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Classification of N=2 Superconformal Field Theories with Two-Dimensional Coulomb Branches, II

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ABSTRACT: We continue the classification of 2-dimensional scale-invariant rigid special Kahler (RSK) geometries. This classification was begun in [1] where singularities corresponding to curves of the form $y^2 = x^6$ with a fixed canonical basis of holomorphic one forms were analyzed. Here we perform the analysis for the $y^2 = x^5$ type singularities. (The final maximal singularity type, $y^2 = x^3(x-1)^3$, will be analyzed in a later paper.) These singularities potentially describe the Coulomb branches of N=2 supersymmetric field theories in four dimensions. We show that there are only 13 solutions satisfying the integrability condition (enforcing the RSK geometry of the Coulomb branch) and the Z-consistency condition (requiring massless charged states at singularities). Of these solutions, one has a marginal deformation, and corresponds to the known solution for certain Sp(2) gauge theories, while the rest correspond to isolated strongly interacting conformal field theories.

1. Introduction

The classification of all possible $N = 2$ superconformal field theories (SCFTs) with one complex dimensional Coulomb branches ("rank one" theories) was carried out almost a decade ago [2, 3, 4]. It corresponds to a classification of all one-dimensional scale-invariant rigid special Kahler (RSK) geometries, and coincides with the Kodaira classification [5] of complex singularities of families of elliptic curves.

The classification of all possible rank 2 $N = 2$ SCFTs turns out to be considerably more complicated, but still amenable to a systematic computation. In this paper we continue this classification, begun in [1]. We are interested in classifying possible interacting SCFTs which are neither IR free nor factor into lower-rank interacting and/or IR free parts. These correspond to the "maximal" singularities of the corresponding genus 2 Seiberg-Witten curve. As discussed in [1], there are just three types of such maximal singularities corresponding to curves of the form $y^2 = x^6$, $y^2 = x^5$, and $y^2 = x^3(x-1)^3$ with a fixed canonical basis of holomorphic one-forms. The first case was analyzed in [1], we analyze the second case here, and we leave the third case to another paper.

As reviewed in [1] the effective action on a two-dimensional Coulomb branch is described by a genus 2 Riemann surface together with basis of holomorphic 1-forms. All genus 2 Riemann surfaces are hyperelliptic, and can be described algebraically as a double-sheeted cover of the complex x-plane (plus infinity) by $y^2 = P(x)$ where P is a polynomial of order 5 or 6 in x. We can choose the complex coordinates x and y such that the basis of holomorphic one-forms has the canonical form $\omega_u = x dx/y$ and $\omega_v = dx/y$.

The central charge, Z, of the $N = 2$ superalgebra depends linearly on the magnetic and electric charges of the $\mathrm{U}(1)^2$ low energy gauge group on the Coulomb branch of the moduli space. It is related to the holomorphic one-forms by

$$\partial_u Z = \oint x dx/y, \qquad \partial_v Z = \oint dx/y. \tag{1.1}$$

Here u and v are global complex coordinates on the Coulomb branch. The integrability of (1.1) gives the partial differential equation for the curve,

$$\partial_u y^{-1} - \partial_v (x y^{-1}) = \partial_x (b y^{-1}), \tag{1.2}$$

where b is an arbitrary meromorphic function of x. (As we discuss in section 3 below, one can show that b is in fact a quadratic polynomial in x.) At genus 2 this integrability equation completely encodes the constraints on the Coulomb branch geometry coming from $N = 2$ supersymmetry (*i.e.*, its RSK geometry).

To search for SCFTs, we look for scaling solutions of the integrability condition (1.2). Scaling means that u, v, x and y can be assigned scaling dimensions (which we denote by square brackets). For the scaling to make sense, all scaling dimensions must have positive real parts. It is convenient to define $r := [v]/[u]$ and $s := 1/[v]$. By (1.1) and (1.2) the scaling dimensions of all quantities can be expressed in terms of $[u]$ and $[v]$, and therefore r and

s. We then define dimensionless variables, $\xi := u^{1-r}x$, $\omega := -ru^{-r}v$, $\eta := u^{1+rs-2r}y$, and $\beta := r^{-1}u^{2-r}b + (1-r^{-1})\xi$, in terms of which the integrability equation becomes

$$[(\xi-\omega)\partial_\omega + (s-1)]\eta^{-1} = \partial_\xi(\beta\eta^{-1}). \tag{1.3}$$

The form of η and β can be greatly constrained by using the remaining freedom to make changes of variables among x, u, and v, as well as by imposing that the solutions are single-valued in u and v. These conditions are outlined in section 2 for the $y^2 = x^5$ type singularities. The result is that they restrict r to take a discretely infinite set of values for each possible value of s, and they allow the differential equation (1.3) to be reduced to a set of polynomial equations.

We solve these polynomial equations for η (and β) in section 3, finding 28 solutions whose form depends only on s. The list of solutions is given in table 1. Note that each entry corresponds to an infinite number of solutions, since r can take infinitely many values.

The modulus of the central charge gives the lower bound on the mass for any state with corresponding electric and magnetic charges. Singularities in the effective action are the result of charged states becoming massless, therefore every singularity must be accompanied by vanishing central charge. This "Z-consistency condition" is an extra physical requirement on our solutions. In the scaling case when the integrability condition (1.3) is satisfied, (1.1) can be integrated to give

$$Z = \frac{u^{rs}}{rs} \oint \frac{(\xi-\omega)}{\eta} d\xi. \tag{1.4}$$

The Z-consistency condition is then evaluated in section 4 by evaluating (1.4) at the various singularities of the η's found from solving the integrability equation. The result is that only 13 solutions survive; they are recorded in table 2 below.

We conclude in section 5 with a discussion of these scale-invariant solutions for the possible $N = 2$ supersymmetric low energy effective action on 2-dimensional Coulomb branches. All are consistent with $N = 2$ superconformal invariance. One has an exactly marginal operator, and coincides with the known curve [6] for certain Sp(2) scale invariant $N = 2$ supersymmetric gauge theories. Of the remaining 12 solutions, only one has been previously identified [7, 8, 9] as an $N = 2$ superconformal fixed point theory found by appropriately tuning vevs, masses, and couplings in other $N = 2$ field theories. The other 11 are new isolated fixed point theories.

2. $\mathbf{y^2 = x^5}$ type singularities

The degenerations of the general genus 2 curve $y^2 = P(x)$, where P is an order six polynomial in x, can be classified according to how the six branch points collide with one another. These correspond to all the ways of partitioning the six branch points into colliding subsets. Of these, there are just three maximal degenerations, which have the property that every cycle on the Riemann surface is homologous to a vanishing cycle. They are the partitions (6), (5,1), and (3,3). The first corresponds to degenerations where all six branch points collide at a single point in the projective x-plane; the second to 5 branch points colliding at a single $x = x_1$ while the sixth remains separate at $x = x_2$; and the third to three colliding at x_1 and the other three colliding at x_2. As mentioned above, this paper is devoted to the second, or (5,1), case.

Even after fixing the basis of holomorphic one-forms, there is left unfixed a group of reparametrizations of x, u, and v, called the "holomorphic reparameterizations" in [1], which act by general holomorphic reparameterizations on u and v, together with a fractional linear transformation on x. This can be used to send any three distinct points on the x projective plane to chosen values. We can partially fix this freedom by choosing the maximal (5,1) degeneration to be at $u = v = 0$, and by choosing $x_1 = 0$ and $x_2 = \infty$. This leaves a singularity of the form $y^2 \sim x^5$ at $u = v = 0$. Therefore, we can write the curve as

$$y^2 = a(u,v)\left(f_6(u,v)x^6 + x^5 + \sum_{k=0}^{4} f_k(u,v)x^k\right), \tag{2.1}$$

for some unknown coefficients a and f_k, where the f_k vanish when $u = v = 0$.

Single-valuedness of the physics as a function of the good coordinates (u, v) on the moduli space imply [1] that the f_k are single-valued functions on the Coulomb branch; a, however, need not be single-valued.

The scaling hypothesis together with our other coordinate choices mentioned above leaves only a three-parameter subgroup of the holomorphic reparameterizations unfixed. One of these is simply an overall rescaling, which can be used to set the overall coefficient of a to 1. Another is the freedom to shift $v \to v + Cu^r$ with an associated shift of $x \to x - rCu^{r-1}$. This freedom can be completely fixed by setting $f_4(u, v) = 0$ by an argument analogous to one used in [1]. The remaining unfixed holomorphic reparametrizations involves a rescaling of v and x keeping u unchanged; it can be fixed by setting any non-zero coefficient in one of the f_k's to 1.

A happy simplification in for the (5,1) type singularities is that scaling, regularity, and the vanishing of the f_k at $u = v = 0$ implies that f_6 must vanish identically. Thus, in the dimensionless scaling variables introduced in section 1, the (5,1) curve can be taken to be of the form

$$\eta^2 := \alpha(\omega)\phi(\xi,\omega) \quad \text{with} \quad \phi(\xi,\omega) = \xi^5 + \sum_{k=0}^{3} \phi_k(\omega)\xi^k. \tag{2.2}$$

Scaling plus regularity on the moduli space imply that the functions $\phi_k(\omega)$ have the following dependences on ω:

$$\begin{aligned}
\phi_3 &= a_3\omega + b_3 \\
\phi_2 &= a_2\omega^2 + b_2\omega + c_2 \\
\phi_1 &= a_1\omega^3 + b_1\omega^2 + c_1\omega + d_1 \\
\phi_0 &= a_0\omega^4 + b_0\omega^3 + c_0\omega^2 + d_0\omega + e_0.
\end{aligned} \tag{2.3}$$

A further simplification compared to the $y^2 = x^6$ case is in the argument determining the x-dependence in the unknown integration function $b(u, v, x)$ that appears in the integrability condition (1.2). In [1] the highest order in x of y^2 was six; in the present case it is only five. Since the highest order of b in x was shown to be no more than the highest order of y^2 divided by two, and since b is single-valued in x, we find that b is at most quadratic in x: $b = \sum_{k=0}^{2} b_k(u, v)\, x^k$, or, in terms of the dimensionless scaling variables, $\beta = \sum_{k=0}^{2} \beta_k(\omega)\, \xi^k$.

3. Solutions to the integrability equation

Substituting the first equation in (2.2) into (1.3) gives the scale invariant integrability equation

$$(\xi - \omega)\partial_\omega \phi = \beta \partial_\xi \phi - \phi[2(1-s) + 2\partial_\xi \beta + (\xi - \omega)\partial_\omega \ln \alpha]. \tag{3.1}$$

Substituting the ξ expansions of ϕ and β then give a series of ordinary differential equations in ω for the coefficient functions α, ϕ_k and β_k. In this case we find that the first non-identically zero coefficient of ξ gives

$$\partial_\omega \ln \alpha = \beta_2, \tag{3.2}$$

allowing us to eliminate α. Two of the resulting equations allow us to simply solve for β_0 and β_1 in terms of s, β_2, and ϕ_3:

$$5\beta_0 = 2\beta_2\phi_3 + \phi_3', \qquad 3\beta_1 = 2(1-s) - \omega\beta_2. \tag{3.3}$$

Upon substituting these equations into the integrability equations we are left with the following 4 equations:

$$\begin{aligned}
0 &= \beta_2[\qquad\qquad 25\omega\phi_0 + \ \ 6\phi_1\phi_3] + [-50(1-s)\phi_0 \qquad\qquad + 15\omega\phi_0' + 3\phi_1\phi_3'] \\
0 &= \beta_2[-75\phi_0 + 20\omega\phi_1 + 12\phi_2\phi_3] + [-40(1-s)\phi_1 - 15\phi_0' + 15\omega\phi_1' + 6\phi_2\phi_3'] \\
0 &= \beta_2[-60\phi_1 + 15\omega\phi_2 + 18\phi_3\phi_3] + [-30(1-s)\phi_2 - 15\phi_1' + 15\omega\phi_2' + 9\phi_3\phi_3'] \\
0 &= \beta_2[-45\phi_2 + 10\omega\phi_3 \qquad\qquad] + [-20(1-s)\phi_3 - 15\phi_2' + 15\omega\phi_3' \qquad\qquad].
\end{aligned} \tag{3.4}$$

Finally, eliminating β_2, substituting (2.3), and expanding in powers of ω gives a set of polynomial equations for the coefficients $\{a_i, b_i, c_i, d_i, e_0\}$ and s.

Solving this system of polynomial equations and using (3.2) and (3.3) then determines α, β, η^2, and s. It turns out that this polynomial system, unlike the one found for the $y^2 = x^6$ type singularity in [1], can be solved completely in a reasonable amount of time on a desktop computer. The method we employed was simply judicious elimination of variables using (at worst) resultants of pairs of polynomials. This resulted in a large (about 10^3 node) tree of possibilities. (The interested reader can request a copy of a *Mathematica*™ notebook with the computation from the authors.)

Table 1 lists the solutions, $\eta^2 = \alpha\phi$, of the integrability equation together with their associated value of $[v]$. In addition to the curves in this table, there are also formal solutions found with $[v] = -8$, -5, $4/5$, 1, $4/3$, $3/2$, 2, 3, 4, ∞, and one with arbitrary $[v]$. The ones with

#	$[v]$	ϕ	α
1√	1	$\xi^5 + \tau_1\xi^3 + \tau_2\xi^2 + \tau_3\xi + 1$	1
2	8/7	$128(9\xi-10)^4(9\xi-5) + 1440(9\xi-10)^3\omega$ $+ 1620(9\xi-10)^2\omega^2 + 729(9\xi-10)\omega^3$	ω^{-2}
3	5/4	$\xi^5 + \omega$	$\omega^{-1/5}$
4	5/4	$625(9\xi-10)^4(9\xi-5) + 7500(9\xi-10)^3\omega$ $+ 1620(9\xi-10)^2\omega^2 + 7290(9\xi-10)\omega^3 + 2187\omega^4$	ω^{-2}
5	4/3	$\xi(\xi^4 + \omega)$	$\omega^{-1/4}$
6	4/3	$16(9\xi-10)^4(9\xi-5) + 30(9\xi-10)^2(18\xi-35)\omega$ $- 225(54\xi-55)\omega^2$	ω^{-1}
7	4/3	$8(9\xi-10)^4(9\xi-5) + 60(9\xi-10)^2(18\xi-17)\omega$ $+ 45(27\xi-29)\omega^2$	ω^{-1}
8√	10/7	$\xi^5 + 5\xi - 4\omega$	1
9√	8/5	$\xi(\xi^4 + 4\xi - 3\omega)$	1
10	5/3	$\xi^5 + \omega^2$	$\omega^{-2/5}$
11	40/21	$\xi^5 + \omega(5\xi - 4\omega)$	$\omega^{-1/4}$
12	2	$\xi(\xi^4 + \omega^2)$	$\omega^{-1/2}$
13√	20/9	$\xi^5 + 10\xi^2\omega + 15\xi\omega^2 + 6\omega^2$	$\omega^{-2/5}$
14√	12/5	$\xi(\xi^4 + \omega(4\xi - 3\omega))$	$\omega^{-1/3}$
15√	12/5	$(\xi^2 + 2\omega)(\xi^3 + 3\xi\omega + 2\omega)$	$\omega^{-1/3}$
16	5/2	$\xi^5 + \omega^3$	$\omega^{-3/5}$
17√	5/2	$\xi^5 + (5\xi - 3\omega)^2$	1
18	20/7	$\xi^5 + \omega^2(5\xi - 4\omega)$	$\omega^{-1/2}$
19√	15/4	$\xi^5 + \omega(5\xi - 3\omega)^2$	$\omega^{-1/3}$
20	4	$\xi(\xi^4 + \omega^3)$	$\omega^{-3/4}$
21√	4	$\xi[\xi^4 + \tau\xi^2(\xi - \omega/2) + (\xi - \omega/2)^2]$	1
22√	24/5	$\xi(\xi^4 + \omega^2(4\xi - 3\omega))$	$\omega^{-2/3}$
23	5	$\xi^5 + \omega^4$	$\omega^{-4/5}$
24	40/7	$\xi^5 + \omega^3(5\xi - 4\omega)$	$\omega^{-3/4}$
25√	15/2	$\xi^5 + \omega^2(5\xi - 3\omega)^2$	$\omega^{-2/3}$
26√	8	$\xi(\xi^4 + \omega(2\xi - \omega)^2)$	$\omega^{-1/2}$
27√	10	$\xi^5 + (5\xi - 2\omega)^3$	1
28√	20	$\xi^5 + \omega(5\xi - 2\omega)^3$	$\omega^{-1/2}$

Table 1: Potentially physical solutions the integrability equation for curves $\eta^2 = \alpha\phi$ of $y^2 = x^5$ type singularity. A check next to the row number means the corresponding solution passes the Z-consistency condition at $v = 0$.

$[v] = 4/5,\ 1,\ 4/3,\ 3/2,\ 2,\ 3,\ 4$, as well as the curve with arbitrary $[v]$, were discarded because they do not resolve the singularity on the Coulomb branch (*i.e.*, they remained singular for all u and v); Seiberg-Witten curves have a well-defined physical interpretation describing an $N = 2$ U(1)^2 low energy effective action only when they are non-singular. The solutions with $[v]$ = -8, -5, and ∞ were discarded simply because they do not have a consistent scaling interpretation.

4. Z-consistency condition

It is necessary to test the physicality of remaining solutions by applying the condition that every singularity is accompanied by vanishing Z (for some choice of U(1)^2 electric and magnetic charges). This condition follows from the fact that $|Z|$ gives a lower limit on the mass of charged states and that singularities occur when charged states become massless; for more detail see [2]. These singularities occur in three ways.

Singularities along $v = 0$. Only the checked curves in table 1 pass the Z-consistency test along the submanifold $v = 0$ emanating from the singularity at the origin. They pass it either by simply having no singularity at $v = 0$ when $u \neq 0$, or because Z indeed vanishes there for some choice of integration contour. The curves which fail this check categorized into two groups. The first group (curves numbered 3, 5, 10, 11, 12, 16, 18, 20, 23, and 24) have the form $\eta^2 = \omega^{-j/(5-k)}[\xi^5 + c\omega^j\xi^k + \ldots]$ where ξ^k is the highest power of ξ appearing in the curve after ξ^5. In all these curves, the singularity at $\omega = 0$ occurs at $\xi = 0$ on the curve. The central charge can then be approximated around $\omega = 0$ by

$$Z \sim \omega^{-j/(5-k)} \oint dx(\omega^{j/(5-k)}x - \omega)(x^5 + \ldots)^{-1/2}, \tag{4.1}$$

after making the change of variables $x = \omega^{j/(k-5)}\xi$. The leading ω-dependence then exactly cancels, leaving Z finite as $\omega \to 0$.

The second group (curves numbered 2, 4, 6, and 7) fails because of the property that at $\omega = 0$, the singularity occurs at $\xi \neq 0$. This removes the positive contribution from the numerator of the integrand, which prevents the central charge from vanishing. This is identical to what happens in the next section when $\xi_0 \neq \omega_0$ (as it should since we could always use our holomorphic reparameterization freedom to shift ω and ξ to move the singularity away from $\omega = 0$).

Singularities along $v \sim u^r$. Along with singularities at $\omega = 0$, it is possible for singularities to occur at finite $\omega = \omega_0$. As $\omega \to \omega_0$ branch points of ϕ may collide at $\xi = \xi_0$ so that $\phi \sim (\xi - \xi_0)^{2+n}\widetilde{\phi}$ where $\widetilde{\phi}$ is nonsingular and n is a some non-negative integer. The value of Z can be analyzed around (ξ_0, ω_0) by making the change of variables $\xi = \xi_0 + y^{1/2}x$, $\omega = \omega_0 + y$, and taking $y \to 0$. Upon making this transformation,

$$Z \sim y^{-n/4} \oint dx(\xi_0 - \omega_0 + y^{1/2}x - y)\widetilde{\phi}^{-1/2}, \tag{4.2}$$

which generically remains finite or even diverges as $y \to 0$. However if $\xi_0 = \omega_0$ an extra factor of $y^{1/2}$ comes from the numerator of the integrand and $Z \sim y^{(2-n)/4}$. In order for the exponent to be positive we must have $n = 0$ or 1. Examination of the checked solutions in table 1 shows that all have the properties $\xi_0 = \omega_0$ and $n = 0$, and therefore all pass this check.

Singularities along $u = 0$**.** Finally, we must check the Z-consistency condition at any singularities along the $u = 0$ submanifold, which corresponds to $\omega = \infty$ in dimensionless variables. All but two of the checked curves in table 1 have the form $\eta^2 = \omega^{-j/(5-k)}[\xi^5 + c\omega^j \xi^k + \ldots + c'\omega^\ell \xi^m]$ where ξ^k is the highest power of ξ occurring (after ξ^5) and ξ^m is the lowest. Changing variables to $x = \omega^{\ell/(m-5)}\xi$, gives at large ω

$$Z = \frac{u^{rs}}{rs}\omega^{\frac{j}{2(5-k)}+1-\frac{3\ell}{2(5-m)}} \oint dx(x^5 + \ldots)^{-1/2}. \tag{4.3}$$

Recall, that at fixed v, $\omega \sim u^{-r}$. Therefore, the exponent of u in the previous expression is $s - \frac{j}{2(5-k)} - 1 + \frac{3\ell}{2(5-m)}$. Remarkably, for all of these curves this exponent vanishes, implying that Z remains finite as $u \to 0$. Thus all these curves fail the Z-consistency test at $u = 0$ unless they happen to have no singularity along the $u = 0$ submanifold. Whether the curves are singular or not at $u = 0$ is controlled by whether the term $\omega^\ell \xi^m$ with the lowest power of ξ vanishes there or not. Reintroducing dimensionful quantities, this term reads $u^{5r-5}\omega^\ell \xi^m \sim u^{r(5-\ell-m)-(5-m)} v^\ell x^m$, so r must assume the value

$$r = \frac{5-m}{5-\ell-m} \tag{4.4}$$

for the curve not to be singular at $u = 0$. This condition selects a single value of r for each curve. But existence of a scaling limit implies also that $r < 1$ [1]. This is satisfied for this value of r for all the curves except curve number 1 (with $[v] = 1$).

The two checked curves which do not follow the above pattern are numbers 13 and 15. (Their prefactor α is not proportional to $\omega^{-j/(5-k)}$.) Nevertheless, the same argument works for these curves: Z is non-zero at $u = 0$, and only for a single value of r are they non-singular there.

The resulting list of the 13 solutions which pass the Z-consistency test is given below in table 2. Here we have recorded the values of $[v]$ and $[u]$, put back in the explicit u and v dependence in the curve, and used the holomorphic reparametrization rescaling and shift freedom to simplify the form of the curves where possible.

5. Discussion

All of these curves have $[u]$ and $[v]$ real and greater than one, consistent with an interpretation as interacting $N = 2$ superconformal fixed points. Only one, the curve with $[v] = 4$ and $[u] = 2$ has a dimensionless coupling τ. This curve in fact corresponds to the known curve [6] of the Sp(2) $\simeq$ SO(5) scale invariant gauge theory with either six massless hypermultiplets in the **4** or three in the **5** of the gauge group. (Both theories' effective actions are described by the same curve in the massless limit, an occurrence which is known to occur in other theories

Dimensions $[v]$	$[u]$	Curve $y^2 = \ldots$	
10/7	8/7		$[x^5 + (ux + v)]$
8/5	6/5		$[x^5 + x(ux + v)]$
20/9	10/9	$v^{-2/5}$	$[x^5 + v(5ux^2 - 15vx - 6vu)]$
12/5	6/5	$v^{-1/3}$	$[x^5 + vx(2ux + 3v)]$
12/5	6/5	$v^{-1/3}$	$[x^2 - 4v][x^3 - 2v(3x + 2u)]$
5/2	3/2		$[x^5 + (ux + v)^2]$
15/4	3/2	$v^{-1/3}$	$[x^5 + v(2ux + 3v)^2]$
4	2		$[x^5 + \tau x^3(ux + v) + x(ux + v)^2]$
24/5	6/5	$v^{-2/3}$	$[x^5 + v^2 x(ux + 3v)]$
15/2	3/2	$v^{-2/3}$	$[x^5 + v^2(ux + 3v)^2]$
8	2	$v^{-1/2}$	$[x^5 + vx(ux + 2v)^2]$
10	4		$[x^5 + (ux + v)^3]$
20	4	$v^{-1/2}$	$[x^5 + v(ux + 2v)^3]$

Table 2: Solutions of the integrability equation and Z-consistency condition for $y^2 = x^5$ singularities. For all curves the basis of holomorphic one-forms is $\omega_u = xdx/y$ and $\omega_v = dx/y$.

[2].) The weak coupling limit occurs as $\tau \to \pm 2$, in which case v and u can be identified with the adjoint casimirs for Sp(2), explaining their dimensions.

One other curve, the one with $[v] = 10/7$ and $[u] = 8/7$ has been found previously [7, 8, 9] by tuning parameters in higher-rank asymptotically free $N = 2$ theories. Presumably many other curves in table 2 can also be found in this way, but a systematic search along these lines is algebraically prohibitively complicated.

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