Construction of Configuration Space Kähler Geometry from Symmetry Principles: part I

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Abstract

The construction of the configuration space geometry is considered from a purely group theoretic point of view. The basic hypothesis that G/H_i for given values of zero modes is an infinite-dimensional symmetric space with G-invariant Kähler metric. The task is to identify the groups G and H_i appearing in the coset decomposition $CH = \cup_i G/H_i$ as well as the isometry invariants and the zero modes (label i) and to derive detailed information about the symplectic form and Kähler metric of configuration space. If one could neglect the complications caused by the failure of the classical non-determinism of Kähler action, the construction of the configuration space geometry would reduce by $Diff^4$ invariance to the construction of the geometry on the boundary of CH that is in the set of 3-surfaces belonging to $\delta H = \delta M^4_+ \times CP_2$ (the moment of big bang physically).

The failure of the classical non-determinism forces to introduce two kinds of causal determinants (CDs). 7-D light like CDs are unions of the boundaries of future and past directed light cones in M^4 at arbitrary positions (also more general light like surfaces $X^7 = X^3_l \times CP_2$ might be considered). CH is a union of sectors associated with these 7-D CDs playing in a very rough sense the roles of big bangs and big crunches. The creation of pairs of positive and negative energy space-time sheets occurs at $X^3 \subset X^7$ in the sense that negative and positive energy space-time sheet meet at X^3. Also 3-D light like causal determinants $X^3_l \subset X^4$ must be introduced: elementary particle horizons provide a basic example of this kind of CDs.

What I call 7–3 duality can be seen as the analog of field particle duality. 7–3 duality states that the two causal determinants play a dual role in the construction of the theory and implies that 3-surfaces are effectively two-dimensional with respect to the CH metric in the sense that all relevant data about CH geometry is contained by the two-dimensional intersections of 7-D and 3-D CDs. This simplifies dramatically the formulas for configuration space Hamiltonians since they can be expressed as generalized Kähler magnetic or electric fluxes over these 2-surfaces.

The consistency with the vacuum degeneracy of the Kähler metric defined by Kähler action gives strong restriction on the maximal isometry group. By its metric 2-dimensionality $\delta M^4_\pm$ allows both conformal and symplectic structures. The isometry group turns out to be the group of canonical transformations of $\delta M^4_\pm \times CP_2$.

The metric two-dimensionality of the light cone boundary and the related infinite-dimensional groups of isometries (!), conformal transformations and canonical transformations play a decisive role in the complexification. Complexification corresponds to the corresponding operation for the conformal algebra associated with the radial coordinate of the light cone boundary. The conformal weights of the canonical generators are complex such that the real part is half-integer valued. The imaginary part of the conformal weight defines complexification. The algebra h in the standard decomposition $g = t + h$ defining symmetric space corresponds to the sub-algebra of canonical algebra with generators having integer value real part of the conformal and t to its complement at the point of configuration space, which is identified as as the unique maximum of the Kähler action for given values of zero modes.

One can identify infinite families of isometry invariants characterizing the topology, shape and size of 3-surface as well as classical Kähler magnetic fields in X^3. Also one can identify explicit representation for the configuration space counterparts of Hamiltonians of δH generating canonical isometries of configuration space. An explicit general form for the symplectic form, Kähler form and Kähler metric can be deduced in terms of generalized Kähler magnetic fluxes: the metric constructed has the canonical transformations of light cone boundary as its isometries. An alternative representation for the configuration space Hamiltonians is defined by the Kähler electric fluxes instead of magnetic fluxes. The hypothesis that these Hamiltonians are affinely related to the magnetic flux Hamiltonians for the absolute minima of Kähler action implies electric-magnetic duality generalizing self-duality of Euclidian Yang-Mills theory. The coefficients in the affine relation can depend on isometry invariants. The characteristic Lie-algebraic properties of symmetric spaces guarantee that canonical transformations act as isometries and that the metric is Ricci flat. The constructed metric has same general qualitative properties as that given by the Kähler function defined as an absolute minimum of Kähler action.

Super Kac-Moody *resp.* super-canonical symmetries associated with 3-D *resp.* 7-D light like CDs become microscopic symmetries and Poincare invariance is exact. The sub-spaces of the tangent space basis of CH defined by the super-canonical and super Kac-Moody algebras correspond to each other in 1-1 manner as quantal non-zero modes and classical zero modes. Despite the duality super-canonical symmetry differs in many respects from Kac-Moody symmetries of particle physics, which correspond to the conformal invariance associated with the modified Dirac action at 3-D CDs and correspond to the product of Euclidian translation group and electro-weak and color groups.

1 Introduction

The most general expectation is that configuration space can be regarded as a union of coset spaces which are infinite-dimensional symmetric spaces with Kähler structure: $C(H) = \cup_i G/H(i)$. Index i labels 3-topology and zero modes. The group G, which can depend on 3-surface, can be identified as a subgroup of diffeomorphisms of $\delta M^4_+ \times CP_2$ and H must contain as its subgroup a group, whose action reduces to $Diff(X^3)$ so that these transformations leave 3-surface invariant.

The task is to identify plausible candidate for G and to show that the tangent space of the configuration space allows Kähler structure, in other words that the Lie-algebras of G and $H(i)$ allow complexification. One must also identify the zero modes and construct integration measure for the functional integral in these degrees of freedom. Besides this one must deduce information about the explicit form of configuration space metric from symmetry considerations combined with the hypothesis that Kähler function is determined as absolute minimum of Kähler action.

It will be found that in the case of $M^4_+ \times CP_2$ Kähler geometry, or strictly speaking contact Kähler geometry, characterized by a degenerate Kähler form (Diff^4 degeneracy and plus possible other degeneracies) seems possible. Although it seems that this construction must be generalized by allowing all light like 7-surfaces $X^3_l \times CP_2$, at least those for which X^3_l is boundary of light-cone inside M^4_+ or M^4, with the physical interpretation differing dramatically from the original one, the original construction discussed in the sequel involves the most essential aspects of the problem.

1.1 General Coordinate Invariance and generalized quantum gravitational holography

The basic motivation for the construction of configuration space geometry is the vision that physics reduces to the geometry of classical spinor fields in the infinite-dimensional configuration space of 3-surfaces of $M^4_+ \times CP_2$ or of $M^4 \times CP_2$. Hermitian conjugation is the basic operation in quantum theory and its geometrization requires that configuration space possesses Kähler geometry. Kähler geometry is coded into Kähler function.

The original belief was that the four-dimensional general coordinate invariance of Kähler function reduces the construction of the geometry to that for the boundary of configuration space consisting of 3-surfaces on $\delta M^4_+ \times CP_2$, the moment of big bang. The proposal was that Kähler function $K(Y^3)$ could be defined as absolute minimum of so called Kähler action for the unique space-time surface $X^4(Y^3)$ going through given 3-surface Y^3 at $\delta M^4_+ \times CP_2$. For Diff^4 transforms of Y^3 at $X^4(Y^3)$ Kähler function would have the same value so that Diff^4 invariance and degeneracy would be the outcome.

This picture is however too simple.

1. The degeneracy of the absolute minima caused by the classical non-determinism of Kähler action however brings in additional delicacies, and it seems that the reduction to the light cone boundary which in fact corresponds to what has become known as quantum gravitational holography must be replaced with a construction involving more general light like 7-surfaces $X^3_l \times CP_2$.

2. It has also become obvious that the gigantic symmetries associated with $\delta M^4_+ \times CP_2$ manifest themselves as the properties of propagators and vertices, and that M^4 is favored over M^4_+. Cosmological considerations, Poincare invariance, and the new view about energy favor the decomposition of the configuration space to a union of configuration spaces associated with various 7-D causal determinants . The minimum assumption is that all possible unions of future and past light cone boundaries $\delta M^4_\pm \times CP_2 \subset M^4 \times CP_2$ label the sectors of CH: the nice feature of this option is that the considerations of this chapter restricted to $\delta M^3_+ \times CP_2$ generalize almost trivially. This option is beautiful because the center of mass degrees of freedom associated with the different sectors of CH would correspond to M^4 itself and its Cartesian powers. One cannot exclude the possibility that even more general light like surfaces $X^3_l \times CP_2$ of M^4 are important as causal determinants.

The definition of the Kähler function requires that the many-to-one correspondence $X^3 \rightarrow X^4(X^3)$ must be replaced by a bijective correspondence in the sense that X^3 is unique among all its Diff^4 translates. This also allows physically preferred "gauge fixing" allowing to get rid of the mathematical complications due to Diff4 degeneracy. The internal geometry of the space-time sheet $X^4(X^3)$ must define the preferred 3-surface X^3 and also a preferred light like 7-surface $X^3_l \times CP_2$.

This is indeed possible. The possibility of negative Poincare energies inspires the hypothesis that the total quantum numbers and classical conserved quantities of the Universe vanish. This view is consistent with experimental facts if gravitational energy is defined as a difference of Poincare energies of positive and negative energy matter. Space-time surface consists of pairs of positive and negative energy space-time sheets created at some moment from vacuum and branching at that moment. This allows to select X^3 uniquely and define $X^4(X^3)$ as the absolute minimum of Kähler action in the set of 4-surfaces going through X^3. These space-time sheets should also define uniquely the light like 7-surface $X^3_l \times CP_2$, most naturally as the "earliest" surface of this kind. Note that this means that it become possible to assign a unique value of geometric time to the space-time sheet.

The realization of this vision means a considerable mathematical challenge. The effective metric 2-dimensionality of 3-dimensional light-like surfaces X^3_l of M^4 implies generalized conformal and canonical invariances allowing to generalize quantum gravitational holography from light like boundary so that the complexities due to the non-determinism can be taken into account properly.

1.2 Light like 3-D causal determinants, 7-3 duality, and effective 2-dimensionality

Thanks to the non-determinism of Kähler action, also light like 3-surfaces X^3_l of space-time surface appear as causal determinants (CDs). Examples are boundaries and elementary particle horizons at which Minkowskian signature of the induced metric transforms to Euclidian one. This brings in a second conformal symmetry related to the metric 2-dimensionality of the 3-D CD. This symmetry is identifiable as TGD counterpart of the Kac Moody symmetry of string models. The challenge is to understand the relationship of this symmetry to configuration space geometry and the interaction between the two conformal symmetries.

The possibility of spinorial shock waves at X^3_l leads to the hypothesis that they correspond to particle aspect of field particle duality whereas the physics in the interior of space-time corresponds to field aspect. More generally, field particle duality in TGD framework states that 3-D light like CDs and 7-D CDs are dual to each other. In particular, super-canonical and Super Kac Moody symmetries are also dually related.

The underlying reason for 7–3 duality could be understood from a simple geometric picture in which 3-D light like CDs X^3_l intersect 7-D CDs X^7 along 2-D surfaces X^2 and thus form 2-sub-manifolds of the space-like 3-surface $X^3 \subset X^7$. One can regard either canonical deformations of

X^7 or Kac-Moody deformations of X^2 as defining the tangent space of configuration space so that 7–3 duality would relate two different coordinate choices for CH.

The assumption that the data at either X^3 or X_l^3 are enough to determine configuration space geometry implies that the relevant data is contained to their intersection X^2. This is the case if the deformations of X_l^3 not affecting X^2 and preserving light likeness corresponding to zero modes or gauge degrees of freedom and induce deformations of X^3 also acting as zero modes. The outcome is effective 2-dimensionality. One cannot over-emphasize the importance of this conclusion. It indeed stream lines dramatically the earlier formulas for configuration space metric involving 3-dimensional integrals over $X^3 \subset M_+^4 \times CP_2$ reducing now to 2-dimensional integrals. Most importantly, no data about absolute minima of Kähler are needed to construct the configuration space metric so that the construction is also practical.

The reduction of data to that associated with 2-D surfaces conforms with the number theoretic vision about imbedding space as having hyper-octonionic structure [E2]: the commutative sub-manifolds of H have dimension not larger than two and for them tangent space is complex sub-space of complexified octonion tangent space. Number theoretic counterpart of quantum measurement theory forces the reduction of relevant data to 2-D commutative sub-manifolds of X^3. These points are discussed in more detail in the next chapter whereas in this chapter the consideration will be restricted to $X_l^3 = \delta M_+^4$ case which involves all essential aspects of the problem.

1.3 Magic properties of light cone boundary and isometries of configuration space

The special conformal, metric and symplectic properties of the light cone of four-dimensional Minkowski space: δM_+^4, the boundary of four-dimensional light cone is metrically 2-dimensional(!) sphere allowing infinite-dimensional group of conformal transformations and isometries(!) as well as Kähler structure. Kähler structure is not unique: possible Kähler structures of light cone boundary are parametrized by Lobatchevski space $SO(3,1)/SO(3)$. The requirement that the isotropy group $SO(3)$ of S^2 corresponds to the isotropy group of the unique classical 3-momentum assigned to $X^4(Y^3)$ defined as absolute minimum of Kähler action, fixes the choice of the complex structure uniquely. Therefore group theoretical approach and the approach based on Kähler action complement each other.

The allowance of an infinite-dimensional group of isometries isomorphic to the group of conformal transformations of 2-sphere is completely unique feature of the 4-dimensional light cone boundary. Even more, in case of $\delta M_+^4 \times CP_2$ the isometry group of δM_+^4 becomes localized with respect to CP_2! Furthermore, the Kähler structure of δM_+^4 defines also symplectic structure.

Hence any function of $\delta M_+^4 \times CP_2$ would serve as a Hamiltonian transformation acting in both CP_2 and δM_+^4 degrees of freedom. These transformations obviously differ from ordinary local gauge transformations. This group leaves the symplectic form of $\delta M_+^4 \times CP_2$, defined as the sum of light cone and CP_2 symplectic forms, invariant. The group of canonical transformations of $\delta M_+^4 \times CP_2$ is a good candidate for the isometry group of the configuration space.

The approximate canonical invariance of Kähler action is broken only by gravitational effects and is exact for vacuum extremals. This suggests that Kähler function is in a good approximation invariant under the canonical transformations of CP_2 would mean that CP_2 canonical transformations correspond to zero modes having zero norm in the Kähler metric of configuration space.

The groups G and H, and thus configuration space itself, should inherit the complex structure of the light cone boundary. The diffeomorphims of M^4 act as dynamical symmetries of vacuum extremals. The radial Virasoro localized with respect to $S^2 \times CP_2$ could in turn act in zero modes perhaps inducing conformal transformations: note that these transformations lead out from the symmetric space associated with given values of zero modes.

1.4 Canonical transformations of $\delta M^4_+ \times CP_2$ as isometries of configuration space

The canonical transformations of $\delta M^4_+ \times CP_2$ are excellent candidates for inducing canonical transformations of the configuration space acting as isometries. There are however deep differences with respect to the Kac Moody algebras.

1. The conformal algebra of the configuration space is gigantic when compared with the Virasoro + Kac Moody algebras of string models as is clear from the fact that the Lie-algebra generator of a canonical transformation of $\delta M^4_+ \times CP_2$ corresponding to a Hamiltonian which is product of functions defined in δM^4_+ and CP_2 is sum of generator of δM^4_+-local canonical transformation of CP_2 and CP_2-local canonical transformations of δM^4_+. This means also that the notion of local gauge transformation generalizes.

2. The physical interpretation is also quite different: the relevant quantum numbers label the unitary representations of Lorentz group and color group, and the four-momentum labelling the states of Kac Moody representations is not present. Physical states carrying no energy and momentum at quantum level are predicted. The appearance of a new kind of angular momentum not assignable to elementary particles might shed some light to the longstanding problem of baryonic spin (quarks are not responsible for the entire spin of proton). The possibility of a new kind of color might have implications even in macroscopic length scales.

3. The central extension induced from the natural central extension associated with $\delta M^4_+ \times CP_2$ Poisson brackets is anti-symmetric with respect to the generators of the canonical algebra rather than symmetric as in the case of Kac Moody algebras associated with loop spaces. At first this seems to mean a dramatic difference. For instance, in the case of CP_2 canonical transformations localized with respect to δM^4_+ the central extension would vanish for Cartan algebra, which means a profound physical difference. For $\delta M^4_+ \times CP_2$ canonical algebra a generalization of the Kac Moody type structure however emerges naturally.

   The point is that δM^4_+-local CP_2 canonical transformations are accompanied by CP_2 local δM^4_+ canonical transformations. Therefore the Poisson bracket of two δM^4_+ local CP_2 Hamiltonians involves a term analogous to a central extension term symmetric with respect to CP_2 Hamiltonians, and resulting from the δM^4_+ bracket of functions multiplying the Hamiltonians. This additional term could give the entire bracket of the configuration space Hamiltonians at the maximum of the Kähler function where one expects that CP_2 Hamiltonians vanish and have a form essentially identical with Kac Moody central extension because it is indeed symmetric with respect to indices of the canonical group.

1.5 Symmetric space property reduces to conformal and canonical invariance

The idea about symmetric space is extremely beautiful but it millenium had to change before I was ripe to identify the precise form of the Cartan decomposition. The solution of the puzzle turned out to be amazingly simple.

The inspiration came from the finding that quantum TGD leads naturally to an extension of Super Algebras by combining Ramond and Neveu-Schwartz algebras into single algebra. This led to the introduction Virasoro generators and generators of canonical algebra of CP_2 localized with respect to the light cone boundary and carrying conformal weights with a half integer valued real part. Soon came the realization that the conformal weights $h = -1/2 - i\sum_i y_i$, where $z_i = 1/2 + y_i$ are non-trivial zeros of Riemann Zeta, are excellent candidates for the conformal weights. It took some time to answer affirmatively the question whether also the negatives of the trivial zeros

$z = -2n$, $n > 0$ should be included. Thus the conjecture inspired by the work with Riemann hypothesis stating that the zeros of Riemann Zeta appear at the level of basic quantum TGD gets strong support.

The generators whose commutators define the basis of the entire algebra have conformal weights given by the negatives of the zeros of Rieman Zeta. The algebra is a direct sum $g = g_1 \oplus g_2$ such that g_1 has $h = n$ as conformal weights and g_2 $h = n - 1/2 + iy$, where y is sum over imaginary parts y_i of non-trivial zeros of Zeta. Only $h = 2n$, $n > 1$, and $h = -1/2 - iy + n$, such that n is even (odd) if y is sum of odd (even) number of y_i correspond to the weights labelling the generators of t in the Cartan decomposition $g = h + t$. The resulting super-canonical algebra would quite well be christened as Riemann algebra.

The requirement that ordinary Virasoro and Kac Moody generators annihilate physical states corresponds now to the fact that the generators of h vanish at the point of configuration space, which remains invariant under the action of h. The maximum of Kähler function corresponds naturally to this point and plays also an essential role in the integration over configuration space by generalizing the Gaussian integration of free quantum field theories.

The light cone conformal invariance differs in many respects from the conformal invariance of string theories. Finite-dimensional Kac-Moody group is replaced by an infinite-dimensional canonical group. Conformal weights could correspond to zeros of Riemann zeta and suitable superpositions of them in case of trivial zeros, and physical states can have non-vanishing but real conformal weights just as the representations of color group in CP_2 can have non-vanishing color isospin and hyper charge. The conformal weights have also interpretation as quantum numbers associated with unitary representations of Lorentz group: thus there is no conflict between conformal invariance and Lorentz invariance in TGD framework. Complex conformal weights however correspond to complex values of mass squared and super-conformal invariance for physical plays fundamental role in string models. This suggest that 7-3-duality could in TGD framework translate to the statement that the sums of super-canonical and Super Kac-Moody type super-conformal generators annihilate the physical states. This would generalize Goddard-Olive-Kent construction [19].

1.6 Attempts to identify configuration space Hamiltonians

I have made several attempts to identify configuration space Hamiltonians. The first two candidates referred to as magnetic and electric Hamiltonians, emerged in a relatively early stage. The third candidate represents the recent view based on the the formulation of quantum TGD using 3-D light-like surfaces identified as orbits of partons.

1.6.1 Magnetic Hamiltonians

Assuming that the elements of the radial Virasoro algebra of δM^4_+ have zero norm, one ends up with an explicit identification of the symplectic structures of the configuration space. There is almost unique identification for the symplectic structure. Configuration space counterparts of $\delta M^4 \times CP_2$ Hamiltonians are defined by the generalized signed and and unsigned Kähler magnetic fluxes

$$Q_m(H_A, X^2) = Z \int_{X^2} H_A J \sqrt{g_2} d^2x \quad ,$$

$$Q_m^+(H_A, r_M) = Z \int_{X^2} H_A |J| \sqrt{g_2} d^2x \quad ,$$

$$J \equiv \epsilon^{\alpha\beta} J_{\alpha\beta} \quad .$$

H_A is CP_2 Hamiltonian multiplied by a function of coordinates of light cone boundary belonging to a unitary representation of the Lorentz group. Z is a conformal factor depending on canonical invariants. The symplectic structure is induced by the symplectic structure of CP_2.

The most general flux is superposition of signed and unsigned fluxes Q_m and Q_m^+.

$$Q_m^{\alpha,\beta}(H_A, X^2) = \alpha Q_m(H_A, X^2) + \beta Q_m^+(H_A, X^2) \ \ .$$

Thus it seems that symmetry arguments fix the form of the configuration space metric apart from the presence of a conformal factor Z multiplying the magnetic flux and the degeneracy related to the signed and unsigned fluxes.

The notion of 7–3-duality described in the introduction implies that the relevant data about configuration space geometry is contained by 2-D surfaces X^2 at the intersections of 3-D light like CDs (causal determinants) and 7-D CDs such as $M_+^4 \times CP_2$. In this case the entire Hamiltonian could be defined as the sum of magnetic fluxes over surfaces $X_i^2 \subset X^3$. The maximally optimistic guess would be that it is possible to fix both X_i^2 and 7-D CDs freely with X_i^2 possibly identified as commutative sub-manifold of hyper-octonionic H [E2].

1.6.2 Electric Hamiltonians and electric-magnetic duality

Absolute minimization of Kähler action in turn suggests that one can identify configuration space Hamiltonians as classical charges $Q_e(H_A)$ associated with the Hamiltonians of the canonical transformations of the light cone boundary, that is as variational derivatives of the Kähler action with respect to the infinitesimal deformations induced by $\delta M_+^4 \times CP_2$ Hamiltonians. Alternatively, one might simply replace Kähler magnetic field J with Kähler electric field defined by space-time dual $*J$ in the formulas of previous section. These Hamiltonians are analogous to Kähler electric charge and the hypothesis motivated by the experience with the instantons of the Euclidian Yang Mills theories and 'Yin-Yang' principle, as well as by the duality of CP_2 geometry, is that for the absolute minima of the Kähler action these Hamiltonians are affinely related:

$$Q_e(H_A) = Z\left[Q_m(H_A) + q_e(H_A)\right] \ \ .$$

Here Z and q_e are constants depending on canonical invariants only. Thus the equivalence of the two approaches to the construction of configuration space geometry boils down to the hypothesis of a physically well motivated electric-magnetic duality.

The crucial technical idea is to regard configuration space metric as a quadratic form in the entire Lie-algebra of the isometry group G such that the matrix elements of the metric vanish in the sub-algebra H of G acting as $Diff^3(X^3)$. The Lie-algebra of G with degenerate metric in the sense that H vector fields possess zero norm, can be regarded as a tangent space basis for the configuration space at point X^3 at which H acts as an isotropy group: at other points of the configuration space H is different. For given values of zero modes the maximum of Kähler function is the best candidate for X^3. This picture applies also in symplectic degrees of freedom.

1.6.3 Configuration space Hamiltonians as Noether charges for the partonic dynamics

The formulation of quantum TGD using light-like 3-surfaces identified as orbits of partons led to a third proposal in which Hamiltonians correspond to Noether charges being nearly identical with the magnetic Hamiltonians.

There are good reasons to regard this approach as the most plausible one. A rather detailed and rigorous understanding of various super-conformal symmetries and their mutual relationships emerges with a direct connection to p-adic mass calculations. The super-Hamiltonians identifiable as configuration space gamma matrices give rise to the matrix elements of the configuration space Kähler metric. The anti-commutation relations for second quantized induced spinor fields can be written explicitly and one ends up also to a relatively detailed view about the construction of S-matrix. The exponent of Kähler function is tentatively identified as an appropriately defined

Dirac determinant for the modified Dirac operator in the spirit of quantum gravitational holography implied by general coordinate invariance. Also number theoretic universality can be realized in this approach.

This identification is not considered in this chapter further since the discussion would require a detailed construction of partonic quantum dynamics. In [B4] this approach is discussed in more detail. It must be emphasized that this chapter provides a general conceptual framework allowing to follow the arguments of [B4] and is by no means obsolete stuff.

2 Identification of the isometry group

In this section the identification of the isometry group of the configuration space will be discussed at general level.

2.1 Reduction to the light cone boundary

The reduction to the light cone boundary would occur exactly if Kähler action were strictly deterministic. This is not the case but it seems possible to generalize the construction at light cone boundary to the general case.

The identification of the groups G and H follows as a consequence of 4-dimensional Diff invariance. The right question to ask is the following one. How could one coordinatize the physical(!) vibrational degrees of freedom for 3-surfaces in Diff^4 invariant manner: coordinates should have same values for all Diff^4 related 3-surfaces belonging to the orbit of X^3? The answer is following:

1. Fix some 3-surface (call it Y^3) on the orbit of X^3 in Diff^4 invariant manner.
2. Use as configuration space coordinates of X^3 and all its diffeomorphs the coordinates parameterizing small deformations of Y^3. This kind of replacement is physically acceptable since metrically the configuration space is equivalent with $Map/Diff^4$.
3. Require that the fixing procedure is Lorentz invariant, where Lorentz transformations in question leave light M^4_+ invariant and thus act as isometries.

The simplest choice of Y^3 is the intersection of the orbit of 3-surface (X^4) with the set $\delta M^4_+ \times CP_2$, where δM^4_+ denotes the boundary of the light cone (moment of big bang):

$$Y^3 \quad = \quad X^4 \cap \delta M^4_+ \times CP_2 \tag{1}$$

Lorentz invariance allows also the choice $X \times CP_2$, where X corresponds to the hyperboloid $a = \sqrt{(m^0)^2 - r_M^2} = constant$ but only the proposed choice ($a = 0$) leads to a natural complexification in M^4 degrees of freedom. This choice is also cosmologically very natural and completely analogous to the quantum gravitational holography of string theories.

Configuration space has a fiber space structure. Base space consists of 3-surfaces $Y^3 \subset \delta M^4_+ \times CP_2$ and fiber consists of 3-surfaces on the orbit of Y^3 , which are Diff^4 equivalent with Y^3. The distance between the surfaces in the fiber is vanishing in configuration space metric. An elegant manner to avoid difficulties caused by Diff^4 degeneracy in configuration space integration is to *define* integration measure as integral over the reduced configuration space consisting of 3-surfaces Y^3 on light cone boundary.

Situation is however quite not so simple. The vacuum degeneracy of Kähler action suggests strongly classical non-determinism so that there are several, possibly, infinite number of absolute minimum space-time surfaces $X^4(Y^3)$ associated with given Y^3 on light cone boundary. This implies additional degeneracy, One might hope that the reduced configuration space could be replaced by its covering space so that given Y^3 corresponds to several points of the covering space and configuration space has many-sheeted structure. Obviously the copies of Y^3 have identical geometric properties. Configuration space integral would decompose into a sum of integrals over different sheets of the reduced configuration space. Note that configuration space spinor fields are in general different on different sheets of the reduced configuration space.

Even this is probably not enough: it is quite possible that all light like surfaces of M^4 possessing Hamilton Jacobi structure (and thus interpretable as light fronts) are involved with the construction of the configuration space geometry. Because of their metric two-dimensionality the proposed construction should generalize. This would mean that configuration space geometry has also local laboratory scale aspects and that the general ideas might allow testing.

2.2 Identification of the coset space structure

In finite-dimensional context globally symmetric spaces are of form G/H and connection and curvature are independent of the metric, provided it is left invariant under G. The hope is that same holds true in infinite-dimensional context. The most one can hope of obtaining is the decomposition $C(H) = \cup_i G/H_i$ over orbits of G. One could allow also symmetry breaking in the sense that G and H depend on the orbit: $C(H) = \cup_i G_i/H_i$ but it seems that G can be chosen to be same for all orbits. What is essential is that these groups are infinite-dimensional. The basic properties of the coset space decomposition give very strong constraints on the group H, which certainly contains the subgroup of G, whose action reduces to diffeomorphisms of X^3.

2.2.1 Consequences of the decomposition

If the decomposition to a union of coset spaces indeed occurs, the consequences for the calculability of the theory are enormous since it suffices to find metric and curvature tensor for single representative 3-surface on a given orbit (contravariant form of metric gives propagator in perturbative calculation of matrix elements as functional integrals over the configuration space). The representative surface can be choosen to correspond to the maximum of Kähler function on a given orbit and one obtains perturbation theory around this maximum (Kähler function is not isometry invariant).

The task is to identify the infinite-dimensional groups G and H and to understand the zero mode structure of the configuration space. Only seven years after the discovery of the candidate for the Kähler function defining the metric, it became finally clear that these identifications follow quite nicely from $Diff^4$ invariance and $Diff^4$ degeneracy as well as special properties of the Kähler action.

2.2.2 Configuration space isometries as a subgroup of $Diff(\delta M^4_+ \times CP_2)$

The reduction to light cone boundary leads to the identification of the isometry group as some subgroup of for the group G for the diffeomorphisms of $\delta M^4_+ \times CP_2$. These diffeomorphisms indeed act in a natural manner in δCH, the the space of 3-surfaces in $\delta M^4_+ \times CP_2$. Configuration space is expected to decompose to a union of the coset spaces G/H_i, where H_i corresponds to some subgroup of G containing the transformations of G acting as diffeomorphisms for given X^3. Geometrically the vector fields acting as diffeomorphisms of X^3 are tangential to the 3-surface. H_i could depend on the topology of X^3 and since G does not change the topology of 3-surface each 3-topology defines separate orbit of G. Therefore, the union involves sum over all topologies of X^3 plus possibly other 'zero modes'. Different topologies are naturally glued together since singular 3-surfaces intermediate between two 3-topologies correspond to points common to the two sectors with different topologies.

2.2.3 Coset space structure for a symmetric space

The key ingredient in the theory of symmetric spaces is that the Lie-algebra of G has the following decomposition

$$g = h + t \ ,$$
$$[h,h] \subset h \ , \quad [h,t] \subset t \ , \quad [t,t] \subset h \ .$$

In present case this has highly nontrivial consequences. The commutator of *any* two infinitesimal generators generating nontrivial deformation of 3-surface belongs to h and thus vanishing norm in the configuration space metric at the point which is left invariant by H. In fact, this same condition follows from Ricci flatness requirement and guarantees also that G acts as isometries of the configuration space. These conditions can be satisfied if one extends Super Algebras to contain also generators having half-odd integer valued conformal weight and symmetric space property states nothing but the conformal invariance of the theory. This generalization is supported by the properties of the unitary representations of Lorentz group at the light cone boundary and by number theoretical considerations.

2.3 Isometries of configuration space geometry as canonical transformations of $\delta M^4_+ \times CP_2$

During last decade I have considered several candidates for the group G of isometries of the configuration space as the sub-algebra of the subalgebra of $diff(\delta M^4_+ \times CP_2)$. To begin with let us write the general decomposition of $diff(\delta M^4_+ \times CP_2)$:

$$diff(\delta M^4_+ \times CP_2) \quad = \quad S(CP_2) \times diff(\delta M^4_+) \oplus S(\delta M^4_+) \times diff(CP_2) \ . \tag{2}$$

Here $S(X)$ denotes the scalar function basis of space X. This Lie-algebra is the direct sum of light cone diffeomorphisms made local with respect to CP_2 and CP_2 diffeomorphisms made local with respect to light cone boundary.

The idea that entire diffeomorphism group would act as isometries looks unrealistic since the theory should be more or less equivalent with topological field theory in this case. Consider now the various candidates for G.

1. The fact that canonical transformations of CP_2 and M^4_+ diffeomorphisms are dynamical symmetries of the vacuum extremals suggests the possibility that the diffeomorphisms of the light cone boundary and canonical transformations of CP_2 could leave Kähler function invariant and thus correspond to zero modes. The canonical transformations of CP_2 localized with respect to light cone boundary acting as canonical transformations of CP_2 have interpretation as local color transformations and are a good candidate for the isometries. The fact that local color transformations are not even approximate symmetries of Kähler action is not a problem: if they were exact symmetries, Kähler function would be invariant and zero modes would be in question.

2. CP_2 local conformal transformations of the light cone boundary act as isometries of δM^4_+. Besides this there is a huge group of the canonical symmetries of $\delta M^4_+ \times CP_2$ if light cone boundary is provided with the symplectic structure. Both groups must be considered as candidates for groups of isometries. $\delta M^4_+ \times CP_2$ option exploits fully the special properties of $\delta M^4_+ \times CP_2$, and one can develop simple argument demonstrating that $\delta M^4_+ \times CP_2$ canonical invariance is the correct option. Also the construction of configuration space gamma matrices as super-canonical charges supports $\delta M^4_+ \times CP_2$ option.

To sum up, the most realistic identification of the isometries is as the $\delta M^4_+ \times CP_2$ canonical transformations and identifiable as local color transformations accompanying also deformation of 3-surfaces. The conformal transformations of the light cone boundary would act as transformations affecting zero modes and possibly inducing conformal scaling of the configuration space metric.

3 Complexification

3.1 Why complexification is needed?

A necessary prerequisite for the Kähler geometry is the complexification of the tangent space in vibrational degrees of freedom. The Minkowskian signature of M^4 metric seems however to represent an insurmountable obstacle for the complexification of M^4 type vibrational degrees of freedom. On the other hand, complexification seems to have deep roots in the actual physical reality.

1. In the perturbative quantization of gauge fields one associates to each gauge field excitation polarization vector e and massless four-momentum vector p ($p^2 = 0$, $p \cdot e = 0$). These vectors define the decomposition of the tangent space of M^4: $M^4 = M^2 \times E^2$, where M^2 type polarizations correspond to zero norm states and E^2 type polarizations correspond to physical states with non-vanishing norm. Same type of decomposition occurs also in the linearized theory of gravitation. The crucial feature is that E^2 allows complexification! The general conclusion is that the modes of massless, linear, boson fields define always complexification of M^4 (or its tangent space) by effectively reducing it to E^2. Also in string models similar situation is encountered. For a string in D-dimensional space only D-2 transversal Euclidian degrees of freedom are physical.

2. Since symplectically extended isometry generators are expected to create physical states in TGD approach same kind of physical complexification should take place for them, too: this indeed takes place in string models in critical dimension. Somehow one should be able to associate polarization vector and massless four momentum vector to the deformations of a given 3-surface so that these vectors define the decomposition $M^4 = M^2 \times E^2$ for each mode. Configuration space metric should be degenerate: the norm of M^2 deformations should vanish as opposed to the norm of E^2 deformations.

Consider now the implications of this requirement.

1. In order to associate four-momentum and polarization (or at least the decomposition $M^4 = M^2 \times E^2$) to the deformations of the 3-surface one should have field equations, which determine the time development of the 3-surface uniquely. Furthermore, the time development for small deformations should be such that it makes sense to associate four momentum and polarization or at least the decomposition $M^4 = M^2 \times E^2$ to the deformations in suitable basis.

   The solution to this problem is afforded by the proposed definition of the Kähler function. The definition of the Kähler function indeed associates to a given 3-surface a unique four-surface as the absolute minimum of the Kähler action. Therefore one can associate a unique time development to the deformations of the surface X^3 and if TGD describes the observed world this time development should describe the evolution of photon, gluon, graviton, etc. states and so we can hope that tangent space complexification could be defined.

2. We have found that M^2 part of the deformation should have zero norm. In particular, the time like vibrational modes have zero norm in configuration space metric. This is true if

Kähler function is not only $Diff^3$ invariant but also Diff^4 invariant in the sense that Kähler function has same value for all 3-surfaces belonging to the orbit of X^3 and related to X^3 by diffeomorphism of X^4. This is indeed the case.

3. Even this is not enough. One expects the presence of massive modes having also longitudinal polarization and for these states the number of physical vibrational degrees of freedom is 3 so that complexification seems to be impossible by odd dimension.

The reduction to the light cone boundary implied by $Diff^4$ invariance makes possible to identify the complexification. Crucial role is played by the special properties of the boundary of 4-dimensional light cone, which is metrically two-sphere and thus allows generalized complex and Kähler structure.

3.2 The metric, conformal and symplectic structures of the light cone boundary

The special metric properties of the light cone boundary play a crucial role in the complexification. The point is that the boundary of the light cone has degenerate metric: although light cone boundary is topologically 3-dimensional it is metrically 2-dimensional: effectively sphere. In standard spherical Minkowski coordinates light cone boundary is defined by the equation $r_M = m^0$ and induced metric reads

$$ds^2 = -r_M^2 d\Omega^2 = -r_M^2 dz d\bar{z}/(1+z\bar{z})^2 \ , \tag{3}$$

and has Euclidian signature. Since S^2 allows complexification and thus also Kähler structure (and as a by-product also symplectic structure) there are good hopes of obtaining just the required type of complexification in non-degenerate M^4 degrees of freedom: configuration space would effectively inherit its Kähler structure from $S^2 \times CP_2$.

By its effective two-dimensionality the boundary of the four-dimensional light cone has infinite-dimensional group of (local) conformal transformations. Using complex coordinate z for S^2 the general local conformal transformation reads

$$\begin{aligned} r &\rightarrow f(r_M, z, \bar{z}) \ , \\ z &\rightarrow g(z) \ , \end{aligned} \tag{4}$$

where f is an arbitrary real function and g is an arbitrary analytic function with a finite number of poles. The infinitesimal generators of this group span an algebra, call it C, analogous to Virasoro algebra. This algebra is semidirect sum of two algebras L and R given by

$$\begin{aligned} C &= L \oplus R \ , \\ [L,R] &\subset R \ , \end{aligned} \tag{5}$$

where L denotes standard Virasoro algebra of the two- sphere generated by the generators

$$L_n = z^{n+1} d/dz \tag{6}$$

and R denotes the algebra generated by the vector fields

$$R_n = f_n(z,\bar{z},r_M)\partial_{r_M} \ , \tag{7}$$

where $f(z,\bar{z},r_M)$ forms complete real scalar function basis for light cone boundary. The vector fields of R have the special property that they have vanishing norm in M^4 metric.

This modification of conformal group implies that the Virasoro generator L_0 becomes $L_0 = zd/dz - r_M d/dr_M$ so that the scaling momentum becomes the difference $n-m$ or S^2 and radial scaling momenta. One could achieve conformal invariance by requiring that S^2 and radial scaling quantum numbers compensate each other.

Of crucial importance is that light cone boundary allows infinite dimensional group of isometries! An arbitrary conformal transformation $z \rightarrow f(z)$ induces to the metric a conformal factor given by $|df/dz|^2$. The compensating radial scaling $r_M \rightarrow r_M/|df/dz|$ compensates this factor so that the line element remains invariant.

The Kähler structure of light cone boundary defines automatically symplectic structure. The symplectic form is degenerate and just the area form of S^2 given by

$$J = r_M^2 sin(\theta) d\theta \wedge d\phi,$$

in standard spherical coordinates, there is infinite-dimensional group of canonical transformations leaving the symplectic form of the light cone boundary (that is S^2) invariant. These transformations are local with respect to the radial coordinate r_M. The symplectic and Kähler structures of light cone boundary are not unique: different structures are labelled by the coset space $SO(3,1)/SO(3)$. One can however associate with a given 3-surface Y^3 a unique structure by requiring that the the corresponding subgroup $SO(3)$ of Lorentz group acts as the isotropy group of the conserved classical four-momentum assigned to Y^3 by absolute minimization of Kähler action.

In case of $\delta M^4_+ \times CP_2$ both the conformal transformations, isometries and canonical transformations of the light cone boundary can be made local also with respect to CP_2. The idea that the infinite-dimensional algebra of canonical transformations of $\delta M^4_+ \times CP_2$ act as isometries of the configuration space and that radial vector fields having zero norm in the metric of light cone boundary possess zero norm also in configuration space metric, looks extremely attractive.

In the case of $\delta M^4_+ \times CP_2$ one could combine the symplectic and Kähler structures of δM^4_+ and CP_2 to single symplectic/Kähler structure. The canonical transformations leaving this symplectic structure invariant would be generated by the function algebra of $\delta M^4_+ \times CP_2$ such that a arbitrary function serves as a Hamiltonian of a canonical transformation. This group serves as a candidate for the isometry group of the configuration space. An alternative identification for the isometry algebra is as canonical symmetries of CP_2 localized with respect to the light cone boundary. Hamiltonians would be also now elements of the function algebra of $\delta M^4_+ \times CP_2$ but their Poisson brackets would be defined using only CP_2 symplectic form.

The problem is to decide which option is correct. There is a simple argument fixing the latter option. The canonically imbedded CP_2 would be left invariant under δM^4_+ local canonical transformations of CP_2. This seems strange. Under canonical algebra of $\delta M^4_+ \times CP_2$ also canonically imbedded CP_2 is deformed and this sounds more realistic. The isometry algebra is therefore assumed to be the group $can(\delta M^4_+ \times CP_2)$ generated by the scalar function basis $S(\delta M^4_+ \times CP_2) = S(\delta M^4_+) \times S(CP_2)$ of the light cone boundary using the Poisson brackets to be discussed in more detail later.

There are some no-go theorems associated with higher-dimensional Abelian extensions [16], and although the contexts are quite different, it is interesting to consider the recent situation in light of these theorems.

1. Conformal invariance is an essentially 2-dimensional notion. Light cone boundary is however metrically and conformally 2-sphere, and therefore the conformal algebra is effectively that

associated with the 2-sphere. In the same manner, the quaternion conformal algebra associated with the metrically 2-dimensional elementary particle horizons surrounding wormhole contacts allows the usual Kac Moody algebra and actually also contributes to the configuration space metric.

2. In dimensions $D > 2$ Abelian extensions of the gauge algebra are extensions by an infinite-dimensional Abelian group rather than central extensions by the group $U(1)$. This result has an analog at the level of configuration space geometry. The extension associated with the symplectic algebra of CP_2 localized with respect to the light cone boundary is analogous a symplectic extension defined by Poisson bracket $\{p, q\} = 1$. The central extension is the function space associated with δM^4_+ and indeed infinite-dimensional if only only CP_2 symplectic structure induces the Poisson bracket but one-dimensional if $\delta M^4_+ \times CP_2$ Poisson bracket induces the extension. In the latter case the symmetries fix the metric completely at the point corresponding to the origin of symmetric space (presumably the maximum of Kähler function for given values of zero modes).

3. $D > 2$ extensions possess no unitary faithful representations (satisfying certain well motivated physical constraints) [16]. It might be that the degeneracy of the configuration space metric is the analog for the loss of faithful representations.

3.3 Complexification and the special properties of the light cone boundary

In case of Kähler metric G and H Lie-algebras must allow complexification so that the isometries can act as holomorphic transformations. Since G and H can be regarded as subalgebras of the vector fields of $\delta M^4_+ \times CP_2$, they inherit in a natural manner the complex structure of the light cone boundary.

There are two candidates for the configuration space complexification. The simplest, and also the correct, alternative is that complexification is induced by natural complexification of vector field basis on $\delta M^4_+ \times CP_2$. In CP_2 degrees of freedom there is natural complexification

$$\xi \rightarrow \bar{\xi} \ .$$

In δM^4_+ degrees of freedom this could involve the transformation

$$z \rightarrow \bar{z}$$

and certainly involves complex conjugation for complex scalar function basis in the radial direction:

$$f(r_M) \rightarrow \overline{f(r_M)} \ ,$$

which turns out to play same role as the function basis of circle in the Kähler geometry of loop groups [17].

The requirement that the functions are eigen functions of radial scalings favors functions $(r_M/r_0)^k$, where k is in general a complex number. The function can be expressed as a product of real power of r_M and logarithmic plane wave. It turns out that the radial complexification alternative is the correct manner to obtain Kähler structure. The reason is that canonical transformations leave the value of r_M invariant. Radial Virasoro invariance plays crucial role in making the complexification possible.

One could consider also a second alternative assumed in the earlier formulation of the configuration space geometry. The close analogy with string models and conformal field theories suggests that for Virasoro generators the complexification must reduce to the hermitian conjugation of the conformal field theories: $L_n \rightarrow L_{-n} = L_n^\dagger$. Clearly this complexification is induced from the transformation $z \rightarrow \frac{1}{z}$ and differs from the complexification induced by complex conjugation $z \rightarrow \bar{z}$. The basis would be polynomial in z and $\bar{z}$. Since radial algebra could be also seen as Virasoro algebra localized with respect to $S^2 \times CP_2$ one could consider the possibility that also in radial direction the inversion $r_M \rightarrow \frac{1}{r_M}$ is involved.

The essential prerequisite for the Kähler structure is that both G and H allow same complexification so that the isometries in question can be regarded as holomorphic transformations. In finite-dimensional case this essentially what is needed since metric can be constructed by parallel translation along the orbit of G from H-invariant Kähler metric at a representative point. The requirement of H-invariance forces the radial complexification based on complex powers r_M^k: radial complexification works since canonical transformations leave r_M invariant.

Some comments on the properties of the proposed complexification are in order.

1. The proposed complexification, which is analogous to the choice of gauge in gauge theories is not Lorentz invariant unless one can fix the coordinates of the light cone boundary apart from $SO(3)$ rotation not affecting the value of the radial coordinate r_M (if the imaginary part of k in r_M^k is always non-vanishing). This is possible as will be explained later.

2. It turns out that the function basis of light-cone boundary multiplying CP_2 Hamiltonians corresponds to unitary representations of the Lorentz group at light cone boundary so that the Lorentz invariance is rather manifest.

3. There is a nice connection with the proposed physical interpretation of the complexification. At the moment of the big bang all particles move with the velocity of light and therefore behave as massless particles. To a given point of the light cone boundary one can associate a unique direction of massless four-momentum by semiclassical considerations: at the point $m^k = (m^0, m^i)$ momentum is proportional to the vector $(m^0, -m^i)$. Since the particles are massless only two polarization vectors are possible and these correspond to the tangent vectors to the sphere $m^0 = r_M$. Of course, one must always fix polarizations at some point of tangent space but since massless polarization vectors are not physical this doesn't imply difficulties: different choices correspond to different gauges.

4. Complexification in the proposed manner is not possible except in the case of four-dimensional Minkowski space. Non-zero norm deformations correspond to vector fields of the light cone boundary acting on the sphere S^{D-2} and the decomposition to $(1,0)$ and $(0,1)$ parts is possible only when the sphere in question is two-dimensional since other spheres do allow neither complexification nor Kähler structure.

3.4 How to fix the complex and symplectic structures in a Lorentz invariant manner?

One can assign to light-cone boundary a symplectic structure since it reduces effectively to S^2. The possible symplectic structures of δM_+^4 are parameterized by the coset space $SO(3,1)/SO(3))$, where H is the isotropy group $SO(3)$ of a time like vector. Complexification also fixes the choice of the spherical coordinates apart from rotations around the quantization axis of angular momentum.

The selection of some preferred symplectic structure in an ad hoc manner breaks manifest Lorentz invariance but is possible if physical theory remains Lorentz invariant. The more natural possibility is that 3-surface Y^3 itself fixes in some natural manner the choice of the symplectic structure so that there is unique subgroup $SO(3)$ of $SO(3,1)$ associated with Y^3. If configuration space Kähler function corresponds to absolute minimum of Kähler action, this is indeed the case. One can associate unique conserved four-momentum $P^k(Y^3)$ to the absolute minimum $X^4(Y^3)$ of the Kähler action and the requirement that the rotation group $SO(3)$ leaving the symplectic structure invariant leaves also $P^k(Y^3)$ invariant, fixes the symplectic structure associated with Y^3 uniquely.

Therefore configuration space decomposes into a union of symplectic spaces labelled by $SO(3,1)/SO(3)$ isomorphic to $a = constant$ hyperboloid of light cone. The direction of the classical angular momentum vector $w^k = \epsilon^{klmn} P_l J_{mn}$ determined by the classical angular momentum tensor of associated with Y^3 fixes one coordinate axis and one can require that $SO(2)$ subgroup of $SO(3)$ acting as rotation around this coordinate axis acts as phase transformation of the complex coordinate z of S^2. Other rotations act as nonlinear holomorphic transformations respecting the complex structure.

Clearly, the coordinates are uniquely fixed modulo $SO(2)$ rotation acting as phase multiplication in this case. If $P^k(Y^3)$ is light like, one can only require that the rotation group $SO(2)$ serving as the isotropy group of 3-momentum belongs to the group $SO(3)$ characterizing the symplectic structure and it seems that symplectic structure cannot be uniquely fixed without additional constraints in this case. Probably this has no practical consequences since the 3-surfaces considered have actually infinite size and 4-momentum is most probably time like for them. Note however that the direction of 3-momentum defines unique axis such that $SO(2)$ rotations around this axis are represented as phase multiplication.

Similar almost unique frame exists also in CP_2 degrees of freedom and corresponds to the complex coordinates transforming linearly under $U(2)$ acting as isotropy group of the Lie-algebra element defined by classical color charges Q_a of Y^3. One can fix unique Cartan subgroup of $U(2)$ by noticing that $SU(3)$ allows completely symmetric structure constants d_{abc} such that $R_a = d_a^{\ bc} Q_b Q_c$ defines Lie-algebra element commuting with Q_a. This means that R_a and Q_a span in generic case $U(1) \times U(1)$ Cartan subalgebra and there are unique complex coordinates for which this subgroup acts as phase multiplications. The space of nonequivalent frames is isomorphic with $CP(2)$ so that one can say that cm degrees of freedom correspond to Cartesian product of $SO(3,1)/SO(3)$ hyperboloid and CP_2 whereas coordinate choices correspond to the Cartesian product of $SO(3,1)/SO(2)$ and $SU(3)/U(1) \times U(1)$.

Canonical transformations leave the value of δM^4_+ radial coordinate r_M invariant and this implies large number of additional zero modes characterizing the size and shape of the 3-surface. Besides this Kähler magnetic fluxes through the $r_M = constant$ sections of X^3 as a function of r_M provide additional invariants, which are functions rather than numbers. The Fourier components for the magnetic fluxes provide infinite number of canonical invariants. The presence of these zero modes imply that 3-surfaces behave much like classical objects in the sense that neither their shape nor form nor classical Kähler magnetic fields, are subject to Gaussian fluctuations. Of course, quantum superpositions of 3-surfaces with different values of these invariants are possible.

There are reasons to expect that at least certain infinitesimal canonical transformations correspond to zero modes of the Kähler metric (canonical transformations act as dynamical symmetries of the vacuum extremals of the Kähler action). If this is indeed the case, one can ask whether it is possible to identify an integration measure for them.

If one can associate symplectic structure with zero modes, the symplectic structure defines integration measure in a standard manner (for 2n-dimensional symplectic manifold the integration measure is just the n-fold wedge power $J \wedge J ... \wedge J$ of the symplectic form J). Unfortunately, in infinite-dimensional context this is not enough since divergence free functional integral analogous to a Gaussian integral is needed and it seems that it is not possible to integrate in zero modes and that this relates in a deep manner to state function reduction. If all canonical transformations of $\delta M^4_+ \times CP_2$ are represented as symplectic transformations of the configuration space, then the existence of symplectic structure decomposing into Kähler (and symplectic) structure in complexified degrees of freedom and symplectic (but not Kähler) structure in zero modes, is an automatic consequence.

3.5 The general structure of the isometry algebra

There are three options for the isometry algebra of configuration space

1. Isometry algebra as the algebra of CP_2 canonical transformations leaving invariant the symplectic form of CP_2 localized with respect to δM^4_+.
2. Certainly the configuration space metric in δM^4_+ must be non-trivial and actually given by the magnetic flux Hamiltonians defining canonical invariants. Furthermore, the super-canonical generators constructed from quarks automatically give as anti-commutators this part of the configuration space metric. One could interpret these canonical invariants as configuration space Hamiltonians for δM^4_+ canonical transformations obtained when CP_2 Hamiltonian is constant.
3. Isometry algebra consists of $\delta M^4_+ \times CP_2$ canonical transformations. In this case a local color transformation involves necessarily a local S^2 transformation. Unfortunately, it is difficult to decide at this stage which of these options is correct.

The eigen states of the rotation generator and Lorentz boost in the same direction defining a unitary representation of the Lorentz group at light cone boundary define the most natural function basis for the light cone boundary. The elements of this bases have also well defined scaling quantum numbers and define also a unitary representation of the conformal algebra. The product of the basic functions is very simple in this basis since various quantum numbers are additive.

Spherical harmonics of S^2 provide an alternative function basis for the light cone boundary:

$$H^m_{jk} \equiv Y_{jm}(\theta,\phi) r_M^k \ . \tag{8}$$

One can criticize this basis for not having nice properties under Lorentz group.

The product of basis functions is given by Glebch-Gordan coefficients for symmetrized tensor product of two representation of the rotation group. Poisson bracket in turn reduces to the Glebch-Gordans of anti-symmetrized tensor product. The quantum numbers m and k are additive. The basis is eigen-function basis for the imaginary part of the Virasoro generator L_0 generating rotations around quantization axis of angular momentum. In fact, only the imaginary part of the Virasoro generator $L_0 = zd/dz = \rho\partial_\rho - \frac{2}{2}\partial_\phi$ has global single valued Hamiltonian, whereas the corresponding representation for the transformation induced by the real part of L_0, with a compensating radial scaling added, cannot be realized as a global canonical transformation.

The Poisson bracket of two functions $H^m_{j_1k_1}$ and $H^m_{j_2k_2}$ can be calculated and is of the general form

$$\{H^{m_1}_{j_1k}, H^{m_2}_{j_2k_2}\} \equiv C(j_1m_1j_2m_2|j,m_1+m_2)_A H^{m_1+m_2}_{j,k_1+k_2} \ . \tag{9}$$

The coefficients are Glebch-Gordan coefficients for the anti-symmetrized tensor product for the representations of the rotation group.

The isometries of the light cone boundary correspond to conformal transformations accompanied by a local radial scaling compensating the conformal factor coming from the conformal transformations having parametric dependence of radial variable and CP_2 coordinates. It seems however that isometries cannot in general be realized as canonical transformations. The first difficulty is that canonical transformations cannot affect the value of the radial coordinate. For rotation algebra the representation as canonical transformations is however possible.

In CP_2 degrees of freedom scalar function basis having definite color transformation properties is desirable. Scalar function basis can be obtained as the algebra generated by the Hamiltonians of color transformations by multiplication. The elements of basis can be typically expressed as monomials of color Hamiltonians H_c^A

$$H_D^A = \sum_{\{B_j\}} C^A_{DB_1B_2....B_N} \prod_{B_i} H_c^{B_i} \quad , \tag{10}$$

where summation over all index combinations $\{B_i\}$ is understood. The coefficients $C^A_{DB_1B_2....B_N}$ are Glebch-Gordan coefficients for completely symmetric N:th power 8⊗8...⊗8 of octet representations. The representation is not unique since $\sum_A H_c^A H_c^A = 1$ holds true. One can however find for each representation D some minimum value of N.

The product of Hamiltonians $H_A^{D_1}$ and $H_{D_2}^B$ can be decomposed by Glebch-Gordan coefficients of the symmetrized representation $(D_1 \otimes D_2)_S$ as

$$H^A_{D_1} H^B_{D_2} = C^{ABD}_{D_1D_2DC}(S) H^C_D \quad , \tag{11}$$

where $'S'$ indicates that the symmetrized representation is in question. In the similar manner one can decompose the Poisson bracket of two Hamiltonians

$$\{H^A_{D_1}, H^B_{D_2}\} = C^{ABD}_{D_1D_2DC}(A) H^C_D \quad . \tag{12}$$

Here $'A'$ indicates that Glebch-Gordan coefficients for the anti-symmetrized tensor product of the representations D_1 and D_2 are in question.

One can express the infinitesimal generators of CP_2 canonical transformations in terms of the color isometry generators J_c^B using the expansion of the Hamiltonian in terms of the monomials of color Hamiltonians:

$$\begin{aligned} j^A_{DN} &= F^A_{DB} J^B_c \quad , \\ F^A_{DB} &= N \sum_{\{B_j\}} C^A_{DB_1B_2...B_{N-1}B} \prod_j H_c^{B_j} \quad , \end{aligned} \tag{13}$$

where summation over all possible $\{B_j\}$:s appears. Therefore, the interpretation as a color group localized with respect to CP_2 coordinates is valid in the same sense as the interpretation of space-time diffeomorphism group as local Poincare group. Thus one can say that TGD color is localized with respect to the entire $\delta M^4_+ \times CP_2$.

A convenient basis for the Hamiltonians of $\delta M^4_+ \times CP_2$ is given by the functions

$$H^{mA}_{jkD} = H^m_{jk} H^A_D \quad .$$

The canonical transformation generated by H^{mA}_{jkD} acts both in M^4 and CP_2 degrees of freedom and the corresponding vector field is given by

$$J^r = H^A_D J^{rl}(\delta M^4_+) \partial_l H^m_{jk} + H^m_{jk} J^{rl}(CP_2) \partial_l H^A_D \quad . \tag{14}$$

The general form for their Poisson bracket is:

$$\{H^{m_1A_1}_{j_1k_1D_1},H^{m_2A_2}_{j_2k_2D_2}\} = H^{A_1}_{D_1}H^{A_2}_{D_2}\{H^{m_1}_{j_1k_1},H^{m_2}_{j_2k_2}\} + H^{m_1}_{j_1k_1}H^{m_2}_{j_2k_2}\{H^{A_1}_{D_1},H^{A_2}_{D_2}\}$$
$$= \left[C^{A_1A_2A}_{D_1D_2D}(S)C(j_1m_1j_2m_2|jm)_A + C^{A_1A_2A}_{D_1D_2D}(A)C(j_1m_1j_2m_2|jm)_S\right] H^{mA}_{j,k_1+k_2,D} \quad . \tag{15}$$

What is essential that radial 'momenta' and angular momentum are additive in δM^4_+ degrees of freedom and color quantum numbers are additive in CP_2 degrees of freedom.

3.6 Representation of Lorentz group and conformal symmetries at light cone boundary

A guess deserving testing is that the representations of the Lorentz group at light cone boundary might provide natural building blocks for the construction of the configuration space Hamiltonians. In the following the explicit representation of the Lorentz algebra at light cone boundary is deduced, and a function basis giving rise to the representations of Lorentz group and having very simple properties under modified Poisson bracket of δM^4_+ is constructed.

3.6.1 Explicit representation of Lorentz algebra

It is useful to write the explicit expressions of Lorentz generators using complex coordinates for S^2. The expression for the $SU(2)$ generators of the Lorentz group are

$$\begin{aligned} J_x &= (z^2-1)d/dz + c.c. = L_1 - L_{-1} + c.c. \quad , \\ J_y &= (iz^2+1)d/dz + c.c. = iL_1 + iL_{-1} + c.c. \quad , \\ J_z &= iz\frac{d}{dz} + c.c. = iL_z + c.c. \quad . \end{aligned} \tag{16}$$

The expressions for the generators of Lorentz boosts can be derived easily. The boost in m^3 direction corresponds to an infinitesimal transformation

$$\begin{aligned} \delta m^3 &= -\varepsilon r_M \quad , \\ \delta r_M &= -\varepsilon m^3 = -\varepsilon\sqrt{r_M^2-(m^1)^2-(m^2)^2} \quad . \end{aligned} \tag{17}$$

The relationship between complex coordinates of S^2 and M^4 coordinates m^k is given by stereographic projection

$$\begin{aligned} z &= \frac{(m^1+im^2)}{(r_M-\sqrt{r_M^2-(m^1)^2-(m^2)^2})} \\ &= \frac{sin(\theta)(cos\phi+isin\phi)}{(1-cos\theta)} \quad , \\ cot(\theta/2) &= \rho = \sqrt{z\bar{z}} \quad , \\ tan(\phi) &= \frac{m^2}{m^1} \quad . \end{aligned} \tag{18}$$

This implies that the change in z coordinate doesn't depend at all on r_M and is of the following form

$$\delta z = -\frac{\varepsilon}{2}(1+\frac{z(z+\bar{z})}{2})(1+z\bar{z}) \ . \tag{19}$$

The infinitesimal generator for the boosts in z-direction is therefore of the following form

$$L_z = [\frac{2z\bar{z}}{(1+z\bar{z})} - 1]r_M\frac{\partial}{\partial_{r_M}} - iJ_z \ . \tag{20}$$

Generators of L_x and L_y are most conveniently obtained as commutators of $[L_z, J_y]$ and $[L_z, J_x]$. For L_y one obtains the following expression:

$$L_y = 2\frac{(z\bar{z}(z+\bar{z}) + i(z-\bar{z}))}{(1+z\bar{z})^2}r_M\frac{\partial}{\partial_{r_M}} - iJ_y \ , \tag{21}$$

For L_x one obtains analogous expressions. All Lorentz boosts are of the form $L_i = -iJ_i +$ *local radial scaling* and of zeroth degree in radial variable so that their action on the general generator $X^{klm} \propto z^k\bar{z}^l r_M^m$ doesn't change the value of the label m being a mere local scaling transformation in radial direction. If radial scalings correspond to zero norm isometries this representation is metrically equivalent with the representations of Lorentz boosts as Möbius transformations.

3.6.2 Representations of the Lorentz group reduced with respect to $SO(3)$

The ordinary harmonics of S^2 define in a natural manner infinite series of representation functions transformed to each other in Lorentz transformations. The inner product defined by the integration measure $r_M^2 d\Omega dr_M/r_M$ remains invariant under Lorentz boosts since the scaling of r_M induced by the Lorentz boost compensates for the conformal scaling of $d\Omega$ induced by a Lorentz transformation represented as a Möbius transformation. Thus unitary representations of Lorentz group are in question.

The unitary main series representations of the Lorentz group are characterized by half-integer m and imaginary number $k_2 = i\rho$, where ρ is any real number [18]. A natural guess is that $m = 0$ holds true for all representations realizable at the light cone boundary and that radial waves are of form r_M^k, $k = k_1 + ik_2 = -1 + i\rho$ and thus eigen states of the radial scaling so that the action of Lorentz boosts is simple in the angular momentum basis. The inner product in radial degrees of freedom reduces to that for ordinary plane waves when $log(r_M)$ is taken as a new integration variable. The complexification is well-defined for non-vanishing values of ρ.

It is also possible to have non-unitary representations of the Lorentz group and the realization of the symmetric space structure suggests that one must have $k = k_1 + ik_2$, k_1 half-integer. For these representations unitarity fails because the inner product in the radial degrees of freedom is non-unitary. A possible physical interpretation consistent with the general ideas about conformal invariance is that the representations $k = -1 + i\rho$ correspond to the unitary ground state representations and $k = -1 + n/2 + i\rho$, $n = \pm 1, \pm 2, ...$, to non-unitary representations. The general view about conformal invariance suggests that physical states constructed as tensor products satisfy the condition $\sum_i n_i = 0$ completely analogous to Virasoro conditions.

3.6.3 Representations of the Lorentz group with $E^2 \times SO(2)$ as isotropy group

One can construct representations of Lorentz group and conformal symmetries at the light cone boundary. Since $SL(2, C)$ is the group generated by the generators L_0 and $L_\pm$ of the conformal algebra, it is clear that infinite-dimensional representations of Lorentz group can be also regarded as representations of the conformal algebra. One can require that the basis corresponds to eigen functions of the rotation generator J_z and corresponding boost generator L_z. For functions which do not depend on r_M these generators are completely analogous to the generators L_0 generating scalings and iL_0 generating rotations. Also the generator of radial scalings appears in the formulas and one must consider the possibility that it corresponds to the generator L_0.

In order to construct scalar function eigen basis of L_z and J_z, one can start from the expressions

$$\begin{aligned} L_3 &\equiv i(L_z + L_{\bar{z}}) = 2i[\frac{2z\bar{z}}{(1+z\bar{z})} - 1]r_M \frac{\partial}{\partial_{r_M}} + i\rho\partial_\rho \ , \\ J_3 &\equiv iL_z - iL_{\bar{z}} = i\partial_\phi \ . \end{aligned} \tag{22}$$

If the eigen functions do not depend on r_M, one obtains the usual basis z^n of Virasoro algebra, which however is not normalizable basis. The eigenfunctions of the generators L_3, J_3 and $L_0 = ir_M d/dr_M$ satisfying

$$\begin{aligned} J_3 f_{m,n,k} &= m f_{m,n,k} \ , \\ L_3 f_{m,n,k} &= n f_{m,n,k} \ , \\ L_0 f_{m,n,k} &= k f_{m,n,k} \ . \end{aligned} \tag{23}$$

are given by

$$f_{m,n,k} = e^{im\phi}\frac{\rho^{n-k}}{(1+\rho^2)^k} \times (\frac{r_M}{r_0})^k \ . \tag{24}$$

$n = n_1 + in_2$ and $k = k_1 + ik_2$ are in general complex numbers. The condition

$$n_1 - k_1 \geq 0$$

is required by regularity at the origin of S^2 The requirement that the integral over S^2 defining norm exists (the expression for the differential solid angle is $d\Omega = \frac{\rho}{(1+\rho^2)^2}d\rho d\phi$) implies

$$n_1 < 3k_1 + 2 \ .$$

From the relationship $(cos(\theta), sin(\theta)) = (\rho^2-1)/(\rho^2+1), 2\rho/(\rho^2+1))$ one can conclude that for $n_2 = k_2 = 0$ the representation functions are proportional to f $sin(\theta)^{n-k}(cos(\theta)-1)^{n-k}$. Therefore they have in their decomposition to spherical harmonics only spherical harmonics with angular momentum $l < 2(n-k)$. This suggests that the condition

$$|m| \leq 2(n-k) \tag{25}$$

is satisfied quite generally.

The emergence of the three quantum numbers (m,n,k) can be understood. Light cone boundary can be regarded as a coset space $SO(3,1)/E^2 \times SO(2)$, where $E^2 \times SO(2)$ is the group leaving the light like vector defined by a particular point of the light cone invariant. The natural choice of the Cartan group is therefore $E^2 \times SO(2)$. The three quantum numbers (m,n,k) have interpretation as quantum numbers associated with this Cartan algebra.

The representations of the Lorentz group are characterized by one half-integer valued and one complex parameter. Thus k_2 and n_2, which are Lorentz invariants, might not be independent parameters, and the simplest option is $k_2 = n_2$.

The nice feature of the function basis is that various quantum numbers are additive under multiplication:

$$f(m_a, n_a, k_a) \times f(m_b, n_b, k_b) = f(m_a + m_b, n_a + n_b, k_a + k_b) \ .$$

These properties allow to cast the Poisson brackets of the canonical algebra of the configuration space into an elegant form.

The Poisson brackets for the δM^4_+ Hamiltonians defined by f_{mnk} can be written using the expression $J^{\rho\phi} = (1+\rho^2)/\rho$ as

$$\begin{aligned} \{f_{m_a,n_a,k_a}, f_{m_b,n_b,k_b}\} &= i\left[(n_a - k_a)m_b - (n_b - k_b)m_a\right] \times f_{m_a+m_b,n_a+n_b-2,k_a+k_b} \\ &+ 2i\left[(2-k_a)m_b - (2-k_b)m_a\right] \times f_{m_a+m_b,n_a+n_b-1,k_a+k_b-1} \ . \end{aligned} \quad (26)$$

3.6.4 Can one find unitary light-like representations of Lorentz group?

It is interesting to compare the representations in question to the unitary representations of Lorentz group discussed in [18].

1. The unitary representations discussed in [18] are characterized by are constructed by deducing the explicit representations for matrix elements of the rotation generators J_x, J_y, J_z and boost generators L_x, L_y, L_z by decomposing the representation into series of representations of $SU(2)$ defining the isotropy subgroup of a time like momentum. Therefore the states are labelled by eigenvalues of J_z. In the recent case the isotropy group is $E^2 \times SO(2)$ leaving light like point invariant. States are therefore labelled by three different quantum numbers.

2. The representations of [18] are realized the space of complex valued functions of complex coordinates ξ and $\overline{\xi}$ labelling points of complex plane. These functions have complex degrees $n_+ = m/2 - 1 + l_1$ with respect to ξ and $n_- = -m/2 - 1 + l_1$ with respect to $\overline{\xi}$. l_0 is complex number in the general case but for unitary representations of main series it is given by $l_1 = i\rho$ and for the representations of supplementary series l_1 is real and satisfies $0 < |l_1| < 1$. The main series representation is derived from a representation space consisting of homogenous functions of variables z^0, z^1 of degree n_+ and of $\overline{z}^0$ and $\overline{z}^1$ of degrees $n_\pm$. One can separate express these functions as product of $(z^1)^{n^+}$ $(\overline{z}^1)^{n_-}$ and a polynomial of $\xi = z^1/z^2$ and $\overline{\xi}$ with degrees n_+ and n_-. Unitarity reduces to the requirement that the integration measure of complex plane is invariant under the Lorentz transformations acting as Moebius transformations of the complex plane. Unitarity implies $l_1 = -1 + i\rho$.

3. For the representations at δM^4_+ formal unitarity reduces to the requirement that the integration measure of $r_M^2 d\Omega dr_M/r_M$ of δM^4_+ remains invariant under Lorentz transformations. The action of Lorentz transformation on the complex coordinates of S^2 induces a conformal scaling which can be compensated by an S^2 local radial scaling. At least formally the function space of δM^4_+ thus defines a unitary representation. For the function basis f_{mnk} $k = -1 + i\rho$ defines a candidate for a unitary representation since the logarithmic waves in the radial coordinate are completely analogous to plane waves for $k_1 = -1$. This condition would be completely analogous to the vanishing of conformal weight for the physical states of super conformal representations. The problem is that for $k_1 = -1$ guaranteing square integrability in S^2 implies $-2 < n_1 < -2$ so that unitarity is possible only for the function basis consisting of spherical harmonics.

   There is no deep reason against non-unitary representations and symmetric space structure indeed requires that k_1 is half-integer valued. First of all, configuration space spinor fields are analogous to ordinary spinor fields in M^4, which also define non-unitary representations of Lorentz group. Secondly, if 3-surfaces at the light cone boundary are finite-sized, the integrals defined by f_{mnk} over 3-surfaces Y^3 are always well-defined. Thirdly, the continuous spectrum of k_2 could be transformed to a discrete spectrum when k_1 becomes half-integer valued.

Hermitian form for light cone Hamiltonians involves also the integration over S^2 degrees of freedom and the non-unitarity of the inner product reflects itself as non-orthogonality of the the eigen function basis. Introducing the variable $u = \rho^2 + 1$ as a new integration variable, one can express the inner product in the form

$$\begin{aligned} \langle m_a, n_a, k_a | m_b, n_b, k_b \rangle &= \pi\delta(k_{2a} - k_{2b}) \times \delta_{m_1,m_2} \times I \ , \\ I &= \int_1^\infty f(u) du \ , \\ f(u) &= \frac{(u-1)^{\frac{(N-K)+i\Delta}{2}}}{u^{K+2}} \ . \end{aligned} \tag{27}$$

The integrand has cut from $u = 1$ to infinity along real axis. The first thing to observe is that for $N = K$ the exponent of the integral reduces to very simple form and integral exists only for $K = k_{1a} + k_{1b} > -1$. For $k_{1i} = -1/2$ the integral diverges.

The discontinuity of the integrand due to the cut at the real axis is proportional to the integrand and given by

$$\begin{aligned} f(u) - f(e^{i2\pi}u) &= \left[1 - e^{-\pi\Delta}\right] f(u) \ , \\ \Delta &= n_{1a} - k_{1a} - n_{1b} + k_{1b} \ . \end{aligned} \tag{28}$$

This means that one can transform the integral to an integral around the cut. This integral can in turn completed to an integral over closed loop by adding the circle at infinity to the integration path. The integrand has $K + 1$-fold pole at $u = 0$.

Under these conditions one obtains

$$\begin{aligned} I &= \frac{2\pi i}{1 - e^{-\pi\Delta}} \times R \times (R-1) .... \times (R - K - 1) \times (-1)^{\frac{N-K}{2} - K - 1} \ , \\ R &\equiv \frac{N-K}{2} + i\Delta \ . \end{aligned} \tag{29}$$

This expression is non-vanishing for $\Delta \neq 0$. Thus it is not possible to satisfy orthogonality conditions without the un-physical $n = k, k_1 = 1/2$ constraint. The result is finite for $K > -1$ so that $k_1 > -1/2$ must be satisfied and if one allows only half-integers in the spectrum, one must have $k_1 \geq 0$, which is very natural if real conformal weights which are half integers are allowed.

3.6.5 Light-like representations corresponding to the trivial zeros of Riemann Zeta

The previous considerations leave still one loophole open. If the light-like state basis satisfies the condition $(m = m(k), n = n(k))$ such that $m(k_1) = m(k_2)$ implies $k_1 = k_2$, the state basis reduces to

$$f_{\pm m(k),n(k),k} = e^{\pm im(k)\phi} \times \frac{\rho^{n(k)-k}}{(1+\rho^2)^k} \times (\frac{r_M}{r_0})^k \quad . \tag{30}$$

and orthogonality conditions are trivially satisfied. Only the radial integration over the light cone is divergent but since state basis involves only integrals over 3-surfaces, this might be allowed. Furthermore, if the configuration space decomposes into M^4 translates of configuration space sector defined by $\delta M^4_+ \times CP_2$, then non-unitary representations of Lorentz group can define unitary representations of Poincare group and Virasoro algebra extended by additional M^4 degrees of freedom just like the ordinary fields based on finite-dimensional non-unitary irreps of Lorentz group do.

The non-trivial zeros $s = 1/2 + iy$ of Riemann Zeta are an excellent candidate for the conformal weights $k = -1/2 - iy$ of $SO(3)$ state basis. This state basis is however discrete and allows addition of new basis elements. Perhaps the trivial zeros $s = -2r$, $r > 1, 2, ...$, could correspond to the state basis $k = 2r$, $r > 0$, orthogonal to this state basis. The work with the construction of quantum TGD has provided strong support for this view [C5], and in fact forced to seriously reconsider the earlier conclusion that light-like representations are not physically interesting.

1. The orthogonality of the $SO(3)$ and $SO(2)$ state basis results from orthogonality in the radial degrees of freedom if only zeros of Riemann Zeta are allowed.

2. The condition

$$m = n - k$$

is natural since it implies that the basis functions behave as z^m or $\overline{z}^m$ near the origin of S^2 as expected on basis of conformal invariance. The non-vanishing value of m means that complexification of the configuration space tangent space can be performed in angular degrees of freedom instead of radial degree of freedom. The values of $m(k)$ define a natural grading of the tangent space basis. One can decompose the basis defined by all values of k as $g = t + h$, such that t corresponds to odd values of m and h to even values of m. Symmetric space structure results if even values of m (and odd values of k) correspond to gauge degrees of freedom.

3. Even values $k = 2r$, $r > 0$ of k results if they correspond to odd values of m. This is achieved if the condition

$$n(k) = 2k - 1 \tag{31}$$

is satisfied. Together with the condition $m = n - k$ it implies

$$m(k) = k - 1 \tag{32}$$

so that $m(k)$ is indeed odd for even k. The conditions $k \leq n < 3k + 2$ and $m(k) \leq k$ are satisfied. The choice implies that m is odd for even values of k.

4. It should be noticed that in the old fashioned string model the angular momenta J at Regge trajectories were evenly spaced: $J = 2n$. Perhaps conformal invariance forces the angular excitations to have even parity also now. Even parity excitations could be also attributed to the spin two character of graviton.

3.6.6 Logarithmic waves and possible connections with number theory and fundamental physics

Logarithmic plane waves labelled by eigenvalues of the scaling momenta appear also in the definition of the Riemann Zeta defined as $\zeta(z) = \sum_n n^{-z}$, n positive integer [E8]. Riemann Zeta is expressible as a product of partition function factors $1/(1+p^{-x-iy})$, p prime and the powers n^{-x-iy} appear as summands in Riemann Zeta. Riemann hypothesis states that the non-trivial zeros of Zeta reside at the line $x = 1/2$. There are indeed intriguing connections. $Log(p)$ corresponds now to the $log(r_M/r_{min})$ and -x-iy corresponds to the scaling momentum $k_1 + ik_2$ so that the special physical role of the conformal weights $k_1 = 1/2 + iy$ corresponds to Riemann hypothesis. The appearance of powers of p in the definition of the Riemann Zeta corresponds to p-adic length scale hypothesis, ($r_M/r_0 = p$ in ζ and corresponds to a secondary p-adic length scale).

The assumption that the logarithmic plane waves are algebraically continuable from the rational points $r_M/r_{min} = m/n$ to p-adic plane waves using a finite-dimensional extension of p-adic numbers leads to the condition that the $log(p)\pi$ belongs to an extension of rational numbers containing some root of e for every prime. Similar hypothesis is inspired by the hypothesis that Riemann Zeta is a universal function existing simultaneously in all number fields. This inspires several interesting observations.

1. p-adic length scale hypothesis stating that $r_{max}/r_{min} = p^n$ is consistent with the number theoretical universality of the logarithmic waves. The universality of Riemann Zeta inspires the hypothesis that the zeros of Riemann Zeta correspond to rational numbers and to preferred values $k_1 + ik_2$ of the scaling momenta appearing in the logarithmic plane waves. In the recent context the most general hypothesis would be that the allowed momenta k_2 correspond to the linear combinations of the zeros of Riemann Zeta with integer coefficients.

2. Hardmuth Mueller [21] claims on basis of his observations that gravitational interaction involves logarithmic radial waves $exp(iklog(r/r_{min}))$ for which the nodes come as $u = r/r_{min} = e^n$. This is true if the scaling momenta k_2 satisfy the condition $k_2/\pi \in Z$. Perhaps Mueller's logarithmic waves really could be seen as a direct signature of the fundamental symmetries of the configuration space. The waves exist p-adically also for $u = qe^m$ if $log(p)\pi$ satisfies the constraint already mentioned.

3. The special role of Golden Mean $\Phi = (1 + \sqrt{5})/2$ in Nature could be understood of also $log(\Phi)\pi$ satisfies the same constraint as $log(p)$ (also other powers of π might appear here). This would imply that the nodes of logarithmic waves can correspond also to the powers of Φ.

   One could of course argue that the number theory at the moment of Big Bang cannot have strong effects on what is observed in laboratory. This might be the case. On the other hand, the non-determinism of the Kähler action however strongly suggests that the construction of the configuration space geometry involves all possible light like 3-surfaces of the future light cone so that logarithmic waves would appear in all length scales. Be as it may, it would be amazing if such an abstract mathematical structure as configuration space geometry would have direct implications to cosmology and to the physics of living systems.

4. The hypothesis that the trivial zeros $z = -2n$, $n > 0$, and non-trivial zeros $z = 1/2 + iy$ of Riemann Zeta define the allowed radial conformal weights of the super-canonical algebra is very elegant but can be criticized. One could add also other radial complex conformal weights without losing the orthogonality. The considerations of [C5] encourage to think that the zeros of Riemann Zeta correspond to the spectrum of conformal weights in the single particle sector whereas for many particle bound states zeros of polyzetas $\zeta(z_1, ..., z_k)$ appear [20]. "Bound state conformal weights" would also appear as off mass shell values

of conformal weights in perturbation theory. At the level of configuration space geometry bound state conformal weights would be associated with 3-surface which serve as space-time correlates of bound states.

4 Magnetic and electric representations of the configuration space Hamiltonians and electric-magnetic duality

Symmetry considerations lead to the hypothesis that configuration space Hamiltonians are apart from a factor depending on canonical invariants equal to magnetic flux Hamiltonians. On the other hand, the hypothesis that Kähler function corresponds to absolute minimum of Kähler action leads to the hypothesis that configuration space Hamiltonians corresponds to classical charges associated with the Hamiltonians of the light cone boundary. These charges are very much like electric charges. The requirement that two approaches are equivalent leads to the hypothesis that magnetic and electric Hamiltonians are identical apart from a factor depending on isometry invariants. At the level of CP_2 corresponding duality corresponds to the self-duality of Kähler form stating that the magnetic and electric parts of Kähler form are identical.

4.1 Radial canonical invariants

All $\delta M^4_+ \times CP_2$ canonical transformations leave invariant the value of the radial coordinate r_M. Therefore the radial coordinate r_M of X^3 regarded as a function of $S^2 \times CP_2$ coordinates serves as height function. The number, type, ordering and values for the extrema for this height function in the interior and boundary components are isometry invariants. These invariants characterize not only the topology but also the size and shape of the 3-surface. The result implies that configuration space metric indeed differentiates between 3-surfaces with the size of Planck length and with the size of galaxy. The characterization of these invariants reduces to Morse theory. The extrema correspond to topology changes for the two-dimensional (one-dimensional) $r_M = constant$ section of 3-surface (boundary of 3-surface). The height functions of sphere and torus serve as a good illustrations of the situation. A good example about non-topological extrema is provided by a sphere with two horns.

There are additional canonical invariants. The 'magnetic fluxes' associated with the δM^4_+ symplectic form

$$J_{S^2} = r_M^2 sin(\theta) d\theta \wedge d\phi$$

over any $X^2 \subset X^3$ are canonical invariants. In particular, the integrals over $r_M = constant$ sections (assuming them to be 2-dimensional) are canonical invariants. They give simply the solid angle $\Omega(r_M)$ spanned by $r_M = constant$ section and thus $r_M^2\Omega(r_M)$ characterizes transversal geometric size of the 3-surface. A convenient manner to discretize these invariants is to consider the Fourier components of these invariants in radial logarithmic plane wave basis discussed earlier:

$$\Omega(k) = \int_{r_{min}}^{r_{max}} (r_M/r_{max})^k \Omega(r_M) \frac{dr_M}{r_M} \quad , \quad k = k_1 + ik_2 \quad , \quad perk_1 \geq 0 \quad . \tag{33}$$

One must take into account that for each section in which the topology of $r_M = constant$ section remains constant one must associate invariants with separate components of the two-dimensional section. For a given value of r_M, r_M constant section contains several components (to visualize the situation consider torus as an example).

Also the quantities

$$\Omega^+(X^2) = \int_{X^2} |J| \equiv \int |\epsilon^{\alpha\beta} J_{\alpha\beta}| \sqrt{g_2} d^2x$$

are canonical invariants and provide additional geometric information about 3-surface. These fluxes are non-vanishing also for closed surfaces and give information about the geometry of the boundary components of 3-surface (signed fluxes vanish for boundary components unless they enclose the dip of the light cone).

Since zero norm generators remain invariant under complexification, their contribution to the Kähler metric vanishes. It is not at all obvious whether the configuration space integration measure in these degrees of freedom exists at all. A localization in zero modes occurring in each quantum jump seems a more plausible and under suitable additional assumption it would have interpretation as a state function reduction. In string model similar situation is encountered; besides the functional integral determined by string action, one has integral over the moduli space.

If the 7–3 duality discussed in the introduction is accepted, there is no need to integrate over the variable r_M and just the fluxes over the 2-surfaces X_i^2 identified as intersections of light like 3-D CDs with X^3 contain the data relevant for the construction of the configuration space geometry. Also the canonical invariants associated with these surfaces are enough.

4.2 Kähler magnetic invariants

The Kähler magnetic fluxes defined both the normal component of the Kähler magnetic field and by its absolute value

$$
\begin{aligned}
Q_m(X^2) &= \int_{X^2} J_{CP_2} = J_{\alpha\beta}\epsilon^{\alpha\beta}\sqrt{g_2}d^2x \ , \\
Q_m^+(X^2) &= \int_{X^2} |J_{CP_2}| \equiv \int_{X^2} |J_{\alpha\beta}\epsilon^{\alpha\beta}|\sqrt{g_2}d^2x \ ,
\end{aligned} \tag{34}
$$

over suitably defined 2-surfaces are invariants under both Lorentz isometries and the canonical transformations of CP_2 and can be calculated once X^3 is given.

For a closed surface $Q_m(X^2)$ vanishes unless the homology equivalence class of the surface is nontrivial in CP_2 degrees of freedom. In this case the flux is quantized. $Q_M^+(X^2)$ is non-vanishing for closed surfaces, too. Signed magnetic fluxes over non-closed surfaces depend on the boundary of X^2 only:

$$
\begin{array}{l}
\int_{X^2} J = \int_{\delta X^2} A \ . \\
J = dA \ .
\end{array}
$$

Un-signed fluxes can be written as sum of similar contributions over the boundaries of regions of X^2 in which the sign of J remains fixed.

$$
\begin{aligned}
Q_m(X^2) &= \int_{X^2} J_{CP_2} = J_{\alpha\beta}\epsilon^{\alpha\beta}\sqrt{g_2}d^2x \ , \\
Q_m^+(X^2) &= \int_{X^2} |J_{CP_2}| \equiv \int_{X^2} |J_{\alpha\beta}\epsilon^{\alpha\beta}|\sqrt{g_2}d^2x \ ,
\end{aligned} \tag{35}
$$

There are also canonical invariants, which are Lorentz covariants and defined as

$$
\begin{aligned}
Q_m(K,X^2) &= \int_{X^2} f_K J_{CP_2} \ , \\
Q_m^+(K,X^2) &= \int_{X^2} f_K |J_{CP_2}| \ , \\
f_{K\equiv(s,n,k)} &= e^{is\phi} \times \frac{\rho^{n-k}}{(1+\rho^2)^k} \times (\frac{r_M}{r_0})^k
\end{aligned} \tag{36}
$$

These canonical invariants transform like an infinite-dimensional unitary representation of Lorentz group.

There must exist some minimal number of canonically non-equivalent 2-surfaces of X^3, and the magnetic fluxes over these surfaces give thus good candidates for zero modes.

1. If 7–3 duality is accepted, the surfaces X_i^2 defined by the intersections of light like 3-D CDs X_l^3 and X^3 provide a natural identification for these 2-surfaces.

2. Without 7-duality situation is more complex. Since canonical transformations leave r_M invariant, a natural set of 2-surfaces X^2 appearing in the definition of fluxes are separate pieces for $r_M = constant$ sections of 3-surface. For a generic 3-surface, these surfaces are 2-dimensional and there is continuum of them so that discrete Fourier transforms of these invariants are needed. One must however notice that $r_M = constant$ surfaces could be be 3-dimensional in which case the notion of flux is not well-defined.

4.3 Isometry invariants and spin glass analogy

The presence of isometry invariants implies coset space decomposition $\cup_i G/H$. This means that quantum states are characterized, not only by the vacuum functional, which is just the exponential $exp(K)$ of Kähler function (Gaussian in lowest approximation) but also by a wave function in vacuum modes. Therefore the functional integral over the configuration space decomposes into an integral over zero modes for approximately Gaussian functionals determined by $exp(K)$. The weights for the various vacuum mode contributions are given by the probability density associated with the zero modes. The integration over the zero modes is a highly problematic notion and it could be eliminated if a localization in the zero modes occurs in quantum jumps. The localization would correspond to a state function reduction and zero modes would be effectively classical variables correlated in one-one manner with the quantum numbers associated with the quantum fluctuating degrees of freedom.

For a given orbit K depends on zero modes and thus one has mathematical similarity with spin glass phase for which one has probability distribution for Hamiltonians appearing in the partition function $exp(-H/T)$. In fact, since TGD:eish Universe is also critical, exact similarity requires that also the temperature is critical for various contributions to the average partition function of spin glass phase. The characterization of isometry invariants and zero modes of the Kähler metric provides a precise characterization for how TGD Universe is quantum analog of spin glass.

The spin glass analogy has been the basic starting point in the construction of p-adic field theory limit of TGD. The ultra-metric topology for the free energy minima of spin glass phase motivates the hypothesis that effective quantum average space-time possesses ultra-metric topology. This approach leads to excellent predictions for elementary particle masses and predicts even new branches of physics [F5, F7]. As a matter fact, an entire fractal hierarchy of copies of standard physics is predicted.

4.4 Magnetic flux representation of the canonical algebra

Accepting 7-3 duality, configuration space Hamiltonians correspond to the fluxes of $X_l^3 \times CP_2$ associated with various 2-surfaces X_i^2 defined by the intersections of light like CDs $X_{l,i}^3$ with X^3.

4.4.1 Generalized magnetic fluxes

Isometry invariants are just special case of fluxes defining natural coordinate variables for the configuration space. Canonical transformations of CP_2 act as $U(1)$ gauge transformations on the Kähler potential of CP_2 (similar conclusion holds at the level of $\delta M_+^4 \times CP_2$).

One can generalize these transformations to local canonical transformations by allowing the Hamiltonians to be products of the CP_2 Hamiltonians with the real and imaginary parts of the functions $f_{m,n,k}$ (see Eq. 24) defining the Lorentz covariant function basis H_A, $A \equiv (a,m,n,k)$ at the light cone boundary: $H_A = H_a \times f(m,n,k)$, where a labels the Hamiltonians of CP_2.

One can associate to any Hamiltonian H^A of this kind both signed and unsigned magnetic flux via the following formulas:

$$\begin{aligned} Q_m(H_A|X^2) &= \int_{X^2} H_A J \ , \\ Q_m^+(H_A|X^2) &= \int_{X^2} H_A |J| \ . \end{aligned} \tag{37}$$

Here X^2 corresponds to any surface X_i^2 resulting as intersection of X^3 with $X^3_{l,i}$. Both signed and unsigned magnetic fluxes and their superpositions

$$Q_m^{\alpha,\beta}(H_A|X^2) = \alpha Q_m(H_A|X^2) + \beta Q_m^+(H_A|X^2) \ , \ A \equiv (a,s,n,k) \tag{38}$$

provide representations of Hamiltonians. Note that canonical invariants $Q_m^{\alpha,\beta}$ correspond to $H^A = 1$ and $H^A = f_{s,n,k}$. $H^A = 1$ can be regarded as a natural central term for the Poisson bracket algebra. Therefore, the isometry invariance of Kähler magnetic and electric gauge fluxes follows as a natural consequence.

The obvious question concerns about the correct values of the parameters α and β. One possibility is that the flux is an unsigned flux so that one has $\alpha = 0$. This option is favored by the construction of the configuration space spinor structure involving the construction of the fermionic super charges anti-commuting to configuration space Hamiltonians: super charges contain the square root of flux, which must be therefore unsigned. Second possibility is that magnetic fluxes are signed fluxes so that β vanishes.

One can define also the electric counterparts of the flux Hamiltonians by replacing J in the defining formulas with its dual $*J$

$$*J_{\alpha\beta} = \epsilon_{\alpha\beta}^{\ \ \gamma\delta} J_{\gamma\delta}.$$

For $H_A = 1$ these fluxes reduce to ordinary Kähler electric fluxes. These fluxes are however not canonical covariants since the definition of the dual involves the induced metric, which is not canonical invariant. The electric gauge fluxes for Hamiltonians in various representations of the color group ought to be important in the description of hadrons, not only as string like objects, but quite generally. These degrees of freedom would be identifiable as non-perturbative degrees of freedom involving genuinely classical Kähler field whereas quarks and gluons would correspond to the perturbative degrees of freedom, that is the interactions between CP_2 type extremals.

4.4.2 Poisson brackets

From the canonical invariance of the radial component of Kähler magnetic field it follows that the Lie-derivative of the flux $Q_m^{\alpha,\beta}(H_A)$ with respect to the vector field $X(H_B)$ is given by

$$X(H_B) \cdot Q_m^{\alpha,\beta}(H_A) = Q_m^{\alpha,\beta}(\{H_B, H_A\}) \ . \tag{39}$$

The transformation properties of $Q_m^{\alpha,\beta}(H_A)$ are very nice if the basis for H_B transforms according to appropriate irreducible representation of color group and rotation group. This in turn implies that the fluxes $Q_m^{\alpha,\beta}(H_A)$ as functionals of 3-surface on given orbit provide a representation for the Hamiltonian as a functional of 3-surface. For a given surface X^3, the Poisson bracket for the two fluxes $Q_m^{\alpha,\beta}(H_A)$ and $Q_m^{\alpha,\beta}(H_B)$ can be defined as

$$\begin{aligned} \{Q_m^{\alpha,\beta}(H_A), Q_m^{\alpha,\beta}(H_B)\} &\equiv X(H_B)\cdot Q_m^{\alpha,\beta}(H_A) \\ &= Q_m^{\alpha,\beta}(\{H_A,H_B\}) = Q_m^{\alpha,\beta}(\{H_A,H_B\}) \ . \end{aligned} \tag{40}$$

The study of configuration space gamma matrices identifiable as canonical super charges demonstrates that the supercharges associated with the radial deformations vanish identically so that radial deformations correspond to zero norm degrees of freedom as one might indeed expect on physical grounds. The reason is that super generators involve the invariants $j^{ak}\gamma_k$ which vanish by $\gamma_{r_M} = 0$.

The natural central extension associated with the canonical group of CP_2 ($\{p,q\} = 1!$) induces a central extension of this algebra. The central extension term resulting from $\{H_A, H_B\}$ when CP_2 Hamiltonians have $\{p,q\} = 1$ equals to the canonical invariant $Q_m^{\alpha,\beta}(f(m_a+m_b, n_a+n_b, k_a+k_b))$ on the right hand side. This extension is however anti-symmetric in canonical degrees of freedom rather than in loop space degrees of freedom and therefore does not lead to the standard Kac Moody type algebra. Also the standard Kac Moody type central extension might be possible but is not needed since the expectation is that Kac Moody extension is associated with the quaternion conformal algebra.

Quite generally, the Virasoro and Kac Moody algebras of string models are replaced in TGD context by much larger symmetry algebras. Kac Moody algebra corresponds to the quaternion conformal algebra relevant to the elementary particle physics. A completely new algebra is the CP_2 canonical algebra localized with respect to the light cone boundary and relevant to the configuration space geometry. One can also consider $S^2 \times CP_2$ canonical algebra and this indeed gives the strongest predictions for the configuration space metric. The local radial Virasoro localized with respect to $S^2 \times CP_2$ acts in zero modes and has automatically vanishing norm with respect to configuration space metric defined by super charges. The non-determinism of Kähler action probably necessitates a further generalization by allowing a very general class of light like surfaces of M^4 besides future and past light cone boundaries and their unions.

4.5 The representation of the canonical algebra based on classical charges defined by the Kähler action

Concerning the identification of the electric counterparts of flux Hamiltonians there are two approaches.

1. The simplest electric duals for flux Hamiltonians are obtained simply by replacing J with its dual in various formulas. In this case all the data needed is available assuming that the time derivative of imbedding space coordinates is same for all light like CDs $X^3_{l_i}$ at X^2_i: tangentiality is obviously a highly natural constraint.

2. Configuration space Hamiltonians could be in one-one correspondence with the Hamiltonians of the light cone boundary parameterized by the function basis for $\delta M^4_+ \times CP_2$.

A detailed discussion of the option 2) demonstrates that it is not so plausible as 1). For a given action principle one can associate unique, not necessarily conserved, classical charge to each Hamiltonian. In particular, one can write immediately the expressions for the charges $Q_H(Y^3)$ in the case of Kähler action for surfaces Y^3 at light cone boundary. These charges are essentially the variational derivatives of the Kähler action with respect to the canonical transformations of the light cone boundary. In the ordinary quantum theory these charges are quantized and become 'quantum Hamiltonians'. In TGD framework quantum theory is reduced to a theory of classical spinor fields in CH so that a natural guess is that these classical conserved charges as functionals of the 3-surface Y^3 on light cone boundary could be interpreted as configuration space Hamiltonians.

Classical charges are variational derivatives of the Kähler action and the expression for the classical charge associated with an infinitesimal deformation of 3-surface Y^3 induced by the transformation $h^k \rightarrow h^k + \epsilon j^k$ is given by

$$
\begin{aligned}
Q_j &= Q_j^1 + Q_j^2 \ , \\
Q_j^I &= \int_{Y^3} J_j^I \sqrt{g_4} d^3x \ , \ I = 1,2 \ , \\
J_j^1 &= T^{0\beta} \partial_\beta h^k h_{kl} j^l \ , \\
J_j^2 &= -J^{0\beta} J_{kl} \partial_\beta h^l j^k \ , \\
T^{\alpha\beta} &= -J^{\alpha\mu} J_\mu^{\ \beta} - \frac{1}{4} g^{\alpha\beta} J^{\mu\nu} J_{\mu\nu} \ .
\end{aligned}
\tag{41}
$$

The two terms correspond to the variations of the Kähler action with respect to induced metric and induced Kähler form, which are both expressible in terms of the imbedding space coordinates and their gradients.

Consider now the case, when j corresponds to a canonical transformation of light cone boundary. As found earlier, these transformations are generated by the Hamiltonians H_A belonging to the function algebra of $\delta M_+^4 \times CP_2$. One can assume that the choice of the radial coordinate r_M uniquely determined from the requirement that the rotations leaving the value of r_M invariant leave also invariant the classical 4-momentum associated with Y^3. In this case Q_k^2 reduces to the form

$$
\begin{aligned}
J_j^2 &= -J^{0\beta} \partial_\beta^{CP_2} H \ , \\
j^k &= J^{kl} \partial_l H \ .
\end{aligned}
\tag{42}
$$

involving contraction of the Kähler electric field with the gradient of the Hamiltonian with respect to CP_2 coordinate only (as expressed by superscript).

Is is useful to study the behavior of these charges for various extremals of the Kähler action.

i) If space-time surface is representable as a graph of a map $M_+^4 \rightarrow CP_2$ and if j is infinitesimal transformation acting only in CP_2 degrees of freedom, the dominant term to the charge comes from Q_j^2 when classical gravitational field described by induced metric is weak: this is the case unless CP_2 coordinates as functions of the space-time coordinates vary significantly in CP_2 length scale. For the canonical transformations of δM_+^4 the sole contribution comes from O_j^1.

ii) Covariant metric and Kähler form of CH vanish identically for 3-surfaces Y^3 having 1-dimensional CP_2 projection. For CP_2 type extremals the only contribution to the metric comes from Q_J^2 (energy momentum tensor vanishes) and can be transformed to a boundary term, which is essentially the electric gauge flux and large as compared to the gravitational contributions.

iii) For free cosmic string only Kähler magnetic field is present and for the canonical transformations of CP_2 the only contribution comes from Q_j^1 and is small as compared to the corresponding contribution for the transformations of the light cone boundary.

The expression for Q_j differs from the magnetic flux representations in that the Hamiltonian vector field rather than Hamiltonian appears in it. In the special case that j generates a canonical transformation of CP_2, the expression for Q^2_j can be transformed to a form involving Hamiltonian using partial integration.

$$\begin{aligned} Q^2_e(H_A) &= \int_{Y^3} j^0 H_A \sqrt{g_4} d^3x - \int_{\delta Y^3} J^{0n} H_A \sqrt{g_4} d^2x \ , \\ j^\alpha &\equiv D_\beta J^{\alpha\beta} \ . \end{aligned} \tag{43}$$

The first term contains Kähler current j^0. Boundary term resembles electric flux Hamiltonian H: this term cannot be compensated by a term coming from the variation of the Kähler action on the dynamical boundary of X^4 (as opposed to Y^3 which is kept fixed in variations). This means that configuration space Hamiltonians given by this formula are determined uniquely unlike the magnetic flux Hamiltonians, which contain additive constants giving rise making possible central extension in Poisson algebra.

Electric flux Hamiltonians defined as charges have some features which look problematic.

1. The presence of the 3-D term is not consistent with the notion of 7–3 duality nor with the simple idea of just replacing Kähler magnetic field with Kähler electric field in the formulas for the magnetic flux Hamiltonians. For a non-vanishing Kähler charge density j^0 the calculation of electric flux Hamiltonians requires data about the time derivatives of the imbedding space coordinates which is not available.

2. If the field equations force the Kähler charge density j^0 to vanish, the two definitions are equivalent if the surfaces X^2_i correspond to boundary components of X^3 only. It is not clear whether this is the case however.

4.6 Electric-magnetic duality

The requirement that magnetic and electric representations of the Hamiltonians are equivalent, is satisfied if electric Hamiltonians are affinely related to magnetic Hamiltonians. Proportionality constant can depend on the isometry invariants, which are also canonical invariants. This boils down to the following alternative requirements

$$Q_e(H_A) = Z\left[Q^{\alpha,\beta}_m(H_A) + q_e(H_A)\right] \ . \tag{44}$$

$Q^{\alpha,\beta}_m$ denotes the superposition of signed and unsigned magnetic flux Hamiltonians. The factor Z can depend on canonical invariants, in particular the magnetic fluxes $Q^{\alpha,\beta}_m(H_a = 1, m, k)$ and corresponding magnetic invariants associated with δM^4_+ degrees of freedom. Also the constants q_e can depend on canonical invariants.

Absolute minimization of Kähler action provides an attractive explanation for the duality as analogous to the electric-magnetic duality for the absolute minima of the Euclidian Yang-Mills action. Duality can be understood from the following argument.

1. Consider various extremals of Kähler action going through Y^3 at light cone boundary. Magnetic fluxes depend on Y^3 only and are same for all these extremals. Since the generation of Kähler magnetic/electric fields creates positive/negative Kähler action, there must be a competition between Kähler magnetic and electric contributions to the Kähler action. In the lowest order, these contributions are also expected to be linear in the squares of the magnetic and electric Hamiltonians respectively. The sign of the electric/magnetic contribution is expected to be negative/positive. Furthermore, the magnitudes of the electric charges $Q_e(H_A)$ are expected to be at maximum for the absolute minima.

2. Altogether this suggests that in the set of the extremals of Kähler action going through Y^3 Kähler action can be expressed in the following form

$$S_K = S_K^0 + S_K^1 \ ,$$
$$S_K^1 = \sum_{H_A} c(H_A) \left[|Q_m(H_A) + q_e(H_A)|^2 - \frac{|Q_e(H_A)|^2}{Z^2} \right] \ . \tag{45}$$

S_K^0 corresponds to the Kähler action for absolute minimum. The coefficients $c_(H_A)$ are non-negative. The parameter Z, which is isometry invariant and same for all extremals going through Y^3, is by definition chosen in such a manner that for the absolute minimum the term S_K^1 vanishes.

3. Duality hypothesis means nothing but the generalization of the electric-magnetic duality of instanton solutions of Euclidian YM theories to TGD context. The self-duality of CP_2 Kähler form supports the duality: in particular, the fact that CP_2 type extremals can be regarded either as elementary particles or magnetic monopoles with magnetic flux flowing in internal degrees of freedom, supports self duality at the level of CH geometry.

What is especially nice that the equivalence of the group theoretical and Kähler action based approaches reduces to duality hypothesis. Since the matrix elements of the configuration space symplectic form (and Kähler form and Kähler metric in complex coordinates) are expressible as Poisson brackets of the configuration space Hamiltonians, Z has interpretation as a conformal factor of the configuration space metric. This means that all information about initial values of the time derivatives of H-coordinates at light cone boundary and about classical gravitation, in particular the dependence of the configuration space metric on CP_2 length scale, is contained in the conformal factor Z. The remaining part is completely independent of the metric: the reduction of all information about scales to the conformal factor is clearly consistent with generalized conformal invariance.

This duality relates in an interesting manner to the conjecture about number theoretic spontaneous compactification [E2] in the sense that space-time surfaces in $M^4 \times CP_2$ could be equivalently regarded as hyper-quaternionic 4-surfaces in M^8 possessing hyper-octonionic structure. The point is that one can consider also the dual definition for which the 4-D normal space defines 4-D sub-algebra of 8-D algebra at each point of the space-time surface. Future-past duality could basically reduce to this purely geometric duality and would basically reflect bra-ket duality.

It is also possible to imagine a second variant of electric-magnetic duality, not in fact a genuine duality.

1. This duality like transformation is defined formally via the replacement $\alpha_K \rightarrow -\alpha_K$ [B3, E3] in Kähler action, and could be formally regarded as a logarithmic version of $g \rightarrow 1/g$ duality, and makes sense in TGD framework since g_K does not appear as a coupling constant. The requirement that the Kähler metric of CH defined by Kähler function is positive definite, poses very strong conditions and might exclude this kind duality. Certainly, the space-time sheet of one phase cannot correspond to that for another phase. For instance, electric *resp.* magnetic flux tubes would be favored in the two phases if absolute minimization of Kähler action defines the variational principle. TGD inspired cosmology encourages to consider this picture seriously [D5].

2. A possible interpretation is that this duality allows to distinguish between positive energy particle propagating to the geometric future and negative energy particle propagating to the geometric past (phase conjugate photons would represent an example of negative energy

particles propagating in the direction of geometric past). Quantum TGD indeed allows to make this distinction at the level of quantum states: the super-canonical conformal weights are complex and the sign of the imaginary part distinguishes between partons and their phase conjugates. The two directions of geometric time and two possible signs of inertial energy and Kähler action could correspond to the two signs g_K^2. At parton level this difference would correspond to two different sign of Chern-Simons action. Both the sign of the imaginary part of the conformal weight (quantum level) and of g_K^2 (classical space-time correlates) would characterize the direction of inherent time arrow of particle. The imaginary part of the conformal weight would also correspond to irreversibility and p-adic mass calculations suggest interpretation in terms of a decay rate.

3. On the other hand, the geometric construction recipe for S-matrix [C2] suggests that $g_K^2 \rightarrow -g_K^2$ duality could relate the incoming and outgoing lines of particle reaction represented by space-time sheets. Just as one assigns to the incoming and outgoing lines positive and negative energies one could assign to them an opposite sign of Kähler action. As a matter fact, the definition of action in mechanical systems is as $S = \int_0^t dtLdt$ so that the sign of S depends on time orientation. In the recent case action is general coordinate invariant and its sign does not depend on time orientation. The different choice of sign of α_K or equivalently different inherent time orientation, for incoming and outgoing particles would have same net effect. This would however imply a profound difference between incoming and and outgoing particles of particle reaction. This interpretation is consistent with the previous one if outgoing particles are always phase conjugate particles.

4. One could argue that the change of sign of α_K breaks unitarity (this is not related to the naive expectation that g_K becomes imaginary since the absolute minima would be different and tend to be dominated by Kähler magnetic fields). The defining property of hyper-finite factors of type II_1 is that infinite-dimensional unit matrix has unit and this makes possible to identify S-matrix as unitary entanglement coefficients between positive and negative energy components of a zero energy state. Unitary S-matrix is thus a property of zero energy state now and the counter argument does not bite.

4.7 Canonical transformations of $\delta M_+^4 \times CP_2$ as isometries and electric-magnetic duality

According to the construction of Kähler metric, canonical transformations of $\delta M_+^4 \times CP_2$ act as isometries whereas radial Virasoro algebra localized with respect to CP_2 has zero norm in the configuration space metric.

As already noticed, the $[t,t] \subset h$ property of the complexified Lie-algebra of G can be satisfied if the radial Virasoro algebra can be interpreted as an extended algebra containing both N-S and Ramond (h) type algebras and thus possessing also Virasoro generators with half-odd integer conformal weight. This requires that all commutators $\{H_{a,m_a,n_a,k_a}, H_{B,m_b,n_b,k_b}\}$ possess vanishing (non-vanishing) norm when $k_{1a} + k_{1b}$ is integer (half-odd integer) at the point of configuration space corresponding to zero values of t-coordinates serving as coordinates for the configuration space, which is most naturally the maximum of Kähler action. These vanishing conditions, which are completely analogous to the Super Virasoro conditions, hold true only at the maximum of Kähler action.

Hamiltonians can be organized into light like unitary representations of $so(3,1) \times su(3)$ and the symmetry condition $Zg(X,Y) = 0$ requires that the component of the metric is $so(3,1) \times su(3)$ invariant and this condition is satisfied if the component of metric between two different representations D_1 and D_2 of $so(3,1) \times su(3)$ is proportional to Glebch-Gordan coefficient $C_{D_1 D_2, D_S}$

between $D_1 \otimes D_2$ and singlet representation D_S. In particular, metric has components only between states having identical $so(3,1) \times su(3)$ quantum numbers.

Electric-magnetic duality implies that the action of the canonical transformations of the light cone boundary as configuration space isometries is an intrinsic property of the light cone boundary so that the absolute minimization of the Kähler action only determines the conformal factor of the metric. Actually this is precisely as it should be if the group theoretical construction fixes the metric completely. Hence it should be possible by a direct calculation check whether the metric defined by the magnetic flux Hamiltonians is invariant under isometries. Canonical invariance of the metric means that matrix elements of the metric are left translates of the metric along geodesic lines starting from the origin of coordinates, which now naturally corresponds to extremum of Kähler action. Since metric derives from symplectic form this means that the matrix elements of symplectic form given by Poisson brackets of Hamiltonians must be left translates of their values at origin along geodesic line. The matrix elements in question are given by flux Hamiltonians and since canonical transforms of flux Hamiltonian is flux Hamiltonian for the canonical transform of Hamiltonian, it seems that the conditions are satisfied.

5 General expressions for the symplectic and Kähler forms

One can derive general expressions for symplectic and Kähler forms as well as Kähler metric of the configuration space. The fact that these expressions involve only first variation of the Kähler action implies huge simplification of the basic formulas. Duality hypothesis leads to further simplifications of the formulas.

5.1 Closedness requirement

The fluxes of Kähler magnetic and electric fields for the Hamiltonians of $\delta M^4_+ \times CP_2$ suggest a general representation for the components of the symplectic form of the configuration space. The basic requirement is that Kähler form satisfies the defining condition

$$X \cdot J(Y,Z) + J([X,Y],Z) + J(X,[Y,Z]) \quad = \quad 0 \ , \tag{46}$$

where X, Y, Z are now vector fields associated with Hamiltonian functions defining configuration space coordinates.

5.2 Matrix elements of the symplectic form as Poisson brackets

Quite generally, the matrix element of $J(X(H_A), X(H_B))$ between vector fields $X(H_A))$ and $X(H_B))$ defined by the Hamiltonians H_A and H_B of $\delta M^4_+ \times CP_2$ isometries is expressible as Poisson bracket

$$J^{AB} \quad = \quad J(X(H_A), X(H_B)) = \{H_A, H_B\} \ . \tag{47}$$

J^{AB} denotes contravariant components of the symplectic form in coordinates given by a subset of Hamiltonians. The magnetic flux Hamiltonians $Q^{\alpha,\beta}_m(H_{A,k})$ of Eq. 37 provide an explicit representation for the Hamiltonians at the level of configuration space so that the components of the symplectic form of the configuration space are expressible as classical charges for the Poisson brackets of the Hamiltonians of the light cone boundary:

$$J(X(H_A),X(H_B)) = Q_m^{\alpha,\beta}(\{H_A,H_B\}) = Q_m^{\alpha,\beta}(\{H_A,H_B\}) \ . \quad (48)$$

Note that $Q_m^{\alpha,\beta}$ contains unspecified conformal factor depending on canonical invariants characterizing Y^3 and is unspecified superposition of signed and unsigned magnetic fluxes.

Second representation for the symplectic form follows from the hypothesis that Kähler function corresponds to the absolute minimum of Kähler action. The fact that flux Hamiltonians depend on the first variation of Kähler only makes it possible to deduce explicit formulas for the flux Hamiltonians in terms of classical charges associated with Kähler action. If self duality holds, the matrix elements of symplectic form are apart from conformal factor identical with magnetic flux Hamiltonian

$$\begin{aligned} J(X(H_A),X(H_B)) &= Q_e(\{H_A,H_B\}) \\ &= Z\left[Q_m^{\alpha,\beta}(\{H_A,H_B\}) + q_e\{H_A,H_B\}\right] \ . \end{aligned} \quad (49)$$

Here Z and q_e are constants depending on canonical invariants only.

Configuration space Hamiltonians vanish for the extrema of the Kähler function as variational derivatives of the Kähler action. Hence Hamiltonians are good candidates for the coordinates appearing as coordinates in the perturbative functional integral around extrema (with maxima giving dominating contribution). It is clear that configuration space coordinates around a given extremum include only those Hamiltonians, which vanish at extremum (that is those Hamiltonians which span the tangent space of G/H) In Darboux coordinates the Poisson brackets reduce to the canonical form

$$\begin{aligned} \{P^I,Q^J\} &= J^{IJ} = J_I\delta^{I,J} \ . \\ J_I &= 1 \ . \end{aligned} \quad (50)$$

It is not clear whether Darboux coordinates with $J_I = 1$ are possible in the recent case: probably the unit matrix on right hand side of the defining equation is replaced with a diagonal matrix depending on canonical invariants so that one has $J_I \neq 1$. The integration measure is given by the symplectic volume element given by the determinant of the matrix defined by the Poisson brackets of the Hamiltonians appearing as coordinates. The value of the symplectic volume element is given by the matrix formed by the Poisson brackets of the Hamiltonians and reduces to the product

$$Vol = \prod_I J_I$$

in generalized Darboux coordinates.

Kähler potential (that is gauge potential associated with Kähler form) can be written in Darboux coordinates as

$$A = \sum_I J_I P_I dQ^I \ . \quad (51)$$

5.3 General expressions for Kähler form, Kähler metric and Kähler function

The expressions of Kähler form and Kähler metric in complex coordinates can obtained by transforming the contravariant form of the symplectic form from canonical coordinates provided by Hamiltonians to complex coordinates:

$$J^{Z^i\bar{Z}^j} = iG^{Z^i\bar{Z}^j} \quad = \quad \partial_{H^A}Z^i\partial_{H^B}\bar{Z}^j J^{AB} \quad , \tag{52}$$

where J^{AB} is given by the classical Kahler charge for the light cone Hamiltonian $\{H^A, H^B\}$. Complex coordinates correspond to linear coordinates of the complexified Lie-algebra providing exponentiation of the isometry algebra via exponential mapping. What one must know is the precise relationship between allowed complex coordinates and Hamiltonian coordinates: this relationship is in principle calculable. In Darboux coordinates the expressions become even simpler:

$$J^{Z^i\bar{Z}^j} = iG^{Z^i\bar{Z}^j} \quad = \quad \sum_I J(I)(\partial_{P^i}Z^i\partial_{Q^I}\bar{Z}^j - \partial_{Q^I}Z^i\partial_{P^I}\bar{Z}^j) \quad . \tag{53}$$

Kähler function can be formally integrated from the relationship

$$\begin{aligned} A_{Z^i} &= i\partial_{Z^i}K \quad , \\ A_{\bar{Z}^i} &= -i\partial_{Z^i}K \quad . \end{aligned} \tag{54}$$

holding true in complex coordinates. Kähler function is obtained formally as integral

$$K \quad = \quad \int_0^Z (A_{Z^i}dZ^i - A_{\bar{Z}^i}d\bar{Z}^i) \quad . \tag{55}$$

5.4 $Diff(X^3)$ invariance and degeneracy and conformal invariances of the symplectic form

$J(X(H_A), X(H_B))$ defines symplectic form for the coset space G/H only if it is $Diff(X^3)$ degenerate. This means that the symplectic form $J(X(H_A), X(H_B))$ vanishes whenever Hamiltonian H_A or H_B is such that it generates diffeomorphism of the 3-surface X^3. For 7–3 duality this reduce to the much weaker condition that $J(X(H_A), X(H_B))$ vanishes if H_A or H_B generates two-dimensional diffeomorphism $d(H_A)$ at the surface X_i^2.

One can always write

$$J(X(H_A), X(H_B)) = X(H_A)Q(H_B|X_i^2) \quad .$$

If H_A generates diffeomorphism, the action of $X(H_A)$ reduces to the action of the vector field X_A of some X_i^2-diffeomorphism. Since $Q(H_B|r_M)$ is manifestly invariant under the diffemorphisms of X^2, the result is vanishing:

$$X_A Q(H_B|X_i^2) = 0 \quad ,$$

so that $Diff^2$ invariance is achieved.

The radial diffeomorphisms possibly generated by the radial Virasoro algebra do not produce trouble. The change of the flux integrand X under the infinitesimal transformation $r_M \rightarrow r_M + \epsilon r_M^n$

is given by $r_M^n dX/dr_M$. Replacing r_M with $r_M^{-n+1}/(-n+1)$ as variable, the integrand reduces to a total divergence dX/du the integral of which vanishes over the closed 2-surface X_i^2. Hence radial Virasoro generators having zero norm annihilate all matrix elements of the symplectic form. The induced metric of X_i^2 induces a unique conformal structure and since the conformal transformations of X_i^2 can be interpreted as a mere coordinate changes, they leave the flux integrals invariant.

5.5 Complexification and explicit form of the metric and Kähler form

The identification of the Kähler form and Kähler metric in canonical degrees of freedom follows trivially from the identification of the symplectic form and definition of complexification. The requirement that Hamiltonians are eigen states of angular momentum (and possibly Lorentz boost generator), isospin and hypercharge implies physically natural complexification. In order to fix the complexification completely one must introduce some convention fixing which states correspond to 'positive' frequencies and which to 'negative frequencies' and which to zero frequencies that is to decompose the generators of the canonical algebra to three sets Can_+, Can_- and Can_0. One must distinguish between Can_0 and zero modes, which are not considered here at all. For instance, CP_2 Hamiltonians correspond to zero modes.

The natural complexification relies on the imaginary part of the radial conformal weight whereas the real part defines the $g = t + h$ decomposition naturally. The wave vector associated with the radial logarithmic plane wave corresponds to the angular momentum quantum number associated with a wave in S^1 in the case of Kac Moody algebra. One can imagine three options.

1. It is quite possible that the spectrum of k_2 does not contain $k_2 = 0$ at all so that the sector Can_0 could be empty. This complexification is physically very natural since it is manifestly invariant under $SU(3)$ and $SO(3)$ defining the preferred spherical coordinates. The choice of $SO(3)$ is unique if the classical four-momentum associated with the 3-surface is time like so that there are no problems with Lorentz invariance.

2. If $k_2 = 0$ is possible one could have

$$
\begin{aligned}
Can_+ &= \{H^a_{m,n,k=k1_+ik_2}, k_2 > 0\} \ , \\
Can_- &= \{H^a_{m,n,k}, k_2 < 0\} \ , \\
Can_0 &= \{H^a_{m,n,k}, k_2 = 0\} \ .
\end{aligned} \tag{56}
$$

3. If it is possible to $n_2 \neq 0$ for $k_2 = 0$, one could define the decomposition as

$$
\begin{aligned}
Can_+ &= \{H^a_{m,n,k}, k_2 > 0 \ \text{ or } \ k_2 = 0, n_2 > 0\} \ , \\
Can_- &= \{H^a_{m,n,k}, k_2 < 0 \ \text{ or} k_2 = 0, n_2 < 0\} \ , \\
Can_0 &= \{H^a_{m,n,k}, k_2 = n_2 = 0\} \ .
\end{aligned} \tag{57}
$$

   In this case the complexification is unique and Lorentz invariance guaranteed if one can fix the $SO(2)$ subgroup uniquely. The quantization axis of angular momentum could be chosen to be the direction of the classical angular momentum associated with the 3-surface in its rest system.

The only thing needed to get Kähler form and Kähler metric is to write the half Poisson bracket defined by Eq. 59

$$J_f(X(H_A), X(H_B)) = 2Im\,(iQ_f(\{H_A, H_B\}_{-+})) \ ,$$
$$G_f(X(H_A), X(H_B)) = 2Re\,(iQ_f(\{H_A, H_B\}_{-+})) \ . \quad (58)$$

Symplectic form, and thus also Kähler form and Kähler metric, could contain a conformal factor depending on the isometry invariants characterizing the size and shape of the 3-surface. At this stage one cannot say much about the functional form of this factor.

5.6 Comparison of CP_2 Kähler geometry with configuration space geometry

The explicit discussion of the role of $g = t + h$ decomposition of the tangent space of the configuration space provides deep insights to the metric of the symmetric space. There are indeed many questions to be answered. To what point of configuration space (that is 3-surface) the proposed $g = t + h$ decomposition corresponds to? Can one derive the components of the metric and Kähler form from the Poisson brackets of complexified Hamiltonians? Can one characterize the point in question in terms of the properties of configuration space Hamiltonians? Does the central extension of the configuration space reduce to the symplectic central extension of the canonical algebra or can one consider also other options?

5.6.1 Cartan decomposition for CP_2

A good manner to gain understanding is to consider the CP_2 metric and Kähler form at the origin of complex coordinates for which the sub-algebra $h = u(2)$ defines the Cartan decomposition.

1. $g = t + h$ decomposition depends on the point of the symmetric space in general. In case of CP_2 $u(2)$ sub-algebra transforms as $g \circ u(2) \circ g^{-1}$ when the point s is replaced by gsg^{-1}. This is expected to hold true also in case of configuration space (unless it is flat) so that the task is to identify the point of the configuration space at which the proposed decomposition holds true.

2. The Killing vector fields of h sub-algebra vanish at the origin of CP_2 in complex coordinates. The corresponding Hamiltonians need not vanish but their Poisson brackets must vanish. It is possible to add suitable constants to the Hamiltonians in order to guarantee that they vanish at origin.

3. It is convenient to introduce complex coordinates and decompose isometry generators to holomorphic components $J^a_+ = j^{ak}\partial_k$ and $j^a_- = j^{a\bar{k}}\partial_{\bar{k}}$. One can introduce what might be called half Poisson bracket and half inner product defined as

$$\{H^a, H^b\}_{-+} \equiv \partial_{\bar{k}} H^a J^{\bar{k}l} \partial_l H^b$$
$$= j^{ak} J_{k\bar{l}} j^{b\bar{l}} = -i(j^a_+, j^b_-) \ . \quad (59)$$

One can express Poisson bracket of Hamiltonians and the inner product of the corresponding Killing vector fields in terms of real and imaginary parts of the half Poisson bracket:

$$\{H^a, H^b\} = 2Im\left(i\{H^a, H^b\}_{-+}\right) \ ,$$
$$(j^a, j^b) = 2Re\left(i(j^a_+, j^b_-)\right) = 2Re\left(i\{H^a, H^b\}_{-+}\right) \ . \quad (60)$$

What this means that Hamiltonians and their half brackets code all information about metric and Kähler form. Obviously this is of utmost importance in the case of the configuration space metric whose symplectic structure and central extension are derived from those of CP_2.

Consider now the properties of the metric and Kähler form at the origin.

1. The relations satisfied by the half Poisson brackets can be written symbolically as

$$\begin{aligned} &\{h,h\}_{-+} = 0 \ , \\ &Re\,(i\{h,t\}_{-+}) = 0 \ , \quad Im\,(i\{h,t\}_{-+}) = 0 \ , \\ &Re\,(i\{t,t\}_{-+}) \neq 0 \ , \quad Im\,(i\{t,t\}_{-+}) \neq 0 \ . \end{aligned} \tag{61}$$

2. The first two conditions state that h vector fields have vanishing inner products at the origin. The first condition states also that the Hamiltonians for the commutator algebra $[h,h] = SU(2)$ vanish at origin whereas the Hamiltonian for $U(1)$ algebra corresponding to the color hyper charge need not vanish although it can be made vanishing. The third condition implies that the Hamiltonians of t vanish at origin.

3. The last two conditions state that the Kähler metric and form are non-vanishing between the elements of t. Since the Poisson brackets of t Hamiltonians are Hamiltonians of h, the only possibility is that $\{t,t\}$ Poisson brackets reduce to a non-vanishing $U(1)$ Hamiltonian at the origin or that the bracket at the origin is due to the symplectic central extension. The requirement that all Hamiltonians vanish at origin is very attractive aesthetically and forces to interpret $\{t,t\}$ brackets at origin as being due to a symplectic central extension. For instance, for S^2 the requirement that Hamiltonians vanish at origin would mean the replacement of the Hamiltonian $H = cos(\theta)$ representing a rotation around z-axis with $H_3 = cos(\theta) - 1$ so that the Poisson bracket of the generators H_1 and H_2 can be interpreted as a central extension term.

4. The conditions for the Hamiltonians of $u(2)$ sub-algebra state that their variations with respect to g vanish at origin. Thus $u(2)$ Hamiltonians have extremum value at origin.

5. Also the Kähler function of CP_2 has extremum at the origin. This suggests that in the case of the configuration space the counterpart of the origin corresponds to the maximum of the Kähler function.

5.6.2 Cartan algebra decomposition at the level of configuration space

The discussion of the properties of CP_2 Kähler metric at origin provides valuable guide lines in an attempt to understand what happens at the level of the configuration space. The use of the half bracket for the configuration space Hamiltonians in turn allows to calculate the matrix elements of the configuration space metric and Kähler form explicitly in terms of the magnetic or electric flux Hamiltonians.

1. The algebra g is labelled by color quantum numbers of CP_2 Hamiltonians and by the label (m,n,k) labelling the function basis of the light cone boundary. It is assumed that radial conformal weights for the generators spanning the algebra via commutators correspond to the zeros of Riemann Zeta. Thus one has generators with labels $(m = k-1, n = 2k-1, k)$ for trivial zeros and $(m, J, k = 1/2 + iy)$ for non-trivial zeros plus what one obtains as their Poisson brackets. J refers here to angular momentum.

By the previous considerations related to the trivial zeros of Riemann Zeta, the generators having $k = 2r+1, m = 2r, n = 4r-1$ must correspond to gauge degrees of freedom that is algebra h_t in Cartan decomposition $g_t = t_t + h_t$ whereas the generators $k = 2n$ correspond t_t.

By a trial and error one concludes that the structure of the algebra should be following.

$$\begin{array}{lll} g = g_t + g_{nt} \ , & g_t = t_t + h_t \ , & g_{nt} = t_{nt} + h_{nt} \ , \\ [h_i, h_i] \subset h_i \ , & [t_i, t_i] \subset h_i \ , & [h_i, t_i] \subset t_i \ , \\ [h_t, h_{nt}] \subset h_{nt} \ , & [h_t, t_{nt}] \subset t_{nt} \ , & [t_t, h_{nt}] \subset t_{nt} \ , \end{array} \tag{62}$$

Here the subscript $i = t$ refers to trivial zeros of ζ and $i = nt$ to the non-trivial zeros of zeta.

2. Consider first Cartan decomposition for g_t. g_t has grading by $m(k) = k-1$ and odd values of m must correspond to t_t and even values of m to h_t:

$$h_t = g_{t,odd} \ , \quad t_t = g_{t,even} \ . \tag{63}$$

The Cartan condition $[t_t, t_t] \subset h_t$ implies that the Poisson bracket must be accompanied by a multiplication with a function

$$f_t(r_M) = \frac{1}{r_M} \ , \tag{64}$$

which reduces the conformal weight by one unit so that the conditions defining Cartan decomposition are indeed satisfied.

3. Contrary to the earlier assumption, one must require $[g_{nt,-1/2}, g_{nt,-1/2}] \subset g_{nt,-1/2}$ in order to satisfy all conditions implied by Cartan decomposition. This requires the division of $[g_{nt}, g_{nt}]$ type Poisson brackets by function

$$f_{nt}(r_M) = \frac{1}{r_M^{1/2}} \ . \tag{65}$$

Half-odd integer Cartan algebra closes to itself when the sums of non-trivial zeros of Zeta are allowed as conformal weights:

$$\left[g_{nt,-1/2}, g_{nt,-1/2}\right] \subset g_{nt,-1/2} \ . \tag{66}$$

This result is in conflict with the earlier belief that the real part of conformal weight is additive in Poisson bracket. Since the radial logarithmic waves $r_M^{1/2+iy}$ for continuous value of y define a basis equivalent with plane wave basis, the result is actually very natural. The fact that superpositions of y_i define orthogonal states is consistent with the additivity of y in the Poisson bracket.

4. g_{nt} has a natural grading by the number N of the imaginary parts y_i of zeros of Riemann Zeta appearing in the conformal weight of the multiple commutator generated by a repeated commutation in g_{nt}. N is uniquely determined if there exists no non-trivial zeros which are even multiples of each other in the case that generators with opposite values of y are not present and from the value of y_{tot} one can always deduce whether N is even or odd. One must have

$$g^{N=odd}_{nt,-1/2} \subset t_{nt} \ \ , \ \ g^{N=even}_{nt,-1/2} \subset h_{nt} \ \ . \tag{67}$$

Here the superscript "odd" *resp.* "even" tells that generator corresponds to odd *resp.* even N.

5. For the commutators $[h_i, t_j]$, $i \neq j$ the real parts of the conformal weights are of form $n - 1/2$. Cartan decomposition is achieved if also $[g_t, g_{nt}]$ commutators involve the division of the Poisson bracket by

$$f_{n,nt}(r_M) = \frac{1}{r_M} \ \ . \tag{68}$$

This leads to the identification of the full Cartan decomposition of g_{nt} as

$$t_{nt} = g^{odd}_{nt,even-1/2} \oplus g^{even}_{nt,odd-1/2} \ \ , \ \ h_{nt} = g^{odd}_{nt,odd-1/2} \oplus g^{even}_{nt,even-1/2} \ \ , \ \ . \tag{69}$$

It is easy to verify that all conditions of the Cartan decomposition are obeyed.

6. The examination of the orthogonality conditions shows that orthogonality of the super-canonical basis is achieved if one allows only $n = 0, 1$ in the conformal weights of t_{nt}. This conforms with the assumption that Virasoro generators L_n, $n \geq 0$ act as gauge symmetries in the super-canonical algebra. Due the special features of super-canonical generators with $h = 2n$ gauge invariance allows t_t to have conformal weights $h = 2n$, $n \geq 0$. This means that super-canonical conformal weights corresponding to the configuration space tangent space vector fields reside at the positive real axis at points $h = 2n$, $n > 0$, and at the two vertical lines $Re(h) = \pm 1/2$ with imaginary parts being of form $\sum n_i y_i$, $\sum n_i$ odd at $Re(h) = -1/2$ and $\sum n_i$ even at $Re(h) = 1/2$. These rules apply also to the conformal weights of the physical states with bound state conformal weights assignable to the zeros of polyzetas forming a possible exception.

5.6.3 The form of extension and metric at the maximum of the Kähler function

The best guess for the counterpart of origin of CP_2, call it s, is as the maximum of the Kähler function for given values of zero modes. The isometry invariance of the metric can be used to demonstrate that Kähler function cannot have saddle points so that the maximum is unique. The generators h have zero norm and could even vanish at s. The Hamiltonians of the entire algebra can be made vanishing at s by an addition of a constant term (this might be needed in the case of generators having vanishing color isospin and hypercharge). The Hamiltonians of the algebra have extremum at s and this might help to derive information about the maximum of the Kähler function.

1. The complexification corresponds for g_t to the changes of the sign of angular momentum quantum number $m(k) = k - 1$ and for gnt to $y \rightarrow -y$ for the imaginary part of the zero of Riemann Zeta.

2. The counterpart for the $u(1)$ algebra of CP_2 must be a sub-algebra of the algebra g commuting with it. Since the algebra generators are proportional to all $S^2 \times SU(3)$ Hamiltonians, the counterpart of the $U(1)$ algebra is the Abelian algebra obtained by putting $S^2 \times CP_2$ part constant in g_t and g_{nt}. In g_t Abelian algebra corresponds to functions r_M^{2n-1} and in g_{nt} to the function $r_M^{1/2}$.

3. The analogy with CP_2 encourages the expectation that the matrix elements of the Poisson brackets of the generators of t reduce at the point s corresponding to the maximum of Kähler function to the factorized form

$$
\begin{aligned}
\{Q(H_t^{a,k_1}), Q(H_t^{b,k_2+iy_1})\} &= g_{ab} \times Q^{\alpha,\beta}(r_M^{k_1+k_2-1}) \ , \\
\{Q(H_{nt}^{a,k_1+iy_1}), Q(H_{nt}^{b,k_2+iy_2})\} &= g_{ab}\delta(y_1+y_2) \times Q_m^{\alpha,\beta}(r_M^{k_1+k_2-1/2}) \ , \\
\{Q(H_t^{a,k_1}), Q(H_{nt}^{b,k_2+iy_2})\} &= g_{ab}\delta(y_2) \times Q_m^{\alpha,\beta}(r_M^{k_1+k_2-1/2}) \ .
\end{aligned} \tag{70}
$$

   Here the notations used are

$$
H_t^{a,k} = r_M^k \times H^a \ , \quad H_{nt}^{a,k-1/2-iy} = r_M^{k-1/2-iy} \times H^a \ , \quad y_i = \sum_k y_{i,k} \ . \tag{71}
$$

   The labels $\alpha, beta$ refer to the two parameters appearing in the general definition of configuration space Poisson bracket. The labels a and b correspond to the color and spin quantum numbers labelling $S^2 \times CP_2$ Hamiltonians, and y_i denotes the sum of the imaginary parts for zeros of Riemann Zeta. In the last formula of 70 g_{ab} involves inner product between $SO(3)$ and $SO(2)$ type states.

   The picture is completely analogous to that in the case of CP_2. g_{ab} transforms under $SO(3)$ and color isometries like spin and color singlet and corresponds to the Glebsch-Gordan coefficients associated with the singlet representation in the tensor product of the representations involved. This means diagonality with respect to spin and color isospin and hyper charge.

4. The simplification at the point corresponding to the origin would be rather dramatic since the CP_2 canonical invariants discussed earlier correspond to Poisson brackets of δM_+^4 Hamiltonians and have interpretation as components of the configuration space metric in δM_+^4 degrees of freedom and resulting from an ordinary $U(1)$ central extension. The matrix elements reduce to a form diagonal with respect to the Lorentz and color quantum numbers and are given by Glebch-Gordan coefficients for unitary representations of Lorentz group. This would mean that one can calculate the configuration space metric almost completely at the point in question and apart from the dependence on zero modes.

5.7 Comparison with loop groups

It is useful to compare the recent approach to the geometrization of the loop groups consisting of maps from circle to Lie group G [17], which served as the inspirer of the configuration space geometry approach but later turned out to not apply as such in TGD framework.

In the case of loop groups the tangent space T corresponds to the local Lie-algebra $T(k, A) = exp(ik\phi)T_A$, where T_A generates the finite-dimensional Lie-algebra g and ϕ denotes the angle variable of circle; k is integer. The complexification of the tangent space corresponds to the decomposition

$$T = \{X(k>0,A)\} \oplus \{X(k<0,A)\} \oplus \{X(k=0,A)\} = T_+ \oplus T_- \oplus T_0$$

of the tangent space. Metric corresponds to the central extension of the loop algebra to Kac Moody algebra and the Kähler form is given by

$$J(X(k_1<0,A), X(k_2>0,B)) = k_2\delta(k_1+k_2)\delta(A,B) \ .$$

In present case the finite dimensional Lie algebra g is replaced with the Lie-algebra of the canonical transformations of $\delta M^4_+ \times CP_2$ centrally extended using symplectic extension. The scalar function basis on circle is replaced with the function basis on an interval of length Δr_M with periodic boundary conditions; effectively one has circle also now.

The basic difference is that one can consider two kinds of central extensions now.

1. Central extension is most naturally induced by the natural central extension ($\{p,q\} = 1$) defined by Poisson bracket. This extension is anti-symmetric with respect to the generators of the canonical group: in the case of the Kac Moody central extension it is symmetric with respect to the group G. The canonical transformations of CP_2 might correspond to non-zero modes also because they are not exact symmetries of Kähler action. The situation is however rather delicate since $k = 0$ light cone harmonic has a diverging norm due to the radial integration unless one poses both lower and upper radial cutoffs although the matrix elements would be still well defined for typical 3-surfaces. For Kac Moody group $U(1)$ transformations correspond to the zero modes. Light cone function algebra can be regarded as a local $U(1)$ algebra defining central extension in the case that only CP_2 canonical transformations local with respect to δM^4_+ act as isometries: for Kac Moody algebra the central extension corresponds to an ordinary $U(1)$ algebra. In the case that entire light cone canonical algebra defines the isometries the central extension reduces to a $U(1)$ central extension.

2. In the recent case there is a strong objection against the standard Kac Moody central extension, defining the metric for loop groups. The value of the conformal weight n can have both positive and negative values in case of the Kac Moody algebra. Now however the real part of the conformal weight must be positive and half-integer valued. It seems that this objection alone is enough to exclude Kac Moody extension. This is not bad news since quaternion conformal algebra seems to allow Kac Moody extension.

5.8 Symmetric space property implies Ricci flatness and isometric action of canonical transformations

The basic structure of symmetric spaces is summarized by the following structural equations

$$\begin{aligned} &g = h + t \ , \\ &[h,h] \subset h \ , \quad [h,t] \subset t \ , \quad [t,t] \subset h \ . \end{aligned} \tag{72}$$

In present case the equations imply that all commutators of the Lie-algebra generators of $Can(\neq 0)$ having non-vanishing integer valued radial quantum number n_2, possess zero norm. This condition is extremely strong and guarantees isometric action of $Can(\delta M^4_+ \times CP_2)$ as well as Ricci flatness of the configuration space metric.

The requirement $[t,t] \subset h$ and $[h,t] \subset t$ are satisfied if the generators of the isometry algebra possess generalized parity P such that the generators in t have parity $P = -1$ and the generators belonging to h have parity $P = +1$. Conformal weight n must somehow define this parity. The first possibility to come into mind is that odd values of n correspond to $P = -1$ and even values to $P = 1$. Since n is additive in commutation, this would automatically imply $h \oplus t$ decomposition with the required properties. This assumption looks however somewhat artificial. TGD however forces a generalization of Super Algebras and N-S and Ramond type algebras can be combined to a larger algebra containing also Virasoro and Kac Moody generators labelled by half-odd integers. This suggests strongly that isometry generators are labelled by half integer conformal weight and that half-odd integer conformal weight corresponds to parity $P = -1$ whereas integer conformal weight corresponds to parity $P = 1$. Coset space would structure would state conformal invariance of the theory since Super Canonical generators with integer weight would correspond to zero modes.

Quite generally, the requirement that the metric is invariant under the flow generated by vector field X leads together with the covariant constancy of the metric to the Killing conditions

$$X \cdot g(Y,Z) \quad = \quad 0 = g([X,Y],Z) + g(Y,[X,Z]) \ . \tag{73}$$

If the commutators of the complexified generators in $Can(\neq 0)$ have zero norm then the two terms on the right hand side of Eq. (73) vanish separately. This is true if the conditions

$$Q_m^{\alpha,\beta}(\{H^A,\{H^B,H^C\}\}) \quad = \quad 0 \ , \tag{74}$$

are satisfied for all triplets of Hamiltonians in $Can_{\neq 0}$. These conditions follow automatically from the $[t,t] \subset h$ property and guarantee also Ricci flatness as will be found later.

It must be emphasized that for Kähler metric defined by purely magnetic fluxes, one cannot pose the conditions of Eq. (74) as consistency conditions on the initial values of the time derivatives of imbedding space coordinates whereas in general case this is possible. If the consistency conditions are satisfied for a single surface on the orbit of canonical group then they are satisfied on the entire orbit. Clearly, isometry and Ricci flatness requirements and the requirement of time reversal invariance might well force Kähler electric alternative.

5.9 How to find Kähler function?

If one has found the expansion of configuration space Kähler form in terms of electric fluxes one can solve also the Kähler function from the defining partial differential equations $J_{k\bar{l}} = \partial k \partial_{\bar{l}} K$. The solution is not unique since the equation allows the symmetry

$$K \rightarrow K + f(z^k) + \overline{f(z^k)} \ ,$$

where f is arbitrary holomorphic function of z^k. This non-uniqueness is probably eliminated by the requirement that Kähler function vanishes for vacuum extremals. This in turn makes in principle possible to find the maxima of Kähler function and to perform functional integration perturbatively around them.

Electric-magnetic duality implies that, apart from conformal factor depending on isometry invariants, one can solve Kähler metric without any knowledge on the initial values of the time derivatives of the imbedding space coordinates. Apart from conformal factor the resulting geometry is purely intrinsic to δCH. The role of Kähler action is only to to define $Diff^4$ invariance and give the rule how the metric is translated to metric on arbitrary point of CH. The degeneracy of the absolute minima also implies that configuration space has multi-sheeted structure analogous to that encountered in case of Riemann surfaces.

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