretic approach based on the properties of quaternions and octonions discussed in the chapter [E2] leads to a proposal for the general solution of field equations based on the generalization of the notion of calibration [33] providing absolute minima of volume to that of Kähler calibration. This approach will not be discussed in this chapter.

In this chapter I will first consider the basic properties of the configuration space, discuss briefly the various approaches to the geometrization of the configuration space, and introduce the two complementary strategies based on a direct guess of Kähler function and on the group theoretical approach assuming that configuration space can be regarded as a union of symmetric spaces. After these preliminaries a definition of the Kähler function is proposed and various physical and mathematical motivations behind the proposed definition are discussed. The key feature of the Kähler action is classical non-determinism, and various implications of the classical non-determinism are discussed.

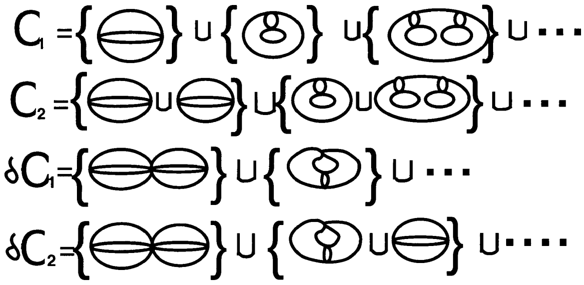


Figure 1: Structure of the configuration space: two-dimensional visualization

# 2 Configuration space

The configuration space of TGD consists of all 3-surfaces of $M^4_+ \times CP_2$ containing sets of
i) surfaces with all possible manifold topologies and arbitrary numbers of components (N-particle sectors)
ii) singular surfaces topologically intermediate between two manifold topologies (see Fig. 2)
We shall use the symbol $C(H)$ to denote the set of 3-surfaces $X^3 \subset H$. It should be emphasized that surfaces related by $Diff^3$ transformations will be regarded as different surfaces in the sequel.

These surfaces form a connected(!) space since it is possible to glue various N-particle sectors to each other along their boundaries consisting of sets of singular surfaces topologically intermediate between corresponding manifold topologies. The connectedness of the configuration space is a necessary prerequisite for the generalization of stringy description of topology changing particle reactions as continuous paths in configuration space (see Fig. 2).

The original view about configuration space geometry was inspired by the stringy generalization of Feynman diagrams. Their space time counterparts would be singular as 4-manifolds but 3-D vertex would represent a singular 3-manifold. During last years however also the direct generalization of the ordinary Feynmann diagram such that lines are replaced with 4-surfaces meeting at their ends has emerged and is favored both by its elegance concerning the treatment of fermion number and physical arguments. For these transitions 4-surfaces would singular but vertices are completely smooth 3-surface. In this case one could say that the transitions between different 3-topologies can occur by a replication of the 3-manifold genuinely quantally. The stringy diagrams would have interpretation as branched paths representing the propagation of a particle along several paths simultaneously as in a double split experiment.

## 2.1 Previous attempts to geometrize configuration space

Concerning the geometrization of the configuration space there is a natural looking strategy. Geometrize first the set of surfaces with single component; the one-particle sector. Furthermore, assume that many particle sectors are more or less equivalent to cartesian products of single particle sectors. This strategy was indeed followed in earlier attempts but it has turned out that this is not quite the correct path to follow.

What made this approach so promising was the observation that any map from a given 3-manifold $X$ to $M^4 \times CP_2$ defines a surface. Maps related by a diffeomorphisms of $X^3$ define

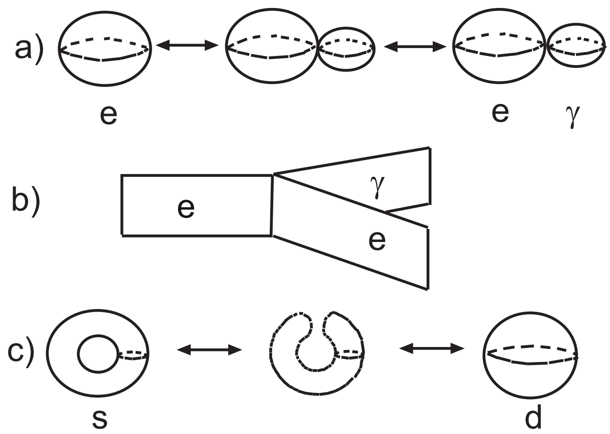


Figure 2: Two-dimensional visualization of topological description of particle reactions. a) Generalization of stringy diagram describing particle decay: 4-surface is smooth manifold and vertex a non-unique singular 3-manifold, b) Topological description of particle decay in terms of a singular 4-manifold but smooth and unique 3-manifold at vertex. c) Topological origin of Cabibbo mixing.

identical surfaces. Thus one can regard the set of the connected 3-surfaces with fixed topology as the space $Map(X^3, H)/Diff^3$, the maps related by diffeomorphism being identified. It soon turned out that the formulation of the theory in the space of 3-surfaces doesn't look simple. Rather, one should use the extended configuration space consisting of the union of the spaces $Map(X^3, H)$ with all possible 3-topologies.

In the mapping space formulation the theory would have looked roughly like following.

Theory is free field theory in the union of the spaces $Map(X^3, H)$ ($X^3$ can have any topology) endowed with Diff$^3$ invariant geometry. Diff$^3$ might act as the isometries of the geometry as has been assumed in earlier attempts to construct geometry of $Map$. The second possibility is that metric is also Diff3 degenerate so that for Diff$^3$ generators interpreted as vector fields in configuration space have zero norm and metrically $Map$ reduces effectively to the space of surfaces. Present approach is based on this alternative. One good reason for this approach is that it gives good hopes to realize Diff3 invariance since the vector fields generating Diff3 transformations correspond to zero norm vector fields already at the level of configuration space geometry. The physically acceptable field configurations are Diff3 invariant and in the case of scalar field they could be regarded as field configurations in $Map(X^3, H)/Diff^3$. A weaker form of Diff3 invariance is based on the requirement that states created by infinitesimal Diff3 generators are zero norm states.

What made the mapping space formulation so attractive at first glance was that in the case of $M^4 \times CP_2$ one can regard the mapping space as a generalization of a coset space $G/H$ of two finite dimensional groups to a local coset space that is a coset space formed by dividing the local gauge group defined by product of $M^4$ and color $SU(3)$ with local gauge group $SU(2) \times U(1)$.

$$\begin{aligned} Map(X^3, M^4 \times CP_2) &= G/H \ , \\ G &= Map(X^3, M^4 \times SU(3)) \ , \\ H &= Map(X^3, SU(2) \times U(1)) \ . \end{aligned} \quad (3)$$

That the representability as a local coset space is indeed very nice feature should become clear from the following arguments.

1. If the finite dimensional coset space concept, or rather the concept of a symmetric space with Kähler structure [17], generalizes then we can expect that metric is invariant under a huge symmetry group: besides diffeomorphisms the group of the local gauge transformations defined by $M^4 \times SU(3)$ act as isometries. Symmetry group might be even larger: one can represent $Map$ also as the coset space of $P \times SU(3)$ divided by $SO(3,1) \times SU(2) \times U(1)$, where $P$ denotes Poincare group. The physical interpretation of the isometry group as the symmetry group of color-gravitational interactions would be natural.

2. The implications of the infinite parameter group of isometries of local coset space structure for the calculability of the theory are expected to be decisive. In finite-dimensional case all the geometrical quantities are left invariant: for example curvature tensor is just covariantly constant. Ricci tensor is proportional to the metric tensor, and so on. In fact, curvature properties of the finite dimensional symmetric spaces are determined purely Lie-algebraically [17]. In present case isometry invariance would mean that one can calculate the whole geometry by restricting the consideration to a single suitably chosen 3-surface! Isometry invariance is expected also to make it possible to solve d'Alembert type free field equations using group theoretical methods.

3. Physical states cannot certainly correspond to representations of the local gauge group $Map(X^3, M^4 \times SU(3))$ since this would lead to catastrophic results concerning the spectrum of, say Dirac operator. If local gauge transformations act as isometries, one obtains an infinite degeneracy for the physical states with given mass. Rather, some Abelian extension of the local gauge group, to be called Kac-Moody group in sequel , should play the role of the spectrum generating group of the field equations as in the case of string models. One possibility to achieve this situation is based on projective representations: true representations of the centrally extended group correspond to the projective representations of $Map$.

There are however grave objections against this kind of scenario.

1. $Map(X^3, M^4 \times SU(3))$ seems in some respects mathematically awkward. First, Lie-algebra generators are not $Diff^3$ invariant so that the realization of $Diff^3$ invariance is expected to be problematic. Secondly, the globalization of gauge symmetry and the choice of the scalar function basis of $X^3$ (generators are products of scalar function basis with the isometry generators of $H$) associated with the algebra are difficult mathematical problems since all 3-topologies are possible. Also it is difficult to understand how one could continue $Map(X^3, H)$ to a global symmetry of the configuration space: the gauge invariance property of the metric is expected to be broken in the sense that different 3-topologies are not related by left invariance.

2. From the case of loop groups [18] it is known that group theory doesn't determine curvature properties uniquely in infinite dimensional case so that all nice properties of the finite dimensional case do not generalize. Loop groups with Kähler metric are however Einstein spaces so that Ricci tensor is indeed covariantly constant quantity and this property might generalize as such. In fact it turns out that Ricci flatness is necessary for the divergence free field theory so that configuration space metric must be vacuum solution of Einstein's equations.

3. The existence of the Abelian extension seems however to be in contradiction with certain no-go theorems about Abelian extensions [20].
i) Abelian extensions of the gauge group $Map(X^3, G)$ (necessary for state construction) are $U(1)$ extensions (central extensions) provided the dimension of the space is smaller than

three(!).
ii) In higher dimensions the group appearing in the extension is infinite parameter Abelian group.
iii) There are indications that no unitary faithful representations (satisfying certain natural physical constraints) for these groups exist [20].

These no-go theorems suggest that $Map(X^3, M^4 \times SU(3))$ is perhaps not the actual isometry group: since Kähler structure is necessary for the existence of the symplectic extension it might happen that isometric action of $Map(X^3, M^4 \times SU(3))$ is not consistent with the existence of the Kähler structure.

4. There are also objections against the reduction of the geometrization to single particle level.

   i) The procedure leads to difficulties with spin and statistics. If N-particle sector is essentially cartesian product of single-particle sectors, the spinors of N-particle sector are tensor products of spinors of 1-particle sectors. Since this holds true also in center of mass degrees of freedom one obtains fermions with integer spin! The correct definition of the metric should avoid this difficulty.

   ii) It would be highly desirable to include the description of interactions into the metric of the configuration space in accordance with the basic ideas of General Theory of Relativity: direct sum metric is however trivial in this respect.

These are not the only difficulties of our previous attempts. The construction of the metric as a naive generalization of loop space metric [18] posing various symmetry requirements leads to a metric, which treats 3-surfaces as essentially one-dimensional objects. Therefore the increase of the dimension from $d = 1$ to $d = 3$ seems to necessitate a completely new approach.

## 2.2 Constraints on the configuration space geometry

The constraints on the configuration space geometry result both from the infinite dimension of the configuration space and from physically motivated symmetry requirements. There are three basic physical requirements on the configuration space geometry: namely four-dimensional general coordinate invariance, Kähler property and the decomposition of configuration space into a union $\cup_i G/H_i$ of symmetric spaces $G/H_i$, each coset space allowing $G$-invariant metric such that $G$ is subgroup of some 'universal group' having natural action on 3-surfaces. Together with the infinite dimensionality of the configuration space these requirements pose extremely strong constraints on the configuration space geometry. In the following we shall consider these requirements in more detail.

### 2.2.1 Diff$^4$ invariance and Diff$^4$ degeneracy

Diff$^4$ plays fundamental role as the gauge group of General Relativity. In string models $Diff^2$ invariance ($Diff^2$ acts on the orbit of the string) plays central role in making possible the elimination of the time like and longitudinal vibrational degrees of freedom of string. Also in the present case the elimination of the tachyons (time like oscillatory modes of 3-surface) is a physical necessity and Diff$^4$ invariance provides an obvious manner to do the job.

In the standard functional integral formulation the realization of Diff$^4$ invariance is an easy task at the formal level. The problem is however that functional integral over four-surfaces is plagued by divergences and doesn't make sense. In the present case the configuration space consists of 3-surfaces and only $Diff^3$ emerges automatically as the group of re-parameterizations of 3-surface. Obviously one should somehow define the action of Diff$^4$ in the space of 3-surfaces. Whatever the action of Diff$^4$ is it must leave the configuration space metric invariant. Furthermore, the elimination of tachyons is expected to be possible only provided the time like deformations of the 3-surface correspond to zero norm vector fields of the configuration space so that 3-surface and its $\mathrm{Diff}^4$ image have zero distance. The conclusion is that configuration space metric should be both $\mathrm{Diff}^4$ invariant and $\mathrm{Diff}^4$ degenerate.

The problem is how to define the action of $\mathrm{Diff}^4$ in $C(H)$. Obviously the only manner to achieve $\mathrm{Diff}^4$ invariance is to require that the very definition of the configuration space metric somehow associates a unique space time surface to a given 3-surface for $\mathrm{Diff}^4$ to act on! The obvious physical interpretation of this space time surface is as "classical space time" so that "Classical Physics" would be contained in configuration space geometry. It is this requirement, which has turned out to be decisive concerning the understanding of the configuration space geometry. Amusingly enough, the historical development was not this: the definition of $\mathrm{Diff}^4$ degenerate Kähler metric was found by a guess and only later it was realized that $\mathrm{Diff}^4$ invariance and degeneracy could have been postulated from beginning!

#### 2.2.2 Decomposition of the configuration space into a union of symmetric spaces $G/H$

The extremely beautiful theory of finite-dimensional symmetric spaces constructed by Elie Cartansuggests that configuration space should possess decomposition into a union of coset spaces $CH = \cup_i G/H_i$ such that the metric inside each coset space $G/H_i$ is left invariant under the infinite dimensional isometry group $G$. The metric equivalence of surfaces inside each coset space $G/H_i$ does not mean that 3-surfaces inside $G/H_i$ are physically equivalent. The reason is that the vacuum functional is exponent of Kähler action which is not isometry invariant so that the 3-surfaces, which correspond to maxima of Kähler function for a given orbit, are in a preferred position physically. For instance, one can calculate functional integral around this maximum perturbatively. The sum of over $i$ means actually integration over the zero modes of the metric (zero modes correspond to coordinates not appearing as coordinate differentials in the metric tensor).

The coset space $G/H$ is a symmetric space only under very special Lie-algebraic conditions. Denoting the decomposition of the Lie-algebra $g$ of $G$ to the direct sum of $H$ Lie-algebra $h$ and its complement $t$ by $g = h \oplus t$, one has

$$[h,h] \subset h \ , \quad [h,t] \subset t \ , \quad [t,t] \subset h \ .$$

This decomposition turn out to play crucial role in guaranteing that $G$ indeed acts as isometries and that the metric is Ricci flat.

The four-dimensional $Diff$ invariance indeed suggests to a beautiful solution of the problem of identifying $G$. The point is that any 3-surface $X^3$ is $Diff^4$ equivalent to the intersection of $X^4(X^3)$ with the light cone boundary. This in turn implies that 3-surfaces in the space $\delta H = \delta M^4_+ \times CP_2$ should be all what is needed to construct configuration space geometry. The group $G$ can be identified as some subgroup of diffeomorphisms of $\delta H$ and $H_i$ contains that subgroup of $G$, which acts as diffeomorphisms of the 3-surface $X^3$. Since $G$ preserves topology, configuration space must decompose into union $\cup_i G/H_i$, where $i$ labels 3-topologies and various zero modes of the metric. For instance, the elements of the Lie-algebra of $G$ invariant under configuration space complexification correspond to zero modes.

The reduction to the light cone boundary, identifiable as the moment of big bang, looks perhaps odd at first. In fact, it turns out that the classical non-determinism of Kähler action does not allow the complete reduction to the light cone boundary: physically this is a highly desirable implication but means a considerable mathematical challenge.

### 2.2.3 Kähler property

Kähler property implies that the tangent space of the configuration space allows complexification and that there exists a covariantly constant two-form $J_{kl}$, which can be regarded as a representation of the imaginary unit in the tangent space of the configuration space:

$$J_k{}^r J_{rl} = -G_{kl} \ . \tag{4}$$

There are several physical and mathematical reasons suggesting that configuration space metric should possess Kähler property in some generalized sense.

1. Kähler property turns out to be a necessary prerequisite for defining divergence free configuration space integration. We will leave the demonstration of this fact later although the argument as such is completely general.

2. Kähler property very probably implies an infinite-dimensional isometry group. The study of the loop groups $Map(S^1, G)$ [18] shows that loop group allows only single Kähler metric with well defined Riemann connection and this metric allows local G as its isometries!

   To see this consider the construction of Riemannian connection for $Map(X^3, H)$. The defining formula for the connection is given by the expression

$$\begin{aligned} 2(\nabla_X Y, Z) &= X(Y,Z) + Y(Z,X) - Z(X,Y) \\ &+ ([X,Y],Z) + ([Z,X],Y) - ([Y,Z],X) \end{aligned} \tag{5}$$

   $X, Y, Z$ are smooth vector fields in $Map(X^3, G)$. This formula defines $\nabla_X Y$ uniquely provided the tangent space of $Map$ is complete with respect to Riemann metric. In the finite-dimensional case completeness means that the inverse of the covariant metric tensor exists so that one can solve the components of connection from the conditions stating the covariant constancy of the metric. In the case of the loop spaces with Kähler metric this is however not the case.

   Now the symmetry comes into the game: if $X, Y, Z$ are left (local gauge) invariant vector fields defined by the Lie-algebra of local $G$ then the first three terms drop away since the scalar products of left invariant vector fields are constants. The expression for the covariant derivative is given by

$$\nabla_X Y = (Ad_X Y - Ad_X^* Y - Ad_Y^* X)/2 \tag{6}$$

   where $Ad_X^*$ is the adjoint of $Ad_X$ with respect to the metric of the loop space.

   At this point it is important to realize that Freed's argument does not force the isometry group of the configuration space to be $Map(X^3, M^4 \times SU(3))$! Any symmetry group, whose Lie algebra is complete with respect to the configuration space metric ( in the sense that any tangent space vector is expressible as superposition of isometry generators modulo a zero norm tangent vector) is an acceptable alternative.

   The Kähler property of the metric is quite essential in one-dimensional case in that it leads to the requirement of left invariance as a mathematical consistency condition and we expect that dimension three makes no exception in this respect. In 3-dimensional case the degeneracy of the metric turns out to be even larger than in 1-dimensional case due to the four-dimensional

Diff degeneracy. So we expect that the metric ought to possess some infinite-dimensional isometry group and that the above formula generalizes also to the 3-dimensional case and to the case of local coset space. Note that in $M^4$ degrees of freedom $Map(X^3, M^4)$ invariance would imply the flatness of the metric in $M^4$ degrees of freedom.

The physical implications of the above purely mathematical conjecture should not be underestimated. For example, one natural looking manner to construct physical theory would be based on the idea that configuration space geometry is dynamical and this approach is followed in the attempts to construct string theories [19]. Various physical considerations (in particular the need to obtain oscillator operator algebra) seem to imply that configuration space geometry is necessarily Kähler. The above result however states that configuration space Kähler geometry cannot be dynamical quantity and is dictated solely by the requirement of internal consistency. This result is extremely nice since it has been already found that the definition of the configuration space metric must somehow associate a unique classical space time and "classical physics" to a given 3-surface: uniqueness of the geometry implies the uniqueness of the "classical physics".

3. The choice of the imbedding space becomes highly unique. In fact, the requirement that configuration space is not only symmetric space but also (contact) Kähler manifold inheriting its (degenerate) Kähler structure from the imbedding space suggests that spaces, which are products of four-dimensional Minkowski space with complex projective spaces $CP_n$, are perhaps the only possible candidates for $H$. The reason for the unique position of the four-dimensional Minkowski space turns out to be that the boundary of the light cone of D-dimensional Minkowski space is metrically a sphere $S^{D-2}$ despite its topological dimension $D-1$: for $D=4$ one obtains two-sphere allowing Kähler structure and infinite parameter group of conformal symmetries!

4. It seems possible to understand the basic mathematical structures appearing in string model in terms of the Kähler geometry rather nicely.

   i) The projective representations of the infinite-dimensional isometry group (not necessarily Map!) correspond to the ordinary representations of the corresponding centrally extended group [23]. The representations of Kac Moody group indeed play central role in string models [24, 25] and configuration space approach would explain their occurrence, not as a result of some quantization procedure, but as a consequence of symmetry of the underlying geometric structure.

   ii) The bosonic oscillator operators of string models would correspond to centrally extended Lie-algebra generators of the isometry group acting on spinor fields of the configuration space.

   iii) The "fermionic" fields ( Ramond fields, [24, 25]) should correspond to gamma matrices of the configuration space. Fermionic oscillator operators would correspond simply to contractions of isometry generators $j^k_A$ with complexified gamma matrices of configuration space

$$\begin{aligned} \Gamma^{\pm}_A &= j^k_A \Gamma^{\pm}_k \\ \Gamma^{\pm}_k &= (\Gamma^k \pm J^k_{\ l}\Gamma^l)/\sqrt{2} \end{aligned} \tag{7}$$

   ($J^k_{\ l}$ is the Kähler form of the configuration space) and would create various spin excitations of the configuration space spinor field. $\Gamma^{\pm}_k$ are the complexified gamma matrices, complexification made possible by the Kähler structure of the configuration space.

This suggests that some generalization of the so called Super Kac Moody algebra of string models [24, 25] should be regarded as a spectrum generating algebra for the solutions of field equations in configuration space.

Although the Kähler structure seems to be physically well motivated there is a rather heavy counter argument against the whole idea. Kähler structure necessitates complex structure in the tangent space of the configuration space. In $CP_2$ degrees of freedom no obvious problems of principle are expected: configuration space should inherit in some sense the complex structure of $CP_2$.

In Minkowski degrees of freedom the signature of the Minkowski metric seems to pose a serious obstacle for complexification: somehow one should get rid of two degrees of freedom so that only two Euclidian degrees of freedom remain. An analogous difficulty is encountered in quantum field theories: only two of the four possible polarizations of gauge boson correspond to physical degrees of freedom: mathematically the wrong polarizations correspond to zero norm states and transverse states span a complex Hilbert space with Euclidian metric. Also in string model analogous situation occurs: in case of $D$-dimensional Minkowski space only $D-2$ transversal degrees of freedom are physical. The solution to the problem seems therefore obvious: configuration space metric must be degenerate so that each vibrational mode spans effectively a 2-dimensional Euclidian plane allowing complexification.

We shall find that the definition of Kähler function to be proposed indeed provides a solution to this problem and also to the problems listed before.

1. The definition of the metric doesn't differentiate between 1- and N-particle sectors, avoids spin statistics difficulty and has the physically appealing property that one can associate to each 3-surface a unique classical space time: classical physics is described by the geometry of the configuration space! And the geometry of the configuration space is determined uniquely by the requirement of mathematical consistency!

2. Complexification is possible only provided the dimension of the Minkowski space equals to four(!).

3. It is possible to identify a unique candidate for the necessary infinite-dimensional isometry group $G$. $G$ is subgroup of the diffeomorphism group of $\delta M^4_+ \times CP_2$. Essential role is played by the fact that the boundary of the four-dimensional light cone, which, despite being topologically 3-dimensional, is metrically two-dimensional(!) Euclidian sphere, and therefore allows infinite-parameter groups of isometries as well as conformal and canonical symmetries and also Kähler structure unlike the higher-dimensional light cone boundaries. Therefore configuration space metric is Kähler only in the case of four-dimensional Minkowski space and allows symplectic $U(1)$ central extension without conflict with the no-go theorems about higher dimensional central extensions.

   The study of the vacuum degeneracy of Kähler function defined by Kähler action forces to conclude that the isometry group must consist of the canonical transformations of $\delta H = \delta M^4_+ \times CP_2$. The corresponding Lie algebra can be regarded as a loop algebra associated with the canonical group of $S^2 \times CP_2$, where $S^2$ is $r_M = constant$ sphere of light cone boundary. Thus the finite-dimensional group $G$ defining loop group in case of string models extends to an infinite-dimensional group in TGD context. This group is a real monster! The radial Virasoro localized with respect to $S^2 \times CP_2$ defines naturally complexification for both $G$ and $H$. The general form of the Kähler metric deduced on basis of this symmetry has same qualitative properties as that deduced from Kähler function identified as the absolute minimum of Kähler action. Also the zero modes, among them isometry invariants, can be identified.

4. The construction of the configuration space spinor structure is based on the identification of the configuration space gamma matrices as linear superpositions of the oscillator operators associated with the second quantized induced spinor fields. The extension of the canonical invariance to super canonical invariance fixes the anti-commutation relations of the induced spinor fields, and configuration space gamma matrices correspond directly to the super generators. Physics as number theory vision suggests strongly that configuration space geometry exists for 8-dimensional imbedding space only and that the choice $M^4_+ \times CP_2$ for the imbedding space is the only possible one.

# 3 Identification of the Kähler function

There are two approaches to the construction of the configuration space geometry: a direct physics based guess of the Kähler function and a group theoretic approach based on the hypothesis that $CH$ can be regarded as a union of symmetric spaces. The rest of this chapter is devoted to the first approach.

## 3.1 Definition of Kähler function

Let $X^3$ be a given 3-surface and let $X^4$ be any four-surface containing $X^3$ as a sub-manifold: $X^4 \supset X^3$. The 4-surface $X^4$ possesses in general boundary. If the 3-surface $X^3$ has nonempty boundary $\delta X^3$ then the boundary of $X^3$ belongs to the boundary of $X^4$: $\delta X^3 \subset \delta X^4$.

The projection of $CP_2$ Kähler form $J$ (induced Kähler form) defines Maxwell field on $X^4$. One can associate to Kähler form Maxwell action and also Chern-Simons anomaly term proportional to $\int_{X^4} J \wedge J$ in well known manner. Chern Simons term is purely topological term and well defined for orientable 4-manifolds, only. Since there is no deep reason for excluding non-orientable space-time surfaces it seems reasonable to drop Chern Simons term from consideration. Therefore Kähler action $S_K(X^4)$ can be defined as

$$S_K(X^4) \quad = \quad k_1 \int_{X^4;X^3 \subset X^4} J \wedge (*J) \ . \tag{8}$$

The sign of the square root of the metric determinant, appearing implicitly in the formula, is defined in such a manner that the action density is negative for the Euclidian signature of the induced metric and such that for a Minkowskian signature of the induced metric Kähler electric field gives a negative contribution to the action density.

The notational convention

$$k_1 \quad \equiv \quad \frac{1}{16\pi\alpha_K} \ , \tag{9}$$

where $\alpha_K$ will be referred as Kähler coupling strength will be used in the sequel. If the preferred extremals minimize/maximize [E2] the absolute value of the action in each region where action density has a definite sign, the value of $\alpha_K$ can depend on space-time sheet.

One can define the Kähler function in the following manner. Consider first the case $H = M^4_+ \times CP_2$ and neglect for a moment the non-determinism of Kähler action. Let $X^3$ be a 3-surface at the light-cone boundary $\delta M^4_+ \times CP_2$. Define the value $K(X^3)$ of Kähler function $K$ as the value of the Kähler action for some preferred extremal in the set of four-surfaces containing $X^3$ as a sub-manifold:

$$K(X^3) \quad = \quad Min\{S_K(X^4)\}_{|X^3 \subset X^4} \ . \tag{10}$$

The original hypothesis was that the intersections of the four-surface with the boundary of the light cone ($\delta M^4_+ \times CP_2$) defined by the condition $a = \sqrt{(m^0)^2 - r_M^2} = 0$ and with the surface $a \rightarrow \infty$ are not subject to variational conditions since this would have meant that all universes have vanishing classical conserved quantities. Define the value $K(Y^3)$ of Kähler function for all Diff$^4$ related 3-surfaces at $X^4(X^3)$ as $K(X^3)$ so that the metric is $Diff^4$ degenerate.

Absolute minimization of Kähler action was the first identification for the principle selecting the preferred extremal. The worst that can happen for this option is that the value of Kähler action is negative and infinite for the entire Universe so that the vacuum functional defined by its exponent vanishes. A more plausible choice of the preferred extremal is based on the assumption that the absolute values of the contributions to Kähler action are separately minimized in regions of definite sign for Kähler action density. This implies the minimization of the absolute value of the net action and extremals are as near as possible to vacuum extremals, and minimize their energy: this gives hopes of constructing the extremals using only data at $X^3$. I ended up to this option from number theoretical vision, which also leads to an explicit proposal for how to construct these extremals of Kähler action [E2].

This simple picture is too simple to be true and must be generalized even in case of $M^4_+$. The reason is the non-determinism of Kähler action implying that besides $\delta M^4_+ \times CP_2$ all light like 7-surfaces $X^3_l \times CP_2$, where $X^3_l$ is light like surface of $M^4_+$ suggest themselves act as causal determinants. The physical reason is that pairs of space-time sheets having opposite time orientation and opposite energies can be created from vacuum at these 7-surfaces.

Even worse (or better), it has however become clear that the gigantic symmetries associated with $\delta M^4_+ \times CP_2$ are also symmetries at the laboratory scale and directly visible at the level of propagators and vertices [C5]. Furthermore, $M^4$ is as a good option as $M^4_+$, and number theoretically even better since it allows interpretation as the space of quaternions with Minkowski metric defined by the imaginary part of $q^2$. Also exact Poincare invariance favors $M^4$ option.

$M^4$ option makes sense only if $X^3$ is selected uniquely by the internal geometry of $X^4$. The possibility of negative Poincare energies inspires the hypothesis that the total quantum numbers and classical conserved quantities of the Universe vanish. By crossing symmetry this view is consistent with elementary particle physics. Consistency with macroscopic physics can be achieved if gravitational energy is defined as the difference of Poincare energies of positive and negative energy matter. This definition indeed resolves the long lasting puzzle created by the fact that Robertson-Walker cosmologies correspond to vacuum extremals with respect to inertial energy and momentum. Space-time surfaces consists of pairs of positive and negative energy space-time sheets created at some moment from vacuum and branching at that moment to separate space-time sheets. This allows to select $X^3$ uniquely and define $X^4(X^3)$ as the absolute minimum of Kähler action. Also a natural fixing of Diff$^4$ gauge becomes possible. This view is also consistent with the non-determinism of Kähler action. This option works for both $M^4_+$ and $M^4$ and is very probably the correct one.

## 3.2 Minkowski space or its future light cone?

The basic question is whether one should choose the imbedding space to be $M^4 \times CP_2$ or $M^4_+ \times CP_2$.

$M^4_+$ option has several nice features.

1. Since future light cone corresponds to vacuum cosmology (cosmic time is Lorentz invariant distance) the latter choice seems to be more physical since it makes big bang cosmology a geometrical necessity and implies the arrow of time naturally. The loss of exact Poincare invariance could be seen as a problem. Even if one accepts light cone alternative as the correct one (as we shall cautiously do) there are two alternative definitions of the Kähler function.

2. For $M^4_+$ option minimizing four-surfaces belong to the future light cone so that the presence of the light cone boundary reflects itself in the properties of minimizing four-surfaces: big bang cosmology is expected to manifest itself in the time development of four-surfaces. This alternative implies the loss of Poincare invariance in cosmological scales: in the laboratory scale Poincare invariance is of course practically exact since Poincare invariance is a symmetry of the extremals of Kähler action and broken only in the set of absolute minima.

3. One could avoid the loss of Poincare invariance without totally giving up the light cone cosmology by defining the metric of $C(M^4_+ \times CP_2)$ as the restriction of the metric of $C(M^4 \times CP_2)$: minimizing four-surfaces would belong to $M^4$ although 3-surfaces belong to light cone. Poincare invariance becomes exact symmetry at the Lie algebra level broken only "kinematically". One can however heavily criticize this alternative: if one wants to interpret four-surface as an actual space-time then it is highly artificial to allow four-surfaces, which do not belong to the actual imbedding space. A second questionable feature is that the presence of the light cone boundary does not reflect itself in the properties of 4-surfaces as it should.

$M^4$ option makes many highly non-trivial and nice predictions which are allowed but not predicted by $M^4_+$ option. The mathematical elegance of $M^4$ option is definitely superior to that of $M^4_+$ alternative.

1. Suppose that the classical non-determinism of Kähler action indeed implies that all light like 7-surfaces $X^3_l \times CP_2$, where $X^3_l$ is light like surface of $M^4_+$, can act as causal determinants. As already noticed, this makes sense if pairs of space-time sheets having opposite time orientation and opposite energies can be created from vacuum at these 7-surfaces.

2. For $M^4$ option the total energy of classical and by quantum-classical correspondence of also quantum universes must vanish and all matter would be created from vacuum. There would be no need to ponder the academic but very nasty question about total fermion numbers of the universe: all states of the universe would be vacua as far net quantum numbers are considered. Of course, also in the case of $M^4_+$ it is possible and natural to postulate that nothing flows out from the future light cone or into it and this would imply vanishing total quantum numbers.

3. $M^4$ option allows both maximal space-time symmetries and forces the fractal hierarchy of cosmologies inside cosmologies defined by light cones inside light cones as does in fact also $M^4_+$ option. These cosmologies would be a result of dynamics rather than of the properties of the imbedding space. If the separation of positive and negative energy densities can be achieved in cosmological length scales, this option might work. The nice feature is that configuration space becomes a union of configuration spaces associated with various light-like causal determinants $X^3_l \times CP_2$ with the most plausible identification of $X^3_l$ being as a union of future and past directed light cone boundaries.

4. Poincare transformations act as symmetries and one can assign to given space-time sheet unique value of geometric time as the moment of geometric time when it was created. This is of utmost importance concerning the understanding of the relationship between subjective and geometric time in TGD inspired theory of consciousness. It makes also possible to assign to S-matrix time parameter identifiable as interaction time without problems with energy conservation.

5. For $M^4$ option the super conformal invariance associated with light like 3-surfaces $X^3_l \times CP_2$ and super-conformal invariance associated with 3-dimensional light-like boundaries and "elementary particle" horizons of space-time surfaces interact very naturally. The super conformal invariance associated with 3-dimensional light-like surfaces corresponds to the

Super Kac Moody symmetries of string models with Poincare symmetry being exact, and determines mass squared formula. The super-canonical invariance associated with $X_l^3 \times CP_2$ is something new and it modifies that the stringy mass formula. The interaction of super Kac-Moody conformal algebra in super-canonical algebra is of special significance in the construction of quantum theory.

6. $M^4$ can be interpreted as the space of quaternions with Minkowski metric identifiable as the imaginary part of $q^2$. The imbedding space can be interpreted as a space having hyper-octonionic tangent space structure [E2], and space-time surfaces as maximal associative sub-manifolds with hyper-quaternionic tangent space structure. Furthermore, the fact that $CP_2$ parameterizes hyper-quaternionic planes of hyper-octonion space, raises $M^4 \times CP_2$ in a completely unique position number theoretically.

Which of this alternatives is correct? At the practical laboratory level there are no testable differences between these options and it is very difficult to test whether the first moments of our cosmology are associated with a cosmology inside cosmology or $M^4_+$. One can however say that whereas $M^4_+$ option allows what seems to be the correct interpretation, $M^4$ option forces it, and its mathematical elegances is superior. For a long time I nearly-believed that $M^4_+$ alternative is the correct one but after a long period of certainty I feel more and more empathy towards $M^4$ option.

## 3.3 The values of the Kähler coupling strength?

Since the vacuum functional of the theory turns out to be essentially the exponent $exp(K)$ of the Kähler function, the dynamics depends on the normalization of the Kähler function. Since the Theory of Everything should be unique it would be highly desirable to find arguments fixing the normalization or equivalently the possible values of the Kähler coupling strength $\alpha_K$. Also a discrete spectrum of values is acceptable.

The quantization of Kähler form could result in the following manner. It will be found that Abelian extension of the isometry group results by coupling spinors of the configuration space to a multiple of Kähler potential. This means that Kähler potential plays role of gauge connection so that Kähler form must be integer valued by Dirac quantization condition for magnetic charge. So, if Kähler form is co-homologically nontrivial it is quantized.

Unfortunately, the exact definition of renormalization group concept is not at all obvious. There is however a much more general but more or less equivalent manner to formulate the condition fixing the value of $\alpha_K$. Vacuum functional $exp(K)$ is analogous to the exponent $exp(-H/T)$ appearing in the definition of the partition function of a statistical system and S-matrix elements and other interesting physical quantities are integrals of type $\langle O \rangle = \int exp(K) O \sqrt{G} dV$ and therefore analogous to the thermal averages of various observables. $\alpha_K$ is completely analogous to temperature. The critical points of a statistical system correspond to critical temperatures $T_c$ for which the partition function is nonanalytic function of $T - T_c$ and according RGE hypothesis critical systems correspond to fixed points of renormalization group evolution. Therefore, a mathematically more precise manner to fix the value of $\alpha_K$ is to require that some integrals of type $\langle O \rangle$ (not necessary S-matrix elements) become nonanalytic at $1/\alpha_K - 1/\alpha_K^c$.

This analogy suggests also a physical motivation for the unique value or value spectrum of $\alpha_K$. Below the critical temperature critical systems suffer something analogous to spontaneous magnetization. At the critical point critical systems are characterized by long range correlations and arbitrarily large volumes of magnetized and non-magnetized phases are present. Spontaneous magnetization might correspond to the generation of Kähler magnetic fields: the most probable 3-surfaces are Kähler magnetized for subcritical values of $\alpha_K$. At the critical values of $\alpha_K$ the most probable 3-surfaces contain regions dominated by either Kähler electric and or Kähler magnetic fields: by the compactness of $CP_2$ these regions have in general outer boundaries.

This suggests that 3-space has hierarchical, fractal like structure: 3-surfaces with all sizes (and with outer boundaries) are possible and they have suffered topological condensation on each other. Therefore the critical value of $\alpha_K$ allows the richest possible topological structure for the most probable 3-space. In fact, this hierarchical structure is in accordance with the basic ideas about renormalization group invariance. This hypothesis has highly nontrivial consequences even at the level of ordinary condensed matter physics.

The assumption about single critical value of $\alpha_K$ is probably too strong. p-Adic length scale hierarchy together with the immense vacuum degeneracy of the Kähler action leads to the hypothesis that different p-adic length scales correspond to different critical values of $\alpha_K$, and that ordinary coupling constant evolution is replaced by a piecewise constant evolution induced by that for $\alpha_K$. One implication is the vanishing of the loop corrections and thus the absence of loop divergences in the perturbation theory. This point is discussed in [C5].

Renormalization group invariance is closely related with criticality. The self duality of the Kähler form and Weyl tensor of $CP_2$ indeed suggest RG invariance. The point is that in $N = 1$ super-symmetric field theories duality transformation relates the strong coupling limit for ordinary particles with the weak coupling limit for magnetic monopoles and vice versa. If the theory is self dual these limits must be identical so that action and coupling strength must be RG invariant quantities. The geometric realization of the duality transformation is easy to guess in the standard complex coordinates $\xi_1, \xi_2$ of $CP_2$ (see Appendix of the book). In these coordinates the metric and Kähler form are invariant under the permutation $\xi_1 \leftrightarrow \xi_2$ having Jacobian $-1$.

Consistency requires that particles of the theory are equivalent with magnetic monopoles: the so called $CP_2$ type extremals identified as elementary particles are isometric imbeddings of $CP_2$ and can be regarded as monopoles. The magnetic flux however flows in internal degrees of freedom (possible by nontrivial homology of $CP_2$) so that no long range $1/r^2$ magnetic field is created. The magnetic contribution to Kähler action is positive and this suggests that ordinary magnetic monopoles are not stable, since they do not minimize Kähler action: a cautious conclusion in accordance with the experimental evidence is that TGD does not predict magnetic monopoles. It must be emphasized that the prediction of monopoles of practically all gauge theories and string theories and follows from the existence of a conserved electromagnetic charge.

# 4 Questions

A good manner to get grasp about the properties of Kähler function is through what might be called frequently asked questions.

## 4.1 Absolute minimization or something else?

The requirement that the 4-surface having given 3-surface as its sub-manifold is absolute minimum of the Kähler action is the most obvious guess for the principle selecting the preferred extremals and has been taken as a working hypothesis for about one and half decades. The principle admittedly looks somewhat ad hoc, and quite recently (I am writing this in the beginning of 2005) it turned out that that absolute minimization principle should be perhaps relaxed in the sense that the absolute values of the contributions to the net Kähler action coming from regions where the action density has definite sign [E2] are separately minimized (or maximed in dual case). This would allow $\alpha_K$ to depend on space-time sheet and allow to understand p-adic evolution of $\alpha_K$. Therefore a critical discussion is in order.

### 4.1.1 Consequences of the absolute minimum property

Absolute minimum property, or any more refined manner to select preferred extremal of Kähler action, has several nontrivial consequences. In fact these consequences are shared also by the modified variational principle.

1. The so called classical theory becomes an essential part of the configuration space geometry and quantum theory. One can associate with a given 3-surface a unique space-time, which might be interpreted as the classical space time associated with the quantum state which is completely localized to a given four-surface. In particular, one can associate definite conserved quantities with 3-surface $X^4$: this however implies that the minimizing four-surface has infinite size as will be found later.

2. Minimization requirement implies that classical space time can be regarded as a generalized Bohr orbit. The point is that the initial value problem associated with the minimization of Kähler action differs from the standard initial value problems of the classical field theories. In the ordinary initial value problem both the values $h^k(x)$ of $H$ coordinates and their time derivatives $\partial_t h^k(x)$ could be fixed arbitrarily as functions of $X^3$ coordinates. Now however the minimizing 4-surface is unique and the values of time derivatives are fixed by the minimization condition. This in turn implies something analogous to the quantization of canonical momenta.

   Therefore the definition of Kähler function seems to catch some quite essential features of quantum theory and seems to point out that Bohr rules are not merely a by product of WKB approximation but exact part of quantum theory. An interesting question is whether these quantization conditions could explain classically the quantization of, say, electric charge and mass.

3. Minimization requirement implies four-dimensional Diff degeneracy of the configuration space metric with all its deep consequences to be discussed shortly.

4. The minimization requirement together with the assumption that vacuum functional is of the form $exp(K)$ implies that theory is well defined in the limit of the infinite system. The stability of the theory results from two opposing tendencies ("Yin" and "Yang"!). Vacuum functional favors large action but minimization principle tends to make action small, in fact negative since Kähler action is not positive definite. Therefore, action is expected to be negative for most configurations. In the limit of infinite system this means that vacuum functional is non-vanishing only for those systems, which have vanishing average action per volume in sufficiently large length scales. This has highly nontrivial cosmological consequences. For obvious reasons we shall refer this stabilization mechanism as "Ying-Yang" principle in the sequel.

5. Absolute minimum property implies the existence of an additional symmetry, not necessarily identifiable as isometries of the configuration space metric. If four-surfaces correspond to sub-manifolds of the light cone (rather than whole Minkowski space) this symmetry leaves Kähler function invariant and therefore implies additional degeneracy of the configuration space metric (besides four-dimensional Diff degeneracy): we shall later consider the identification of this degeneracy.

6. Absolute minimum property fixes the definition of the Kähler function uniquely. The cosmologically natural light cone alternative (four-surfaces are sub-manifolds of the light cone rather than sub-manifolds of the whole Minkowski space) is very probably the only mathematically acceptable alternative.

### 4.1.2 Objection against absolute minimization and variant of Yin-Yang principle

"Yin-Yang" principle has been seen as a nice implication of absolute minimization. One can however argue that Yin-Yang principle need not be consistent with the absolute minimization. Indeed, it might happen that all absolute minima representing the entire Universe have an infinite negative value of Kähler action so that the exponent of Kähler function would vanish identically.

This situation is not encountered for the modified variational principle. If the absolute values of the contributions to the Kähler action from regions where the action density has definite sign, are separately minimized, the absolute value of the net action is minimized. This gives good hopes that net Kähler action is near to zero for the entire Universe.

For this option the extremals are as near as possible to vacuum extremals and Ying-Yang Principle would be realized in a slightly different form favoring extremals with small but positive value of Kähler function: depending on the sign of Kähler function this would slightly favor magnetic or electric configurations.

There are good reasons to expect that the extremals minimize rest energy and possibly also other conserved quantities and it would become possible to deduce the initial values of the time derivatives of the imbedding space coordinates at space-like causal determinant $X^3$ from energy minimization numerically and hence also construct the $X^4(X^3)$ from the data at $X^3$. It is not possible to overemphasize the implications of this for the computability of the theory.

### 4.1.3 Absolute minimum condition explicitly and Diff$^4$ degeneracy

The following argument is tailored for absolute minima but it goes through also for the modified variational principle [E2]. The trivial manner to achieve Diff$^4$ degeneracy and -invariance of the Kähler function is to restrict absolute minimization to the boundary of the light cone and to define the value of the Kähler function for diffeo-related 3-surfaces by requiring Diff$^4$ invariance. The second alternative is to allow absolute minimization for all 3-surfaces: in this case one must prove Diff$^4$ invariance of Kähler function. The following argument suggests that these two definitions indeed are equivalent.

To derive the argument consider what minimization principle actually means for a given 3-surface. Absolute minimum surfaces are assumed to correspond to sub-manifolds of either light cone or Minkowski space.

1. The four-surface associated with $X^3$ is expected to carry non-vanishing four momentum. Action is minimized through the generation of Kähler electric fields and this necessarily leads to a generation of a non-vanishing energy momentum tensor and the energy density associated with the Maxwell action is positive definite. Therefore the minimizing four-surface necessarily has infinite extension with respect to $M^4$ time in future direction. If the four-surface is sub-manifold of $M^4$ instead of $M^4_+$ four-surface must have infinite extension in past also.

2. Absolute minimum condition implies the stationarity of the Kähler action with respect to the local variations of the four-surface satisfying the condition that these variations are trivial, when restricted to $X^3$. Therefore the standard Lagrangian field equations hold true

$$\partial_\alpha(\partial L/\partial h^k_{,\alpha}) - \partial L/\partial h^k \quad = \quad 0 \ , \qquad (11)$$

   where $L$ is Kähler Lagrangian. First variation gives also the following kind of boundary term from future and past infinities.

$$\delta_{\infty}S = \int_{-\infty}^{+\infty} (\partial L/\partial h^k_{,0})\delta h^k d^3x \ , \tag{12}$$

which vanishes identically for the variations considered since these variations vanish at infinity:

$$\delta h^k(x)_{|\pm\infty} = 0 \ . \tag{13}$$

3. Absolute minimum condition can be formulated by considering the second variation of the action that is small deformations $\delta h^k_1(x)$ of the four-surface subject to the condition that they vanish at $X^3$

$$\delta h^k_1(x)_{|X^3} = 0 \ , \tag{14}$$

and lead to a new extremum of the Kähler action. These variations satisfy the equations

$$\begin{aligned} &- [(\partial^2 L/\partial h^k_{,\alpha}\partial h^l_{,\beta})\delta h^l_{,\beta}]_{,\alpha} \\ &+ (\partial^2 L/\partial h^k_{,\alpha}\partial h^l)\delta h^l_{,\alpha} \\ &+ (\partial^2 L/\partial h^k \partial h^l)\delta h^l = 0 \ . \end{aligned} \tag{15}$$

These equations can be derived by expanding action to second order with respect to the variation $\delta h^k_1$ and regarding it as dynamical variable. Since these variations do not necessarily vanish at infinity the above described first order term is in general non-vanishing for these variations. The condition for absolute minimum states that this term vanishes:

$$\delta_{\infty}S = \int_{a}^{+\infty} (\partial L/\partial h^k_{,0})\delta h^k_1 d^3x = 0 \ . \tag{16}$$

Here the lower bound of the substitution (denoted by $a$) corresponds to either to the limit $m^0 \rightarrow -\infty$ or to the intersection of the four-surface with the boundary of the light cone.

4. Minimization condition gives infinite number of conservation laws since the quantities appearing in the minimization condition can be regarded as charges associated with the standard current

$$J^{n)\alpha} = (\partial L/\partial h^k_{,\alpha})\delta h^{n)k}_1 \ . \tag{17}$$

and minimization condition states that corresponding charges are conserved:

$$\Delta Q_n = 0 \tag{18}$$

so that the corresponding symmetry is realized only in the sense that the values of the charges are same at $\infty$ and $a = 0$ or $a = -\infty$. The corresponding currents are not conserved:

$$\partial_\alpha J^\alpha \neq 0 \ . \tag{19}$$

The charges are conserved (but not necessarily nontrivial) not only for the deformations vanishing at $X^3$ but also for the deformations vanishing at an arbitrary Diff$^4$ transform of $X^3$. The point is that the deformations vanishing at Diff$^4$ transform of $X^3$ are obtained from the deformations vanishing at $X^3$ by replacing their argument with Diff$^4$ transformed argument. These deformations certainly satisfy the field equations of the second variation. Furthermore, since Diff$^4$ transformations can be assumed to approach identity at future and past sufficiently rapidly, the values of the conserved charges are same as for the original deformation and are therefore conserved.

The deformations satisfying the charge conservation obey linear superposition and in general the resulting deformations do not vanish for any 3-surface belonging to $X^4$. It is obvious however that this kind of deformations do not correspond to all possible deformations: there are a lot of second variations of Kähler action, which do not obey charge conservation constraint.

5. The conserved charges might be vanishing. If the deformations of 4-surfaces approach to zero at infinity fast enough then the charges at infinity vanish automatically. Also the partial derivatives of the Lagrangian with respect to the time derivatives $\partial_0 h^k$ might vanish so fast that charges vanish asymptotically. In case of the light cone alternative this implies that the charges associated with the intersection of the four-surface with light cone boundary vanish and this in turn implies that Kähler function is invariant under these deformations and therefore an additional degeneracy of the configuration space metric besides the four-dimensional Diff degeneracy. We shall later find that these symmetries might correspond to conformal symmetries (analogous to those of string models and conformal color gauge invariance (the TGD counterpart of the local color symmetry (or rather degeneracy) in the sense of QCD) and that a connection with the mathematical formalism of string models and conformal field theories emerges.

6. Absolute minimum condition implies four-dimensional Diff degeneracy. The point is that charges are conserved not only for $X^3$ but also for the diffeomorphs of $X^3$. Therefore $X^4$ satisfies the absolute minimum conditions for the diffeomorphs of $X^3$, too. The conclusion is that Kähler function is Diff$^4$ invariant and therefore Kähler metric is Diff$^4$ degenerate.

## 4.2 Why non-local Kähler function?

Kähler function is nonlocal functional of 3-surface. Non-locality of the Kähler function seems to be at odds with basic assumptions of local quantum field theories. Why this rather radical departure from the basic assumptions of local quantum field theory? The answer is shortly given: configuration space integration appears in the definition of the inner product for generalized Schrödinger amplitudes and this inner product must be free from perturbative divergences. Consider now the argument more closely.

In the case of finite-dimensional symmetric space with Kähler structure the representations of the isometry group necessitate the modification of the integration measure defining the inner product so that the integration measure becomes proportional to the exponent $exp(K)$ of the Kähler function [26]. The generalization to infinite-dimensional case is obvious. Also the requirement of Kac-Moody symmetry leads to the presence of this kind of vacuum functional as will be found later. The exponent is in fact uniquely fixed by finiteness requirement. Configuration space integral is of the following form

$$\int \bar{S}_1 exp(K) S_1 \sqrt{g} dX \quad . \tag{20}$$

One can develop perturbation theory using local complex coordinates around a given 3-surface in the following manner. The $(1,1)$-part of the second variation of the Kähler function defines the metric and therefore propagagator as contravariant metric and the remaining $(2,0)-$ and $(0,2)$-parts of the second variation are treated perturbatively. The most natural choice for the 3-surface are obviously the 3-surfaces, which correspond to extrema of the Kähler function.

When perturbation theory is developed around the 3-surface one obtains two ill-defined determinants.

1. The Gaussian determinant coming from the exponent, which is just the inverse square root for the matrix defined by the metric defining $(1,1)$-part of the second variation of the Kähler function in local coordinates.
2. The metric determinant

   The matrix representing covariant metric is however same as the matrix appearing in Gaussian determinant by the defining property of the Kähler metric: in local complex coordinates the matrix defined by second derivatives is of type $(1,1)$. Therefore these two ill defined determinants (recall the presence of Diff degeneracy) cancel each other exactly for a unique choice of the vacuum functional!

Of course, the cancellation of the determinants is not enough. For an arbitrary local action one encounters the standard perturbative divergences. Since most local actions (Chern-Simons term is perhaps an exception [27]) for induced geometric quantities are extremely nonlinear there is no hope of obtaining a finite theory. For nonlocal action the situation is however completely different. There are no local interaction vertices and therefore no products of delta functions in perturbation theory.

A further nice feature of the perturbation theory is that the propagator for small deformations is nothing but the contravariant metric. Also the various vertices of the theory are closely related to the metric of the configuration space since they are determined by the Kähler function so that perturbation theory has beautiful geometric interpretation. Furthermore, since four-dimensional Diff degeneracy implies that the propagator doesn't couple to un-physical modes.

It should be noticed that divergence cancellation arguments do not necessarily exclude Chern Simons term from vacuum functional defined as imaginary exponent of $exp(ik_2 \int_{X^4} J \wedge J)$. The term is not well defined for non-orientable space-time surfaces and one must assume that $k_2$ vanishes for these surfaces. The presence of this term might provide first principle explanation for CP breaking. If $k_2$ is integer multiple of $1/(8\pi)$ Chern Simons term gives trivial contribution for closed space-time surfaces since instanton number is in question. By adding a suitable boundary term of form $exp(ik_3 \int_{\delta X^3} J \wedge A)$ it is possible to guarantee that the exponent is integer valued for 4-surfaces with boundary, too.

There are two arguments suggesting that local Chern Simons term would not introduce divergences. First, 3-dimensional Chern Simons term for ordinary Abelian gauge field is known to define a divergence free field theory [27]. The term doesn't depend at all on the induced metric and therefore contains no dimensional parameters ($CP_2$ radius) and its expansion in terms of $CP_2$ coordinate variables is of the form allowed by renormalizable field theory in the sense that only quartic terms appear. This is seen by noticing that there always exist canonical coordinates, where the expression of the Kähler potential is of the form

$$A = \sum_k P_k dQ^k \quad . \tag{21}$$

The expression for Chern-Simons term in these coordinates is given by

$$k_2 \int_{X^3} \sum_{k,l} P_l dP_k \wedge dQ^k \wedge dQ^l \quad , \tag{22}$$

and clearly quartic $CP_2$ coordinates. A further nice property of the Chern Simons term is that this term is invariant under canonical transformations of $CP_2$, which are realized as $U(1)$ gauge transformation for the Kähler potential.

## 4.3 Why Abelian Yang Mills action?

One can consider infinite number of possible definitions for the action defining Kähler function and one can ask two questions. Why YM action? Why Kähler action? The answer to these questions involves two key words: "Weyl invariance" and "Vacuum degeneracy of Kähler action".

### 4.3.1 Weyl invariance

Weyl invariance (invariance of the action under local scalings of the metric) plays decisive role in string theories so that it is natural to generalize Weyl invariance to $d = 3$ dimensions. Of course, now this invariance cannot correspond to an actual symmetry realizable as configuration space transformations: this purely formal invariance might be however closely related with the hoped for invariance of the Kähler function under the conformal transformations of the light cone. The problem is that there are no local Weyl invariants determined by the internal geometry of 3-manifold in 3 dimensions [28]. Chern Simons term for the induced gauge fields defines Weyl invariant local functional of 3-surface, which however fails to be invariant with respect to four-dimensional diffeomorphism group and is therefore excluded.

Chern Simons term associated with the boundary(!) of the four-manifold defines four-dimensional Diff invariant but leads to extremely degenerate geometry. With respect to $M^4$ degrees of freedom metric would be completely degenerate. Furthermore, the term in question seems to be trivial for 3-surfaces without boundaries since it depends on 3-surface only via the boundary of the 3-surface. In fact the earlier proposal for configuration space geometry was based essentially on this term and was indeed found to lead to difficulties. The geometry defined by Chern Simons boundary term would obviously define something resembling closely what might be called "topological quantum field theory.

There is however the possibility to define nonlocal functionals and indeed the absolute minimum of $YM$ action defines Weyl invariant functional in the sense that the corresponding energy momentum tensor is traceless. It is important to notice that dimension $d = 4$ is completely exceptional in the sense that $YM$ action is Weyl invariant only in dimension $d = 4$ so that TGD approach provides an explanation for the dimension of space time.

### 4.3.2 Vacuum degeneracy of the Kähler action

The basic reason for choosing Kähler action is its enormous vacuum degeneracy, which makes long range interactions possible (the well known problem of the membrane theories is the absence of massless particles [29]). The Kähler form of $CP_2$ defines symplectic structure and any 4-surface for which $CP_2$ projection is so called Lagrange manifold (at most two dimensional manifold with vanishing induced Kähler form), is vacuum extremal due to the vanishing of the induced Kähler form. More explicitly, in the local coordinates, where the vector potential $A$ associated with the Kähler form reads

$$A = \sum_k P_k dQ^k \ . \tag{23}$$

Lagrange manifolds are expressible in the following form

$$P_k = \partial_k f(Q^i) \ . \tag{24}$$

where the function $f$ is arbitrary. Notice that for the general $YM$ action surfaces with one-dimensional $CP_2$ projection are vacuum extremals but for Kähler action one obtains additional degeneracy.

The basic consequence of this degeneracy is that the absolute minimum 4-surfaces for arbitrary set of 3-surfaces are expected to be connected. Space-time is connected although 3-space is not! This in turn implies long range correlations between disjoint 3-surfaces and therefore long range interactions.

To see how this is achieved notice that the union of individual single particle 4-surfaces is certainly an extremal of the Kähler action. It is however evident that one can connect the disjoint 4-surfaces using vacuum 4-surfaces so that they form a connected 4-surface with same action as the original set of disjoint 4-surfaces. Furthermore, Kähler action is not positive definite for 4-surfaces with $M^4$ signature: Kähler electric fields give negative contribution to action. To lower the action one can deform the surface so that Kähler-electric fields are generated. Thus it is clear that the minimum of Kähler action must be connected 4-surface.

Vacuum degeneracy has several important consequences.

1. The construction of the metric in N-particle sector differs in no essential manner from that in 1-particle sector. Metric describes naturally the interactions of 3-surfaces. The deviation of the actual Kähler function from the sum of the single particle Kähler functions can be regarded as "interaction" term in the action.

2. Metric is not merely a direct sum of individual metrics for 3-surfaces and in particular there is only single center of mass term in action. This makes it possible to avoid spin statistics difficulty encountered, when N-particle configuration space is metrically cartesian product of single-particle configuration spaces: in this case the spinors of the configuration space are tensor products of single particle spinors. In center of mass degrees of freedom tensor product structure implies a catastrophe: one obtains integer spin fermions.

3. The metric and the exponent $exp(K)$ of the Kähler function defining vacuum functional gives rise to long range interactions most naturally identifiable as gravitation and electromagnetic interaction. In the absence of vacuum degeneracy extremals would be unions of disjoint 4-surfaces and the interactions would reduce to contact interactions.

4. The presence of long range interactions are characteristic for statistical systems at critical point having the property that the dynamics is renormalization group invariant. In the third part of the book we will show that the assumption that Kähler action is renormalization group invariant has extremely powerful consequences: using as input only fine structure constant and Weinberg angle one can predict the values of various couplings at high and low energy limits plus the mysterious number $10^{-19}$ describing the ratio of elementary particle and Planck mass scales. In addition the value of the Kähler coupling should be unique as a fixed point of the coupling coupling constant evolution (or as the counter part of the critical temperature) so that the theory becomes unique.

5. In the third part of the book we shall show that the removal of the degeneracy is achieved by the generation of long range $1/r^2$ Kähler-electric fields: the topological condensation of particle like 3-surface (particle like 3-surface is "glued" to background 3-surface) deforms the background space time so that Reissner-Nordström type spherically symmetric metric results in asymptotic regions. The generation of Kähler electric fields explains Higgs mechanism classically (particle mass results as Kähler field energy) and color confinement (the generation of Kähler electric flux tubes corresponds to generation of color electric flux tubes since color field is proportional to Kähler field).

6. It turns also that vacuum extremals of the Kähler action can be regarded as idealized space times obtained by smoothing out the various topological details of the actual space time and by describing their presence in terms of Yang Mills currents and energy momentum tensor. In particular, Robertson Walker cosmology corresponds to this kind of idealized space time.

# 5 Four-dimensional Diff invariance

We have already proved that the proposed definition of the Kähler function leads to Diff$^4$ degeneracy and -invariance of the Kähler metric: the essential point is that the minimizing four-surface is assumed to be same for all Diff$^4$ related 3-surfaces belonging to the minimizing four-surface $X^4(X^3)$ of a given 3-surface $X^3$. The simplest manner to guarantee $X^4(Y^3) = X^4(X^3)$ for all $Y^3$ diffeo-related to $X^3$ at $X^4(X^3)$ is to assume that standard $X^3$ to belong to the 'light cone boundary' $\delta M^4_+ \times CP_2$ and *define* $X^4(Y^3) = X^4(X^3)$ for all diffeo-related surfaces $Y^3$. This would mean the restriction of absolute minimization of Kähler action to surfaces at light cone boundary. An alternative possibility is to allow absolute minimization for any surface $Y^3$ in $H$ and to *show* that this definition is equivalent with manifestly Diff$^4$ invariant definition. Four-dimensional $Diff$ invariance has several far reaching consequences to be discussed briefly in the sequel.

## 5.1 Resolution of tachyon difficulty

In TGD as in string models the tachyon difficulty is potentially present: unless the time like vibrational excitations possess zero norm they contribute tachyonic term to the mass squared operator of Super Kac Moody algebra. This difficulty is familiar already from string models [24, 25].

The degeneracy of the metric with respect to the time like vibrational excitations guarantees that time like excitations do not contribute to the mass squared operator so that mass spectrum is tachyon free. It also implies the decoupling of the tachyons from physical states: the propagator of the theory corresponds essentially to the inverse of the Kähler metric and therefore decouples from time like vibrational excitations. The experience with string model suggests that if metric is degenerate with respect to diffeomorphisms of $X^4(X^3)$ there are indeed good hopes that time like excitations possess vanishing norm with respect to configuration space metric.

## 5.2 Absence of Diff anomalies

The four-dimensional Diff invariance of the Kähler function implies that Diff invariance is guaranteed in the strong sense since the scalar product of two Diff vector fields given by the matrix associated with $(1,1)$ part of the second variation of the Kähler action vanishes identically. This property gives hopes of obtaining theory, which is free from Diff anomalies: in fact loop space metric is not Diff degenerate and this might be the underlying reason to the problems encountered in string models [24, 25].

This argument can be made more precise. The Diff degeneracy of the Kähler form implies that in general it is not possible to define all symmetry transformations as canonical transformations since this would necessitate the following representation of the vector field $j_A^k$ generating the isometry

$$j_A^k = J^{kl}\partial_l H^A \ . \tag{25}$$

In general this representation is not expected to exist since $J^{kl}$ is degenerate. Representation can hold only modulo some superposition of Diff generators. Diff itself affords an example of symmetry transformations of this kind.

This result implies that the method yielding Abelian extension in TGD approach yields necessarily a trivial Abelian extension in the case of Diff Lie-algebra and the reason is essentially that the matrix elements of the Kähler form between Diff generators vanish and one cannot represent Diff transformations as Hamiltonian flows. This in turn suggest the absence of Diff anomalies in TGD approach.

## 5.3 Complexification of the configuration space

Four-dimensional Diff degeneracy turns out to play a fundamental role in the complexification of the configuration space. The point is that Diff$^4$ invariance effectively reduces the imbedding space from $M_+^4 \times CP_2$ to $\delta M_+^4 \times CP_2$. Light cone boundary in turn is metrically 2-dimensional Euclidian sphere allowing infinite-dimensional group of conformal symmetries and Kähler structure. Therefore one can say in certain sense configuration space metric inherits the Kähler structure of $S^2 \times CP_2$. Furthermore, this mechanism works in case of four-dimensional Minkowski space only: higher-dimensional spheres do not possess even Kähler structure.

## 5.4 Contravariant metric and Diff$^4$ degeneracy

Diff degeneracy implies that the definition of the contravariant metric, which corresponds to the propagator associated to small deformations of minimizing surface is not quite straightforward. We believe that this problem is only technical. Certainly this problem is not new, being encountered in both GRT and gauge theories [30, 31]. In TGD the solution of the problem is provided by the existence of infinite-dimensional isometry group. If the generators of this group form a complete set in the sense that any vector of the tangent space is expressible as as sum of these generators plus some zero norm vector fields then one can restrict the consideration to this subspace and in this subspace the matrix $g(X,Y)$ defined by the components of the metric tensor indeed indeed possesses well defined inverse $g^{-1}(X,Y)$. This procedure is analogous to gauge fixing conditions in gauge theories and coordinate fixing conditions in General Relativity.

## 5.5 Diff$^4$ invariance and generalized Schrödinger amplitudes

Four-dimensional Diff invariance can be generalized so that it applies at the level of physical states also. Generalized Schrödinger amplitudes are Diff$^4$ invariant. This in fact fixes not only classical but also quantum dynamics completely. The point is that the values of the configuration space spinor fields must be essentially same for all Diff$^4$ related 3-surfaces on the orbit $X^4$ associated with a given 3-surface. This means that the time development of Diff$^4$ invariant configuration spinor field is completely determined by its initial value at the moment of the big bang!

## 5.6 Two alternative definitions of classical space time

TGD approach suggests two alternative definitions of the classical space-time either

1. as the absolute minimum of the Kähler action or

2. as an orbit of possibly Kähler charged 'massless' particle in configuration space. In the following we shall consider these alternative and possibly equivalent definitions in more detail.

### 5.6.1 Absolute minimum of the Kähler action as classical space-time

We have already found that Kähler function associates with a given 3-surface a unique space time. In particular, one can associate definite conserved quantitities with this space time. In order to obtain non-vacuum space-times with nontrivial conserved quantities one must however assume that 4-surface extends to infinity in future direction of $M^4_+$ unless 4-surface contains singularities (the orbit of 3-surface degenerates to point). Note that one must also assume that the intersection of the four-surface with the boundary of the light cone is not subject to variational conditions since this would imply the vanishing of four momentum densities. Of course, one could consider also the possibility that space-times become vacuum space-times at the moment of big bang and that non-vanishing four-momentum densities are generated only later: negative energies would be present however in some form in this scenario.

It is indeed easy to see that for 4-surfaces with finite $M^4$ projection conserved charged are indeed trivial. The possible dynamically generated boundary components have vanishing conserved currents by field equations so that in $M^4$ the conserved quantities associated with 3-surface are necessarily vanishing! This doesn't imply the vanishing of conserved quantities for individual 3-surfaces, when $X^4$ contains several components: only their sum vanishes. This result is unsatisfactory if we want to interpret 4-surface as a classical space time.

That minimization principle implies non-vacuum space-times is suggested by the following argument. Kähler action is not positive definite: the generation of Kähler electric fields gives negative contribution to the action and positive contribution to the energy since the energy density associated with any Kähler field is positive definite quantity. Therefore we expect that the minimum of the action tends to be negative and its is clear that the larger the size of the space time the better the possibilities to develop Kähler electric fields to gain negative action. Therefore space-times are most probably of infinite size and non-vacuum.

There is a counter argument against the infinite size of minimizing space-time surface. There is the danger that the value of the Kähler function diverges for space-times of infinite size so that one should pose the finite size of the space-time as consistency condition, which in turn would imply that all absolute minima of the Kähler action are vacuum space times. There are however delicacies related to this argument.

1. The second variation of the Kähler function can be completely well defined quantity also, when the value of the Kähler action diverges.

2. The vacuum functional of the theory is the exponent $exp(K)$ of the Kähler action and vanishes for 3-surfaces corresponding to the infinite value of the Kähler function: therefore these surfaces decouple from dynamics and one perhaps avoids the potential mathematical difficulties related to them. In fact, the vacuum functional might be finite only for very restricted types of 3-surfaces so that classical 3-space is unique to a very high degree!

3. The fact that the generation of too strong Kähler electric fields leads to Euclidian metric and therefore to a positive contribution to the Kähler action might serve as a regulating mechanism keeping Kähler action finite. In any case, for physically interesting 3-surfaces the space time should become for large values of $M^4_+$ time a zero action extremal of the Kähler function. In fact, the elementary particles can be regarded as Euclidian regions of space time as will be found in the third part of the book.

Kähler function associates to every 3-surface unique space-time surface and one might ask how to sharpen the definition of the classical space-time. The first thing to notice is that one can associate to each 3-topology a unique space-time by requiring that vacuum functional and thus Kähler function is maximum for this 3-space. Furthermore, one could define the "physical space-time" as space-time associated with the 3-surface, which corresponds to an absolute maximum of Kähler function allowing the 3-topology to be arbitrary. Also more refined definitions taking into account the dependence of the generalized Schrödinger amplitude on 3-surface are possible.

### 5.6.2 Classical space-time as null geodesic in configuration space

Kähler geometry provides also alternative definitions of the classical space-time:

1. Null geodesics of the configuration space are perhaps the simplest candidates for the classical space-time one can imagine. If $Map(X^3, M^4 \times SU(3))$ were (it is not!) an exact symmetry geodesic lines correspond to one-parameter subgroups of this group and classical theory would be exactly solvable. Despite this one can associate an infinite number of conserved charges to the orbit of 3-surface corresponding to the generators of the isometry group. This implies that geodesics certainly cannot correspond to the extremals of Kähler function, which is not invariant under isometries (metric would be completely degenerate if this were the case).

   The equations for null geodesic reduce to the condition expressing one parameter subgroup property plus the mass shell condition for a "massless" particle

$$\begin{aligned} \dot{m}^k &= p^k \ , \\ \dot{X}^k &= c_A j^{Ak} \ , \\ p^2 - G_{kl} c_A c_B j^{Ak} j^{Bl} &= 0 \ . \end{aligned} \qquad (26)$$

   Here $p^2$ denotes the mass squared associated with the geodesic motion in $M^4$ cm degrees of freedom. $J^{Ak}$ denotes the generator of the isometry group and the quantities $c_A$ are constant coefficients appearing in the representation of the tangent vector of the geodesic line as a superposition of gauge group generators.

   Using the isometry invariance of the inner products of the isometry generators one obtains the expression for the mass squared the expression

$$p^2 = G^{AB} c_A c_B \ . \qquad (27)$$

   The quantities $G^{AB}$ are constant with respect to local gauge transformations although they depend on the point of the configuration space. Semiclassical quantization condition suggests discrete values for the mass squared operator determined by the Kac Moody symmetry so that classical space-times correspond directly to Kac Moody representations.

2. Null geodesics are not the most general orbits one can imagine for a massless particle in configuration space. The Abelian extension of the isometry group to Kac-Moody group turns out to correspond to the coupling of the configuration space spinors to the Kähler potential and this suggests that classical equations of motion contain additional Lorentz force term describing the interactions with the Maxwell field defined by the Kähler form

$$\ddot{X}^k + \{{}^{\,k}_{l\ m}\} \dot{X}^l \dot{X}^m = k J^k_{\ m} \dot{X}^m \ . \qquad (28)$$

   This equation is consistent with the generalized masslessness condition, which implies the previous expression for the mass squared operator. It is not clear whether one can solve equations of motion exactly in this case.

Are the proposed two definitions of classical space time in fact equivalent? That this might be the case is supported by the following arguments:
i) The presence of the Kähler force term implies the loss of the infinite number of the conserved quantities: only Lorentz and color quantum numbers associated with cm degrees of freedom are conserved.
ii) The absolute minima of Kähler action are indeed null lines of the configuration space since the distance between the intersection $Y^3$ of $X^4$ with light cone boundary and any diffeo-related surface on $X^4$ is vanishing by four-dimensional Diff degeneracy.

If the equivalence indeed holds true then the minimization problem of the Kähler action becomes equivalent to the study of orbits of massless Kähler charged particles in configuration space.

# 6 Some properties of Kähler action

In this section some properties of Kähler action and Kähler function are discussed in light of experienced gained during about 15 years after the introduction of the notion.

## 6.1 Consequences of the vacuum degeneracy

The vacuum degeneracy already discussed is perhaps the most characteristic feature of the Kähler action. Although it is associated with the extremals rather than absolute minima of Kähler action, there are good reasons to expect that it has deep consequences concerning the structure of the theury.

### 6.1.1 Approximate canonical and $Diff(M^4)$ invariances

Vacuum extremals have diffeomorphisms of $M^4_+$ and $M^4_+$ local canonical transformations as symmetries. For non-vacuum extremals these symmetries leave induced Kähler form invariant and only induced metric breaks these symmetries. Canonical transformations of $CP_2$ act on the Maxwell field defined by the induced Kähler form in the same manner as ordinary $U(1)$ gauge symmetries. They are however not gauge symmetries since gauge invariance is still present. In fact, the construction of the configuration space geometry relies on the assumption that canonical transformations of $\delta M^4_+ \times CP_2$ which infinitesimally correspond to combinations of $M^4_+$ local $CP_2$ canonical and $CP_2$-local $M^4_+$ canonical transformations act as isometries of the configuration space.

The fact that $CP_2$ canonical transformations do not act as gauge transformations means that $U(1)$ gauge invariance is effectively broken. This has non-trivial implications. The field equations allow purely geometric vacuum 4-currents not possible in Maxwell's electrodynamics. The existence of longitudinal scalar wave pulses carrying electric field parallel to the direction of propagation and claimed by Tesla for more than century ago are a second key implication of the symmetry.

### 6.1.2 $CP_2$ type extremals and conformal invariance

There is also a second kind of vacuum degeneracy, which is relevant to the elementary particle physics. The so called $CP_2$ type extremals are warped imbeddings $X^4$ of $CP_2$ to $H$ such that Minkowski coordinates are functions of a single $CP_2$ coordinate, and the one-dimensional projection of $X^4$ is random light like curve. The conditions stating light likeness are equivalent with the Virasoro conditions of string models and this actually led to the path leading to the realization that conformal invariance is basic symmetry of TGD. These extremals have non-vanishing action but vanishing Poincare charges.

### 6.1.3 Spin glass degeneracy

Vacuum degeneracy means that all surfaces belonging to $M^4_+ \times Y^2$, $Y^2$ any Lagrange sub-manifold of $CP_2$ are vacua irrespective of the topology and that canonical transformations of $CP_2$ generate new surfaces $Y^2$. If absolute minima are obtained as small deformations of vacuum extremals, one expects therefore enormous ground state degeneracy, which could be seen as 4-dimensional counterpart of spin glass degeneracy. This degeneracy corresponds to the hypothesis that configuration space is a union of symmetric spaces labelled by zero modes which do not appear at the line-element of the configuration space metric.

Zero modes define what might be called the counterpart of spin glass energy landscape and the maxima Kähler function as a function of zero modes define a discrete set which might be called reduced configuration space. Spin glass degeneracy turns out to be crucial element for understanding how macro-temporal quantum coherence emerges in TGD framework.

## 6.2 Some implications of the classical non-determinism of Kähler action

The classical non-determinism has turned out to be the most decisive property of Kähler action, and only the construction of TGD inspired theory of consciousness has gradually revealed its implications.

### 6.2.1 Generalized quantum gravitational holography

The original naive belief was that the construction of the configuration space geometry reduces to $\delta H = \delta M^4_+ \times CP_2$. An analogous idea in string model context became later known as quantum gravitational holography. The basic implication of the vacuum degeneracy is classical non-determinism, which is expected to reflect itself as the properties of the Kähler function and configuration space geometry. Obviously classical non-determinism challenges the notion of quantum gravitational holography.

One could try to believe that the generalization of the notion of 3-surface is enough to get rid of the degeneracy and save quantum gravitational holography in its simplest form. This would mean that one just replaces space-like 3-surfaces with "association sequences" consisting of sequences of space-like 3-surfaces with time like separations as causal determinants. This would mean that the absolute minima of Kähler function would become degenerate: same space-like 3-surface at $\delta H$ would correspond to several association sequences with the same value of Kähler function.

It seems that life is more complex than this. $CP_2$ type extremals have Euclidian signature of the induced metric and therefore $CP_2$ type extremals glued to space-time sheet with Minkowskian signature of the induced metric are surrounded by light like surfaces $X^3_l$, which might be called elementary particle horizons. The non-determinism of the $CP_2$ type extremals suggests strongly that also elementary particle horizons behave non-deterministically and must be regarded as causal determinants having time like projection in $M^4_+$. Pieces of $CP_2$ type extremals are good candidates for the wormhole contacts connecting a space-time sheet to a larger space-time sheet and are also surrounded by an elementary particle horizons and non-determinism is also now present. That this non-determinism would allow the proposed simple description seems highly implausible.

Even this is not enough. The assumption that space-times are 4-surfaces resolves the energy problem of general relativity but also implies that negative energies are possible classically since the time orientation of the 4-surface determines the sign of the energy. This means the possibility of classical pair creation for space-time sheets having opposite signs of classical energy. A good guess is that light like surfaces $X^3_l \times CP_2$ can act as causal determinants besides the boundary of the light cone. This destroys all hopes about the reduction of physics to the light cone boundary.

These causal determinants $X^3_l$ however share metric 2-dimensionality as a common element. This implies conformal and even symplectic structure and generalized conformal invariance. It is this symmetry which makes it possible to generalize the construction of configuration space geometry to take into account the complications caused by the classical non-determinism.

### 6.2.2 Classical non-determinism saves the notion of time

Although classical non-determinism represents a formidable mathematical challenge it is a must for several reasons. Quantum classical correspondence, which has become a basic guide line in the development of TGD, states that all quantum phenomena have classical space-time correlates. This is not new as far as properties of quantum states are considered. What is new that also quantum jumps and quantum jump sequences which define conscious existence in TGD Universe, should have classical space-time correlates: somewhat like written language is correlate for the contents of consciousness of the writer. Classical non-determinism indeed makes this possible. Classical non-determinism makes also possible the realization of statistical ensembles as ensembles formed by strictly deterministic pieces of the space-time sheet so that even thermodynamics has space-time representations. Space-time surface can thus be seen as symbolic representations for the quantum existence.

In canonically quantized general relativity the loss of time is fundamental problem. If quantum gravitational holography would work in the most strict sense, time would be lost also in TGD since all relevant information about quantum states would be determined by the moment of big bang. More precisely, geometro-temporal localization for the contents of conscious experience would not be possible. Classical non-determinism together with quantum-classical correspondence however suggests that it is possible to have quantum jumps in which non-determinism is concentrated in space-time region so that also conscious experience contains information about this region only.

## 6.3 Configuration space geometry, generalized catastrophe theory, and phase transitions

The definition of configuration space geometry has nice catastrophe theoretic interpretation. To understand the connection consider first the definition of the ordinary catastrophe theory [32]. In catastrophe theory one considers extrema of the potential function depending on dynamical variables $x$ as function of external parameters $c$. The basic space decomposes locally into cartesian product $E = C \times X$ of control variables $c$, appearing as parameters in potential function $V(c, x)$ and of state variables $x$ appearing as dynamical variables. Equilibrium states of the system correspond to the extrema of the potential $V(x, c)$ with respect to the variables $x$ and in the absence of symmetries they form a sub-manifold of $M$ with dimension equal to that of the parameter space $C$. In some regions of $C$ there are several extrema of potential function and the extremum value of $x$ as a function of $c$ is multivalued. These regions of $C \times X$ are referred to as catastrophes. The simplest example is cusp catastrophe (see Fig. 6.3) with two control parameters and one state variable.

In catastrophe regions the actual equilibrium state must be selected by some additional physical requirement. If system obeys flow dynamics defined by first order differential equations the catastrophic jumps take place along the folds of the cusp catastrophe (delay rule). On the other hand, the Maxwell rule obeyed by thermodynamic phase transitions states that the equilibrium state corresponds to the absolute minimum of the potential function and the state of system changes in discontinuous manner along the Maxwell line in the middle between the folds of the cusp (see Fig. 6.3). As far as discontinuous behavior is considered fold catastrophe is the basic catastrophe: all catastrophes contain folds as there 'satellites' and one aim of the catastrophe theory is to derive all possible manners for the stable organization of folds into higher catastrophes. The fundamental result of the catastrophe theory is that for dimensions $d$ of $C$ smaller than 5 there are only 7 basic catastrophes and polynomial potential functions provide a canonical representation for the catastrophes: fold catastrophe corresponds to third order polynomial (in fold the two real roots become a pair of complex conjugate roots), cusp to fourth order polynomial, etc.

Consider now TGD counterpart of this. The most obvious identification for the parameter space $C$ would be as the space of all 3-surfaces in $H = M^4_+ \times CP_2$. In order to get rid of the difficulties related to $Diff^4$ invariance one must however restrict the consideration to 3-surfaces belonging to $H_a$ : the set of 3-surfaces of $M^4_+ \times CP_2$ with constant $M^4_+$ proper time coordinate. The counterpart of the total space $E = C \times X$ can be identified as the space of the solutions of Euler Lagrange equations associated with Kähler action (one could consider all 4-surfaces but this is not necessary) and decomposes only locally into Cartesian product. Intuitively the space $X$ corresponds to the time derivatives for the variables specifying the space $X$ and in Hamiltonian formalism to canonical momenta. If the initial value problem is well defined the values of $C$ and $X$ coordinates specify the extremum uniquely. In TGD this is not in general true as the vacuum degeneracy of Kähler action demonstrates.

Potential function corresponds to Kähler action restricted to the solutions of Euler Lagrange equations. Catastrophe surface corresponds to the four-surfaces found by extremizing Kähler action with respect to to the variables of $X$ (time derivatives of coordinates of $C$ specifying $X^3$ in $H_a$ ) keeping the variables of $C$ specifying 3-surface $X^3$ fixed. Extremization with respect to time derivatives implies a phenomenon analogous to the Bohr quantization since canonical momenta cannot be chosen freely as in the ordinary initial value problems of the classical physics. When catastrophe occurs there are several extremizing 4-surfaces going through the given 3-surface: otherwise one obtains just the sought for absolute minimum surface.

The requirement that Kähler function corresponds to absolute minimum is just Maxwell's rule in infinite-dimensional context and implies that phase transition type catastrophic quantum jumps are typical for TGD:eish Universe. Cusp catastrophe provides a simple concretization of the situation. The set $M$ ('Maxwell set') of the critical 3-surfaces corresponds to the Maxwell line of the cusp catastrophe and forms codimension one set in configuration space. For 3-surfaces near to the Maxwell set $M$ small one parameter deformation in the direction normal to it can induce large deformation of the 4-surface associated with it. This implies initial value sensitivity with respect to the coordinate $X_n$ associated with the normal direction. Kähler function itself is continuous on Maxwell surface and mathematical consistency requires that also Kähler metric is continuous on Maxwell surface. A good example of a catastrophic jump is provided by a topology changing quantum jump (3-surface decays to two disjoint 3-surfaces) identifiable as 3-particle vertex.

The present situation differs from the ordinary catastrophe theory in several respects.

1. The parameter space $C$ is infinite-dimensional so that there seems to be no hope of having finite classification for catastrophes in TGD:eish Universe. Of course, all lower-dimensional catastrophes are expected to be present in TGD, too.

2. Kähler action possesses vacuum degeneracy and one cannot exclude the possibility that the absolute minima of the Kähler action are degenerate: this implies further modifications to the standard picture of catastrophe theory.

3. Maxwell rule follows as a theorem in Quantum TGD whereas in ordinary catastrophe theory delay rule (jumps takes place along the folds) follows as a theorem. The latter implies that the description of phase transitions is not possible using the catastrophe theory associated with flows. These observations suggests that classical dynamics (for instance the classical dynamics associated with Kähler action) obeys delay rule whereas quantum dynamics obeys Maxwell rule and that the phenomena of super cooling and super heating are related to classical dynamics and ordinary phase transitions are induced by quantum fluctuations.

The existence of the catastrophes is implied by the vacuum degeneracy of the Kähler action. For example, for pieces of Minkowski space in $M^4_+ \times CP_2$ the second variation of the Kähler action vanishes identically and only the fourth variation is non-vanishing: these 4-surfaces correspond to the dip of the cusp catastrophe. There are also space-time surfaces for which second variation is non-vanishing but degenerate and a hierarchy of subsets in the space of extremal 4-surfaces with decreasing degeneracy of the second variation defines the boundaries of the projection of the catastrophe surface to the space of 3-surfaces. The space-times for which second variation is degenerate contain as subset the critical and initial value sensitive absolute minimum space-times.

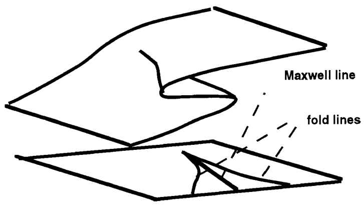


Figure 3: Cusp catastrophe

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